Multistar Decomposition of Complete Multigraphs

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Abstract

In this paper, we consider the problem of decomposing complete multigraphs into multistars (a multistar is a star with multiple edges allowed). We obtain a criterion for the decomposition of the complete multigraph λK_n into multistars with prescribed number of edges, but the multistars in the decomposition with the same number of edges are not necessarily isomorphic. We also consider the problem of decomposing λK_n into isomorphic multistars and propose a conjecture about the decomposition of $2K_n$ into isomorphic multistars.

*Research supported by NSC90-2115-M008-012
†Research supported by NSC90-2115-M424-002

1 Introduction

For a positive integer n, let K_n denote a complete graph on n vertices, and S_n denote a star with n edges. Both K_n and S_n are simple graphs. For a positive integer λ and a graph G, let λG denote the multigraph obtained from G by replacing each edge e of G by λ edges with the same ends as e. Thus λK_n is the multigraph on n vertices such that there are λ edges between every pair of vertices. We call λK_n a *complete multigraph.*

For multigraphs G and H, we say that G has an *H-decomposition* if the edges of G can be partitioned into several subgraphs of G of which each is isomorphic to H. The family of these subgraphs is called an *H-decomposition* of G. M. Tarsi [7] considered the S_m-decomposition of λK_n. An S_m-decomposition $\mathcal{D}$ of λK_n is said to be *center balanced* if the centers of the stars in $\mathcal{D}$ are distributed among the vertices of λK_n as uniformly as possible (i.e., if $\alpha(v)$ denotes the number of stars in $\mathcal{D}$ with centers at $v \in V(\lambda K_n)$, then $|\alpha(x) - \alpha(y)| \leq 1$ for all $x, y \in V(\lambda K_n)$). M. Tarsi obtained the following result.

Theorem 1.1 [7] *Let λ, m, n be positive integers with $n \geq 2$. Then λK_n has an S_m-decomposition if and only if*

$$2m \,|\lambda n(n-1) \quad \text{and}$$

$$\begin{cases} m \leq n/2 & \text{for } \lambda = 1 \\ m \leq n-1 & \text{for even } \lambda \\ m \leq (n-1)/(1+1/\lambda) & \text{for odd } \lambda \geq 3 \,. \end{cases}$$

Furthermore, under the same conditions the S_m-decomposition can be required to be center balanced.

Proof. The center balanced requirement can be seen from the proof of the theorem in [7]. □

C. Lin and T.-W. Shyu considered the problem of decomposing K_n into stars (not necessarily isomorphic) and obtained the following result.

Theorem 1.2 [4] *Let $m_1 \geq m_2 \geq \cdots \geq m_\ell$ be nonnegative integers. Then K_n can be decomposed into stars $S_{m_1}, S_{m_2}, \cdots, S_{m_\ell}$ if and only if*

$$\sum_{i=1}^{\ell} m_i = \binom{n}{2} \quad \text{and}$$

$$\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} (n-i) \qquad \text{for } k = 1, 2, \cdots, n-1. \qquad \square$$

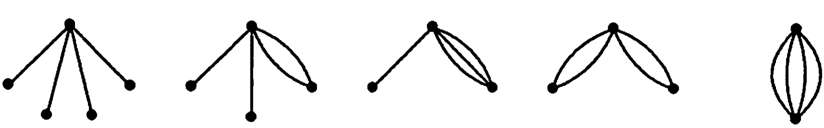
The following more general result concerning the decomposition of uniform hypergraphs into hyperstars is due to Lonc. A *hyperstar* is a nonempty family F of sets with $\bigcap_{X \in F} X \neq \emptyset$.

Theorem 1.3 [5] *Suppose that t and n are positive integers such that $t \leq n-1$. Let $m_1 \geq m_2 \geq \cdots \geq m_\ell$ be nonnegative integers. Then the family of all t-element subsets of an n-element set can be partitioned into l hyperstars of cardinalities $m_1, m_2, \cdots, m_l$ respectively if and only if*

$$\sum_{i=1}^{\ell} m_i = \binom{n}{t} \quad \text{and}$$

$$\sum_{i=1}^{k} m_i \leq \binom{n}{t} - \binom{n-k}{t} \quad \text{for } k = 1, 2, \cdots, n-t+1. \qquad \square$$

Obviously Theorem 1.2 is a special case of Theorem 1.3 with $t = 2$. Our discussions will be restricted to graphs and multigraphs. A *multistar* is a star with multiple edges allowed. We use $S_{1^{n_1}2^{n_2}\cdots k^{n_k}}$ to denote a multistar which has n_1 edges with multiplicity 1, n_2 edges with multiplicity 2, $\cdots$, and n_k edges with multiplicity k. As an illustration, the multistars $S_{1^4}, S_{1^2 2^1}, S_{1^1 2^0 3^1}, S_{1^0 2^2}, S_{1^0 2^0 3^0 4^1}$ are exhibited in Fig. 1. Obviously $S_{1^{n_1}2^{n_2}\cdots k^{n_k}}$ has $n_1 + 2n_2 + \cdots + kn_k$ edges. Also $S_{1^{n_1}} = S_{n_1}$. The multistars with the same number of edges need not be isomorphic. The multistars in Fig. 1 are all the multistars with 4 edges.



S_{1^4} $\quad S_{1^2 2^1}$ $\quad S_{1^1 2^0 3^1}$ $\quad S_{1^0 2^2}$ $\quad S_{1^0 2^0 3^0 4^1}$

Fig. 1

In this paper, we consider the problem of decomposing complete multigraphs into multistars. In Section 2, we obtain a criterion for the decomposition of the complete multigraph λK_n into multistars with prescribed number of edges, but the multistars in the decomposition with the same number of edges need not be isomorphic. In Section 3, we consider the isomorphic multistar decomposition of λK_n and propose a conjecture about that of $2K_n$.

2 Decomposition of complete multigraphs into multistars

In this section we consider the problem of decomposing λK_n into multistars. We extend the results in [4]. Let us begin with the following proposition, which is a criterion for the existence of an orientation of a given graph with prescribed outdegrees. This result was obtained by M. Tarsi [8]. Though [8] deals with simple graphs, Proposition 2.1 holds for multigraphs, the proof of which is exactly the same as that given in [8].

Proposition 2.1 [8] *Let $G = (V, E)$ be a given multigraph and δ a non-negative integer function defined on V. For $A \subseteq V$, let $e(A)$ denote the number of edges with two ends in A, and let $\delta(A)$ denote $\sum \delta(x)$ where the summation is taken over all vertices x's in A. A necessary and sufficient condition for the existence of an orientation of G for which the outdegree of every vertex $x \in V$ is $\delta(x)$ is*

$$\begin{aligned} \delta(V) &= |E| \quad \textit{and} \\ \delta(A) &\geq e(A) \quad \textit{for every } A \subseteq V \textit{ with } 2 \leq |A| \leq |V|. \end{aligned}$$

□

A criterion for the existence of an orientation of λK_n with prescribed outdegrees follows from the above proposition.

Theorem 2.2 *Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be nonnegative integers. A necessary and sufficient condition for the existence of an orientation of λK_n for which the outdegrees of the vertices are $d_1, d_2, \cdots, d_n$ is as follows*

$$\begin{aligned} \sum_{i=1}^{n} d_i &= \lambda\binom{n}{2} \quad \textit{and} \\ \sum_{i=1}^{k} d_i &\geq \lambda\binom{k}{2} \quad \textit{for } k = 2, 3, \cdots, n-1. \end{aligned}$$

Proof. Let $V(\lambda K_n) = \{v_1, v_2, \cdots, v_n\}$. Define a function δ on $V(\lambda K_n)$ by $\delta(v_i) = d_i$, $i = 1, 2, \cdots, n$. The theorem follows from Proposition 2.1 and the following observations. The condition that $\sum_{i=1}^{n} d_i = \lambda\binom{n}{2}$ is equivalent to that $\delta(V(\lambda K_n)) = e(V(\lambda K_n))$. For $2 \leq k \leq n-1$, the condition that $\sum_{i=1}^{k} d_i \geq \lambda\binom{k}{2}$ is equivalent to that $\delta(A) \geq e(A)$ for every $A \subseteq V(\lambda K_n)$ with $|A| = k$. □

The case $\lambda = 1$ in Theorem 2.2 is a well known theorem of Landau [1, 2, 6]. Theorem 2.2 can be rewritten as follows.

Theorem 2.3 *Let $m_1 \geq m_2 \geq \cdots \geq m_n$ be nonnegative integers. A necessary and sufficient condition for the existence of an orientation of λK_n for which the outdegrees of the vertices are $m_1, m_2, \cdots, m_n$ is as follows*

$$\begin{aligned} \sum_{i=1}^{n} m_i &= \lambda\binom{n}{2} \quad \textit{and} \\ \sum_{i=1}^{k} m_i &\leq \lambda\sum_{i=1}^{k}(n-i) \qquad \textit{for } k = 1, 2, \cdots, n-2. \end{aligned}$$

Proof. For $i = 1, 2, \cdots, n$, let $d_i = m_{n-i+1}$. Then $d_1 \leq d_2 \leq \cdots \leq d_n$. (Necessity) Suppose that there exists an orientation of λK_n such that the outdegrees of the vertices are $m_1, m_2, \cdots, m_n$. Equivalently, the outdegrees of the vertices of the oriented λK_n are $d_1, d_2, \cdots, d_n$. By the necessity part of Theorem 2.2, we have

$$\begin{aligned} \sum_{i=1}^{n} d_i &= \lambda\binom{n}{2} \quad \text{and} \\ \sum_{i=1}^{k} d_i &\geq \lambda\binom{k}{2} \qquad \text{for } k = 2, 3, \cdots, n-1. \end{aligned}$$

Then $\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} d_i = \lambda\binom{n}{2}$ and

$$\begin{aligned} \sum_{i=1}^{k} m_i &= \sum_{i=1}^{n} m_i - \sum_{i=k+1}^{n} m_i \\ &= \sum_{i=1}^{n} m_i - \sum_{i=1}^{n-k} d_i \\ &\leq \lambda\binom{n}{2} - \lambda\binom{n-k}{2} \qquad 2 \leq n-k \leq n-1 \\ &= \lambda\sum_{i=1}^{k}(n-i). \end{aligned}$$

Thus $\sum_{i=1}^{k} m_i \leq \lambda\sum_{i=1}^{k}(n-i)$ for $1 \leq k \leq n-2$.
(Sufficiency) Suppose that

$$\sum_{i=1}^{n} m_i = \lambda\binom{n}{2} \quad \text{and}$$

$$\sum_{i=1}^{k} m_i \leq \lambda \sum_{i=1}^{k}(n-i) \qquad \text{for } k = 1, 2, \cdots, n-2.$$

Then $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} m_i = \lambda \binom{n}{2}$ and

$$\begin{aligned}
\sum_{i=1}^{k} d_i &= \sum_{i=n-k+1}^{n} m_i \\
&= \sum_{i=1}^{n} m_i - \sum_{i=1}^{n-k} m_i \\
&\geq \lambda \binom{n}{2} - \lambda \sum_{i=1}^{n-k}(n-i) \qquad 1 \leq n-k \leq n-2 \\
&= \lambda \binom{k}{2}.
\end{aligned}$$

Thus $\sum_{i=1}^{k} d_i \geq \lambda \binom{k}{2}$ for $2 \leq k \leq n-1$. By the sufficiency part of Theorem 2.2, there exists an orientation of λK_n such that the outdegrees of the vertices are $d_1, d_2, \cdots, d_n$; equivalently, the outdegrees are $m_1, m_2, \cdots, m_n$. □

We need the following lemma for our discussions.

Lemma 2.4 *Let n and ℓ be integers such that $n \geq 2$ and $\ell \geq n+1$. Suppose that $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, m_1, m_2, \cdots, m_\ell$ are nonnegative numbers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}$, $m_1 \geq m_2 \geq \cdots \geq m_n \geq \cdots \geq m_\ell$ and*

$$\begin{aligned}
\sum_{i=1}^{\ell} m_i &= \sum_{i=1}^{n-1} \lambda_i (n-i), \\
\sum_{i=1}^{k} m_i &\leq \sum_{i=1}^{k} \lambda_i (n-i) \qquad \textit{for} \quad k = 1, 2, \cdots, n-2.
\end{aligned}$$

Let $m'_1 \geq m'_2 \geq \cdots \geq m'_{\ell-1}$ be a rearrangement of $m_1, m_2, \cdots, m_{n-1}, m_n + m_\ell, m_{n+1}, m_{n+2}, \cdots, m_{\ell-1}$. Then

$$\begin{aligned}
\sum_{i=1}^{\ell-1} m'_i &= \sum_{i=1}^{n-1} \lambda_i (n-i) \quad \textit{and} \\
\sum_{i=1}^{k} m'_i &\leq \sum_{i=1}^{k} \lambda_i (n-i) \qquad \textit{for} \quad k = 1, 2, \cdots, n-2.
\end{aligned}$$

Proof. The required equality follows from the fact that $\sum_{i=1}^{\ell-1} m'_i = \sum_{i=1}^{\ell} m_i$. Now we prove the inequalities. Suppose that $m'_t = m_n + m_\ell$ where $t \le n$. We can see that

$$m'_i = \begin{cases} m_i, & i = 1, 2, \cdots, t-1 \\ m_{i-1}, & i = t+1, t+2, \cdots, n. \end{cases}$$

Suppose, to the contrary of the conclusion, that $\sum_{i=1}^{j} m'_i > \sum_{i=1}^{j} \lambda_i(n-i)$ for some integer j where $t \le j \le n-2$. Then

$$\begin{aligned} 2m_n &\ge m_n + m_\ell \\ &= m'_t \\ &= \sum_{i=1}^{j} m'_i - \sum_{i=1}^{j-1} m_i \\ &> \sum_{i=1}^{j} \lambda_i(n-i) - \sum_{i=1}^{j-1} \lambda_i(n-i) \\ &= \lambda_j(n-j). \end{aligned}$$

Hence $\quad m_n > \lambda_j(n-j)/2.$

Then

$$\begin{aligned} \sum_{i=1}^{n-1} \lambda_i(n-i) &= \sum_{i=1}^{\ell-1} m'_i \\ &\ge \sum_{i=1}^{j} m'_i + \sum_{i=j+1}^{n} m'_i \\ &= \sum_{i=1}^{j} m'_i + \sum_{i=j}^{n-1} m_i \\ &\ge \sum_{i=1}^{j} m'_i + (n-j)m_n \\ &> \sum_{i=1}^{j} \lambda_i(n-i) + \lambda_j(n-j)^2/2 \\ &\ge \sum_{i=1}^{j} \lambda_i(n-i) + \lambda_j \sum_{i=j+1}^{n-1} (n-i) \end{aligned}$$

$$\geq \sum_{i=1}^{n-1} \lambda_i(n-i).$$

Thus we obtain $\sum_{i=1}^{n-1} \lambda_i(n-i) > \sum_{i=1}^{n-1} \lambda_i(n-i)$; this absurdity confirms the lemma. □

Letting $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$ in the above lemma, we have the following.

Lemma 2.5 *Let ℓ and n be integers such that $n \geq 2$ and $\ell \geq n+1$. Suppose that $\lambda, m_1, m_2, \cdots, m_\ell$ are nonnegative numbers such that $m_1 \geq m_2 \geq \cdots \geq m_n \geq \cdots \geq m_\ell$ and*

$$\begin{aligned} \sum_{i=1}^{\ell} m_i &= \lambda \binom{n}{2}, \\ \sum_{i=1}^{k} m_i &\leq \lambda \sum_{i=1}^{k} (n-i) \qquad \text{for} \quad k = 1, 2, \cdots, n-2. \end{aligned}$$

Let $m'_1 \geq m'_2 \geq \cdots \geq m'_{\ell-1}$ be a rearrangement of $m_1, m_2, \cdots, m_{n-1}, m_n + m_\ell, m_{n+1}, m_{n+2}, \cdots, m_{\ell-1}$. Then

$$\begin{aligned} \sum_{i=1}^{\ell-1} m'_i &= \lambda \binom{n}{2} \quad \text{and} \\ \sum_{i=1}^{k} m'_i &\leq \lambda \sum_{i=1}^{k} (n-i) \qquad \text{for} \quad k = 1, 2, \cdots, n-2. \end{aligned}$$

□

Now we prove the main result of this section.

Theorem 2.6 *Let $m_1 \geq m_2 \geq \cdots \geq m_\ell$ be nonnegative integers. Then the complete multigraph λK_n can be decomposed into ℓ multistars, each of which has m_i edges ($i = 1, 2, \cdots, \ell$) if and only if*

$$\begin{aligned} \ell &\geq n-1, \\ \sum_{i=1}^{\ell} m_i &= \lambda \binom{n}{2}, \\ \sum_{i=1}^{k} m_i &\leq \lambda \sum_{i=1}^{k} (n-i) \qquad \text{for} \quad k = 1, 2, \cdots, n-2. \end{aligned}$$

Proof. (Necessity) Let $V(\lambda K_n) = \{v_1, v_2, \cdots, v_n\}$ and $\mathcal{D}$ be the decomposition of λK_n into ℓ multistars. First we prove $\ell \geq n-1$. Suppose, on the contrary, that $\ell \leq n-2$. Then there are two vertices, say v_1, v_2, which are not centers of multistars in $\mathcal{D}$. Thus the edges between v_1 and v_2 belong to none of the multistars in $\mathcal{D}$. This impossibility establishes $\ell \geq n-1$.

The required equality is obvious. To prove the remaining inequalities, orient the edges of λK_n as follows. For an edge belonging to a multistar in $\mathcal{D}$ with the center at v_j, we orient it outward from v_j. Suppose now the outdegree of v_j is $\overline{m}_j$ for $j = 1, 2, \cdots, n$. Obviously $\overline{m}_j = \sum m_i$ where m_i's are the numbers of edges of those multistars in $\mathcal{D}$ with centers at v_j ($\overline{m}_j = 0$ if there is no multistar with the center at v_j). Let $m'_1 \geq m'_2 \geq \cdots \geq m'_n$ be the rearrangement of $\overline{m}_1, \overline{m}_2, \cdots, \overline{m}_n$. By the necessity part of Theorem 2.3,

$$\sum_{i=1}^{k} m'_i \leq \lambda \sum_{i=1}^{k} (n-i) \qquad \text{for } k = 1, 2, \cdots, n-2.$$

Thus

$$\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} m'_i \leq \lambda \sum_{i=1}^{k} (n-i) \qquad \text{for} \quad k = 1, 2, \cdots, n-2.$$

(Sufficiency) We prove the result by induction on ℓ. Since in the case $\ell = n-1$ we can let $m_n = 0$, let us begin with $\ell = n$. By the sufficiency part of Theorem 2.3, there is an orientation of λK_n for which the outdegrees of the vertices are $m_1, m_2, \cdots, m_n$. In a natural way λK_n can be decomposed into n multistars, each of which has m_i edges ($i = 1, 2, \cdots, n$). Now let $\ell \geq n+1$. Suppose that the result holds for $\ell - 1$. Let $m'_1 \geq m'_2 \geq \cdots \geq m'_{\ell-1}$ be a rearrangement of $m_1, m_2, \cdots, m_{n-1}, m_n + m_\ell, m_{n+1}, m_{n+2}, \cdots, m_{\ell-1}$. By Lemma 2.5, $\sum_{i=1}^{\ell-1} m'_i = \lambda\binom{n}{2}$ and $\sum_{i=1}^{k} m'_i \leq \lambda \sum_{i=1}^{k} (n-i)$ for $k = 1, 2, \cdots, n-2$. By the induction hypothesis, λK_n can be decomposed into $\ell - 1$ multistars, each of which has m'_i edges ($i = 1, 2, \cdots, \ell - 1$). A multistar with $m_n + m_l$ edges can easily be decomposed into two multistars with m_n and m_l edges, respectively. Thus λK_n can be decomposed into ℓ multistars, each of which has m_i edges ($i = 1, 2, \cdots, \ell$). This completes the proof. □

Note that Theorem 1.2 is a special case of Theorem 2.6 with $\lambda = 1$. Note also that in Theorem 2.6 the multistars in the decomposition of λK_n with the same number of edges may not be isomorphic. The following theorem follows easily from Theorem 2.6.

Theorem 2.7 *Let λ, n, m be positive integers. Then λK_n can be decomposed into multistars, each of which has m edges (these stars need not be isomorphic) if and only if $m \mid \lambda\binom{n}{2}$ and $m \leq \lambda n/2$.*

Proof. (Necessity) Suppose that λK_n can be decomposed into ℓ multistars, each of which has m edges. By the necessity part of Theorem 2.6, $\ell \geq n-1$ and $m\ell = \lambda\binom{n}{2}$. Thus $m \mid \lambda\binom{n}{2}$ and $m = \lambda n(n-1)/2\ell \leq \lambda n/2$.
(Sufficiency) Let ℓ be the number such that $m\ell = \lambda\binom{n}{2}$. Since $m \leq \lambda n/2$, we have $\ell \geq n-1$. Let $m_1 = m_2 = \cdots = m_\ell = m$. Then

$$\sum_{i=1}^{\ell} m_i = \lambda\binom{n}{2} \qquad \text{and}$$

for $k = 1, 2, \cdots, n-2$,

$$\begin{aligned}\sum_{i=1}^{k} m_i &= mk \\ &\leq \lambda nk/2 \\ &\leq \lambda\sum_{i=1}^{k}(n-i).\end{aligned}$$

By the sufficiency part of Theorem 2.6, λK_n can be decomposed into ℓ multistars, each of which has m edges. □

3 Decomposition of complete multigraphs into isomorphic multistars

The multistars with m edges in Theorem 2.7 may not be isomorphic. Actually the condition in the theorem does not guarantee decompositions into isomorphic multistars. For example, by Theorem 2.7, $3K_5$ can be decomposed into 6 multistars, each of which has 5 edges. But, as shown in the following remark, we can not require that these 6 multistars to be isomorphic.

Remark. Let S be an arbitrary multistar with 5 edges which is contained in $3K_5$. Then $3K_5$ does not have an S-decomposition.

Check. Suppose, on the contrary, that $3K_5$ has an S-decomposition $\mathcal{D}$ where S is a multistar with 5 edges. Since S is contained in $3K_5$, there are four candidates for S, which are $S_{1^3 2^1}, S_{1^2 2^0 3^1}, S_{1^1 2^2}$ and $S_{1^0 2^1 3^1}$. These multistars are given in Fig. 2. Assume that $3K_5$ is decomposed into six S's. Let $V(3K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. An edge with odd multiplicity is said to be an *odd edge*. Since each edge v_iv_j $(1 \leq i < j \leq 5)$ in $3K_5$ has multiplicity 3, v_iv_j is an odd edge. Thus $3K_5$ has ten odd edges. We consider three cases and show that each case leads to a contradiction.

$S_{1^3 2^1}$ $S_{1^2 2^0 3^1}$ $S_{1^1 2^2}$ $S_{1^0 2^1 3^1}$

Fig. 2

Case 1. $S = S_{1^1 2^2}$ or $S_{1^0 2^1 3^1}$.

Since $3K_5$ has ten odd edges and S has only one odd edge, it is impossible for $3K_5$ to be decomposed into six S's.

Case 2. $S = S_{1^3 2^1}$.

Since there are 6 multistars in $\mathcal{D}$ and $3K_5$ has 5 vertices, we see that there are two multistars in $\mathcal{D}$ with centers at the same vertex, say at v_1. Let G be the subgraph of $3K_5$ induced by these two multistars. We see that G is isomorphic to $S_{1^0 2^2 3^2}$, which has v_1 as the center, and v_2, v_3, v_4, v_5 as endvertices. Without loss of generality, assume that G has two edges joining v_1 and v_2, two edges joining v_1 and v_3, three edges joining v_1 and v_4, and three edges joining v_1 and v_5. The graph G is exhibited in Fig. 3. Then it is impossible to have a multistar in $\mathcal{D}$ with its center at v_4, since the multistar in $\mathcal{D}$ (which is isomorphic to $S_{1^3 2^1}$) must be a spanning connected subgraph of $3K_5$ but the edges joining v_4 and v_1 in $3K_5$ have been exhausted. Similarly there is no multistar in $\mathcal{D}$ with its center at v_5. Thus the edges joining v_4 and v_5 do not belong to any multistar in $\mathcal{D}$; this is absurd.

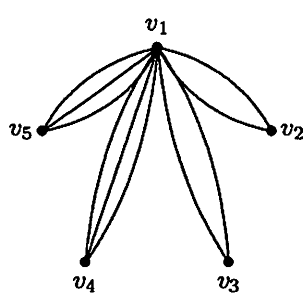


Fig. 3

Case 3. $S = S_{1^2 2^0 3^1}$.

As in Case 2, we may assume that there are two multistars in $\mathcal{D}$ with centers at v_1. Without loss of generality, the subgraph induced by these two multistars is the graph G in case 2, which is exhibited in Fig. 3. For each $i = 2, 3, 4, 5$, at least two edges joining v_i and v_1 have been used for G, so there is at most one multistar in $\mathcal{D}$ with its center at v_i. Then it follows from $|\mathcal{D}| = 6$ that there is exactly one multistar in $\mathcal{D}$ with its center at each v_i $(i = 2, 3, 4, 5)$. The multistar in $\mathcal{D}$ with its center at v_4 has either one edge or three edges joining v_4 and v_5; also the multistar in $\mathcal{D}$ with its center at v_5 has either one edge or three edges joining v_5 and v_4. These contradict the fact that there are exactly three edges joining v_4 and v_5 in $3K_5$. □

Now we consider the problem of decomposing $2K_n$ into isomorphic multistars. Let us begin with a lemma concerning the decomposition of a multistar into isomorphic simple stars.

Lemma 3.1 **([3]; Proposition 1.3)** *Suppose that λ, n are positive integers, and $n_1, n_2, \cdots, n_k$ are nonnegative integers such that $n_1 + 2n_2 + \cdots + kn_k = \lambda n$ and $k \leq \lambda$. Then $S_{1^{n_1}2^{n_2}\cdots k^{n_k}}$ can be decomposed into λ copies of S_n.* □

In the following, a multistar $S_{1^{n_1}2^{n_2}}$ is abbreviated to S_{n_1,n_2}. With $k = 2$ in Lemma 3.1, we have the following.

Lemma 3.2 *Suppose that λ, n are positive integers, and n_1, n_2 are nonnegative integers such that $n_1 + 2n_2 = \lambda n$ and $2 \leq \lambda$. Then S_{n_1,n_2} can be decomposed into λ copies of S_n.* □

An edge in S_{n_1,n_2} with multiplicity 2 is referred to a *2-edge*. Thus S_{n_1,n_2} has n_2 2-edges. The following lemma concerns the decomposition of a multistar into isomorphic multistars.

Lemma 3.3 *Suppose that $\lambda, n_1, n_2, n_1', n_2'$ are nonnegative integers such that $n_1 + 2n_2 = \lambda(n_1' + 2n_2') > 0$, $2 \leq \lambda$ and $n_2 \geq \lambda n_2'$. Then S_{n_1,n_2} can be decomposed into λ copies of $S_{n_1',n_2'}$.*

Proof. For the case $n_1' = 0$, we have $n_1 = 0$ and $n_2 = \lambda n_2'$. It is trivial that S_{0,n_2} can be decomposed into λ copies of $S_{0,n_2'}$; the result follows. For the case $n_2' = 0$, the result follows from Lemma 3.2. Now consider the case $n_1' \geq 1$, $n_2' \geq 1$. Removing $\lambda n_2'$ 2-edges from S_{n_1,n_2}, we obtain the multistar $S_{n_1,n_2-\lambda n_2'}$. Since $n_1 + 2(n_2 - \lambda n_2') = \lambda n_1'$, $2 \leq \lambda$, it follows from Lemma 3.2 that $S_{n_1,n_2-\lambda n_2'}$ can be decomposed into λ copies of $S_{n_1'}$. Then to each of these copies of $S_{n_1'}$ we attach n_2' 2-edges in a natural way to obtain an $S_{n_1',n_2'}$. Thus S_{n_1,n_2} is decomposed into λ copies of $S_{n_1',n_2'}$. □

The case $n_1 = 0$ in the above lemma gives

Lemma 3.4 *Suppose that λ, n_2, n_1', n_2' are nonnegative integers such that $2n_2 = \lambda(n_1' + 2n_2') > 0$, $2 \le \lambda$. Then S_{0,n_2} can be decomposed into λ copies of $S_{n_1',n_2'}$.* □

The case $\lambda = 1$ in Theorem 1.1 gives

Theorem 3.5 *K_n has an S_m-decomposition if and only if $2m|n(n-1)$ and $m \le n/2$. Furthermore the S_m-decomposition can be required to be center balanced.* □

Now we prove the main result of this section.

Theorem 3.6 *Let S be a multistar contained in $2K_n$. Suppose that S has m edges such that*
(1) $m|n(n-1)$ and
(2) $m \le n/2$ or $m = n-1$ or $m = n$.
Then $2K_n$ has an S-decomposition.

Proof. Suppose that $S = S_{n_1,n_2}$ where $n_1 + 2n_2 = m$. Let $V(2K_n) = \{v_1, v_2, \cdots, v_n\}$. Distinguish three cases: Case 1. $m \le n/2$, Case 2. $m = n-1$, Case 3. $m = n$.

Case 1. $m \le n/2$. Consider two subcases.

Subcase 1.1. m is odd or $m = n/2$ for some even integer n.

If m is odd, the assumption $m|n(n-1)$ implies $2m|n(n-1)$. If $m = n/2$, it is obvious that $2m|n(n-1)$. Thus, by Theorem 3.5, K_n has an S_m-decomposition, which implies that $2K_n$ has a $2S_m$-decomposition. Obviously, $2S_m = S_{0,m}$. Since $2m = 2(n_1 + 2n_2)$, it follows from Lemma 3.4 that $S_{0,m}$ can be decomposed into 2 copies of S_{n_1,n_2}. Hence $2K_n$ has an S-decomposition.

Subcase 1.2. m is even and $m < n/2$.

Let $m = 2m'$ where m' is a positive integer. Since $2m'|n(n-1)$ and $m' < n/2$, it follows from Theorem 3.5 that K_n has a center balanced $S_{m'}$-decomposition $\mathcal{D}$. For $v \in V(K_n)$, let $\alpha(v)$ denote the number of stars in $\mathcal{D}$ with centers at v. Then K_n can be decomposed into the following stars: $S_{\alpha(v)m'}$ ($v \in V(K_n)$), which implies that $2K_n$ can be decomposed into the following multistars: $2S_{\alpha(v)m'}$ ($v \in V(K_n)$). Note $2S_{\alpha(v)m'} = S_{0,\alpha(v)m'}$. Since $\mathcal{D}$ is a center balanced $S_{m'}$-decomposition of K_n, we have $\alpha(v) \ge \lfloor |\mathcal{D}|/n \rfloor = \lfloor \binom{n}{2}/m'n \rfloor = \lfloor (n-1)/m \rfloor$. Thus $\alpha(v) \ge 2$ since $m < n/2$. It follows from Lemma 3.4 that $S_{0,\alpha(v)m'}$ can be decomposed into $\alpha(v)$ copies of S_{n_1,n_2}, since $2\alpha(v)m' = \alpha(v)(n_1 + 2n_2)$ and $\alpha(v) \ge 2$. Thus $2K_n$ has an S_{n_1,n_2}-decomposition.

Case 2. $m = n-1$.

We partition the edge set of $2K_n$ into n subsets, each of which induces a subgraph isomorphic to $S = S_{n_1,n_2}$ as follows. For $j = 1, 2, \cdots, n$, let E_j be the set consisting of the following edges: two edges joining v_j and v_{j+i} $(1 \le i \le n_2)$ and one edge joining v_j and v_{j+i} $(n_2+1 \le i \le n_1+n_2)$ where the subscripts of v_{j+i}'s are taken modulo n. Then each induced subgraph $< E_j >$ is isomorphic to S_{n_1,n_2} (as an example, for $n = 8$, $n_1 = 3$, $n_2 = 2$, $< E_1 >$ is exhibited in Fig. 4, and any other $< E_j >$ is a rotation of $< E_1 >$). Also $E(2K_n)$ is partitioned into $E_1, E_2, \cdots, E_n$. Thus $2K_n$ has an S-decomposition.

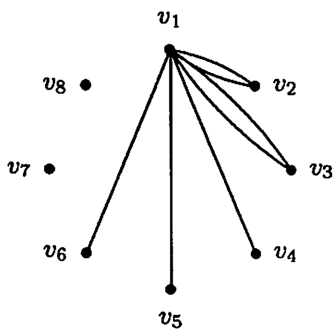


Fig. 4

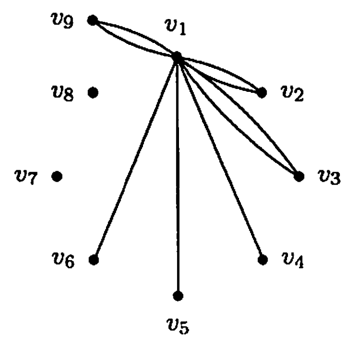


Fig. 5

Case 3. $m = n$.

Since $S = S_{n_1,n_2}$ is contained in K_n, we have $n_1 \le n - 1$. Then it follows from $n = m = n_1 + 2n_2$ that $n_2 \ge 1$. We partition the edge set of $2K_n$ into $n-1$ subsets, each of which induces a subgraph isomorphic to S as follows. For $j = 1, 2, \cdots, n-1$, let E_j be the set consisting of the following edges: two edges joining v_j and v_n, two edges joining v_j and v_{j+i} $(1 \le i \le n_2 - 1)$ and one edge joining v_j and v_{j+i} $(n_2 \le i \le n_1 + n_2 - 1)$ where the subscripts of v_{j+i}'s are taken modulo $n-1$. Then each induced subgraph $< E_j >$ is isomorphic to S_{n_1,n_2} (as an example, for $n = 9$, $n_1 = 3$, $n_2 = 3$, $< E_1 >$ is exhibited in Fig. 5). Also $E(2K_n)$ is partitioned into $E_1, E_2, \cdots, E_{n-1}$. Thus $2K_n$ has an S-decomposition. □

We conclude this paper with the following conjecture.

Conjecture. Let S be a multistar contained in $2K_n$. Suppose S has m edges. Then $2K_n$ has an S-decomposition if and only if $m \mid n(n-1)$ and $m \le n$.

The necessity of the conjecture follows from Theorem 2.7 with $\lambda = 2$, and the sufficiency holds for $m \le \frac{n}{2}$ or $m = n-1$ or $m = n$ by Theorem 3.6.

Acknowledgement

After the publication of [4], we were informed by Professor Z. Lonc of his result (Theorem 1.3 in this paper). We are indebted to him for his kindness.

References

[1] A. Brauer, I.C. Gentry, K. Shaw, A new proof of a theorem by H. G. Landau on tournament matrices, J.C.T. 5 (1968) 289-292.

[2] H.G. Landau, On dominance relations and the structure of animal societies: III. The condition for a score structure. Bull. Math. Biophys. 15 (1953) 143-148.

[3] C. Lin, J.-J. Lin, T.-W. Shyu, Isomorphic star decompositions of multicrowns and the power of cycles, Ars Combinatoria 53(1999) 249-256.

[4] C. Lin, T.-W. Shyu, A necessary and sufficient condition for the star decomposition of complete graphs, J.G.T. 23 (1996) 361-364.

[5] Z. Lonc, Partitions, packings and covering by families with nonempty intersections, J.C.T. A 61 (1992) 263-278.

[6] H.J. Ryser, Matrices of zeros and ones in combinatorial mathematics. Recent Advances in Matrix Theory. University of Wisconsin Press (1964) 103-104.

[7] M. Tarsi, Decomposition of complete multigraphs into stars, Discrete Math. 26 (1979) 273-278.

[8] M. Tarsi, On the decomposition of a graph into stars, Discrete Math. 36 (1981) 299-304.