A Note on a Theorem of Fan Concerning Average Degrees and Long Cycles

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We remedy the gap in the proof of the following theorem stated in [1]:
Theorem. *Let C be a cycle of length c in a graph G, and let H be a component of $G - C$. Suppose that C is locally longest with respect to H, and H is locally k-connected to C, where $2 \leq k \leq 4$, and $|V(H)| \geq k-1$ and, in addition, the average degree of H in G is r. Then $c \geq k(r+2-k)$, with equality only if r is an integer and either H is a complete graph of order $r+1-k$ and every vertex of H has the same k neighbours on C, or H is a complete graph of order $k-1$ and every vertex of H has the same $r+2-k$ neighbours on C.*

1 Introduction

In this paper, we consider only finite, undirected graphs without loops or multiple edges.

Let G be a graph. The vertex set of G is denoted by $V(G)$, and the edge set of G is denoted by $E(G)$. For a subset X of $V(G)$, $G - X$ denotes the subgraph obtained from G by deleting the vertices in X together with the edges incident with them. For $x \in V(G)$, we let $N_G(x)$ denote the set of vertices adjacent to x in G, and set $\deg_G(x) := |N_G(x)|$. For $X \subset V(G)$, we let $N_G(X)$ denote the union of $N_G(x)$ as x ranges over X. For disjoint subsets X, X' of $V(G)$, we define $E_G(X, X') := \{xx' \in E(G) \mid x \in X, x' \in X'\}$. A subgraph of G is often identified with its vertex set. Thus when H is a subgraph of G, we write $G - H$ and $N_G(H)$ for $G - V(H)$ and $N_G(V(H))$; when H and H' are vertex-disjoint subgraphs, we write $E_G(H, H')$ for $E_G(V(H), V(H'))$.

A cycle C is denoted by a sequence $b_1 b_2 \ldots b_m b_1$ of its vertices such that $V(C) = \{b_1, b_2, \ldots, b_m\}$ and $E(C) = \{b_1b_2, b_2b_3, \ldots, b_{m-1}b_m, b_mb_1\}(n =$

$|V(C)|$). Similarly, a path R is denoted by a sequence $b_1 b_2 \dots b_m$ of its vertices such that $V(R) = \{b_1, b_2, \dots, b_m\}$ and $E(R) = \{b_1b_2, b_2b_3, \dots, b_{m-1}b_m\}$. Let $C = b_1 b_2 \dots b_m b_1$ be a cycle. For i, j with $i < j < i+m$, we let $C[b_i, b_j]$ denote the path $b_i b_{i+1} \dots b_j$, and let $C(b_i, b_j)$ denote the path $b_{i+1} b_{i+2} \dots b_{j-1}$ (subscripts are to be read modulo m). Note that if $j = i+1$, then $C(b_i, b_j)$ denotes an empty path.

Let again G be a graph. Let C be a cycle of G, and let H be a component of $G - C$. We say that C is *locally longest with respect to H in G* if we cannot obtain a cycle longer than C by replacing a segment $C[u, v]$ by a uv-path all of whose inner vertices lie in H (a uv-path means a path connecting u and v).

Let C be a subgraph of G, and let x be a vertex in $G - C$. An *(x, C)-path* is a path connecting x to some vertex $v \in V(C)$ such that v is the only vertex of C on the path. Two (x, C)-paths are said to be *disjoint* if they have only the vertex x in common. Let H and C be two subgraphs of G with $V(H) \cap V(C) = \emptyset$. We say that H is *locally k-connected* to C in G if for every vertex $x \in V(H)$, there are k pairwise disjoint (x, C)-paths in G.

Let W be a subset of $V(G)$. The average degree of W in G is the number

$$\frac{1}{|W|} \sum_{z \in W} \deg_G(z).$$

If H is a subgraph of G with vertex set W, we also call this number the average degree of H in G.

In this paper, we are concerned with the following theorem:

Theorem A. *Let C be a cycle of length c in a graph G, and let H be a component of $G - C$. Suppose that C is locally longest with respect to H, and H is locally k-connected to C, where $2 \le k \le 4$, and $|V(H)| \ge k - 1$ and, in addition, the average degree of H in G is r. Then $c \ge k(r + 2 - k)$, with equality only if r is an integer and either H is a complete graph of order $r + 1 - k$ and every vertex of H has the same k neighbours on C, or H is a complete graph of order $k - 1$ and every vertex of H has the same $r + 2 - k$ neighbours on C.*

Theorem A appears as Theorem 2 in Fan [1] but, as we shall describe below, there is a gap in the proof of the theorem given in [1]. In [1], the proof of Theorem 2 is carried out by induction on the number of blocks of H. The problem occurs when H contains at least two blocks, and there is an endblock B of H such that $\sum_{z \in V(B - \{b\})} \deg_G(z) \le (r-1)|V(B - b)|$ where b is the unique cutvertex of H contained in B. That is to say, in [1], it is asserted that if we let $\overline{G}$ denote the graph obtained from G by contracting B, and let $\overline{H}$ denote the subgraph of $\overline{G}$ arising from H through the contraction of B, then we obtain the desired conclusion by applying the induction hypothesis to C and $\overline{H}$ in $\overline{G}$. However, in the case where $k = 4$ and $|V(H)| = |V(B)| + 1$, we cannot apply the induction hypothesis because $|V(\overline{H})| = 2 < k - 1$. We here remedy this gap by proving the following proposition:

Proposition B. *Let C be a cycle of length c in a graph G, and let H be a component of $G - C$. Suppose that C is locally longest with respect to H, H is locally 4-connected to C, and the average degree of H in G is r. Suppose further that there exists an endblock B of H such that $|V(H)| = |V(B)| + 1$. Then $c > 4(r-2)$.*

We conclude this section by defining some more terms which we use in the proof of Proposition B.

Let u and v be two distinct vertices of a graph G. We define the *codistance* $d^*_G(u,v)$ between u and v to be the maximum length of a uv-path in G (a uv-path means a path connecting u and v); if no uv-path exists, we set $d^*_G(u,v) = 0$.

Let C be a cycle of a graph G, and let H be a subgraph of $G - C$. A *strong attachment* of H to C in G is a subset $T = \{u_1, u_2, \ldots, u_t\} \subset N_G(H) \cap V(C)$, where $u_1, u_2, \ldots, u_t$ occur on C in this order, such that either $t \leq 1$, or $t \geq 2$ and for each $1 \leq i \leq t$, there exist $y, z \in V(H)$ with $y \neq z$ such that $u_i y, u_{i+1} z \in E(G)$ (we take $u_{t+1} = u_1$). A strong attachment T of H to C is said to be *maximum* if it has maximum cardinality over all strong attachments of H to C.

2 Preliminary Results

In this section, we collect lemmas which we use in the proof of Proposition B. Most of the lemmas in this section are taken from Fan [1].

Lemma 2.1 [1; Proposition 1]. *Let H and C be two disjoint subgraphs of a graph G, and suppose that H is locally k-connected to C in G. Then $E(C,H)$ contains t independent edges, where $t = \min\{k, |V(H)|\}$.*

Lemma 2.2 [1; Proposition 3]. *Let H and C be two disjoint subgraphs of a graph G, and suppose that H is locally k-connected to C in G. Let B be a block of H. Let $\overline{G}$ be the graph obtained from G by contracting B into a single vertex, and let $\overline{H}$ be the subgraph of $\overline{G}$ arising from H through the contraction of B. Then $\overline{H}$ is locally k-connected to C in $\overline{G}$.*

Lemma 2.3 [1; Lemma 1]. *Let C be a cycle of a graph G, and let H be a subgraph of $G - C$. Let $T = \{u_1, u_2, \ldots, u_t\}$ be a maximum strong attachment of H to C, where $u_1, u_2, \ldots, u_t$ occur on C in this order, and suppose that $t \geq 2$. Set*

$$S = (N_G(H) \cap V(C)) - T.$$

Then the following hold.

(i) *Every vertex in S is joined to exactly one vertex of H.*

(ii) *Let $1 \leq i \leq t$, and write*

$$V(C[u_i, u_{i+1}]) \cap N_G(H) = \{a_0, a_1, \ldots, a_q, a_{q+1}\}$$

with $a_0 = u_i$ and $a_{q+1} = u_{i+1}$ so that $a_0, a_1, \ldots, a_{q+1}$ occur on $C[u_i, u_{i+1}]$ in this order (in the case where $i = t$, we take $u_{t+1} = u_1$). Then there is an index m with $0 \leq m \leq q$ such that

$$V(H) \cap N_G(a_j) = V(H) \cap N_G(a_0) \qquad \text{for all} \quad 0 \leq j \leq m$$

and

$$V(H) \cap N_G(a_j) = V(H) \cap N_G(a_{q+1}) \quad \text{for all} \quad m+1 \leq j \leq q+1.$$

For convenience, we restate (ii) of the above lemma in the following form.

Lemma 2.4 *Let G, C, $H, T = \{u_1, u_2, \ldots, u_t\}$ be as in Lemma 2.3. Then there exists a maximum strong attachment T' of H to C which satisfies the following property:*

if we write $T' = \{v_1, v_2, \ldots, v_t\}$ so that $v_1, v_2, \ldots, v_t$ occur on C in this order, then for each $1 \leq i \leq t$, we have $V(H) \cap N_G(w) = V(H) \cap N_G(v_{i+1})$ for all $w \in V(C[v_i, v_{i+1}] - \{v_i\}) \cap N_G(H)$, where we take $v_{t+1} = v_1$. (1)

Proof. For each $1 \leq i \leq t$, write

$$V(C[u_i, u_{i+1}]) \cap N_G(H) := \{a_{i,0}, a_{i,1}, \ldots, a_{i,q(i)+1}\}$$

with $a_{i,0} = u_i$ and $a_{i,q(i)+1} = u_{i+1}$ so that $a_{i,0}, a_{i,1}, \ldots, a_{i,q(i)+1}$ occur on $C[u_i, u_{i+1}]$ in this order. It follows from Lemma 2.3 (ii) that for each i, there exists an index $m(i)$ with $0 \leq m(i) \leq q(i)$ such that

$$V(H) \cap N_G(a_{i,j}) = V(H) \cap N_G(a_{i,0}) \qquad \text{for all } 0 \leq j \leq m(i), \tag{2}$$

and

$$V(H) \cap N_G(a_{i,j}) = V(H) \cap N_G(a_{i,q(i)+1}) \quad \text{for all } m(i)+1 \le j \le q(i)+1. \tag{3}$$

Set

$$T' := \{a_{1,m(1)}, a_{2,m(2)}, \ldots, a_{t,m(t)}\}.$$

Then we see from (2) that T' is a strong attachment, and (2) and (3) together imply that T' satisfies (1).

Lemma 2.5 [1; Theorem 1]. *Let u and v be two distinct vertices of a nonseparable graph G of order at least 3. Suppose that the average degree of the vertices other than u and v is r. Then the following hold.*

(i) $d^*_G(u,v) \ge r$.

(ii) *Equality holds in* (i) *if and only if r is an integer, each of x and y is joined to all vertices in $V(G) - \{u,v\}$, and each component of $G - \{u,v\}$ is a complete graph of order $r-1$.*

The following lemma follows immediately from the definition of a locally longest cycle:

Lemma 2.6. *Let C be a cycle of a graph G, and H be a component of $G - C$, and suppose that C is locally longest with respect to H. Let u, v be distinct vertices on C, and suppose that there exist $y \in V(H) \cap N_G(u)$ and $z \in V(H) \cap N_G(v)$ such that $y \ne z$. Then $|E(C[u,v])| \ge d^*_H(y,z) + 2$.*

3 Proof of Proposition B

Let G, C, H, c, r, B be as in Proposition B. Write $V(H) = V(B) \cup \{x\}$, and let b be the unique cutvertex of H; thus $E(H) = E(B) \cup \{bx\}$. We divide the proof into two cases according to the order of B.

Case 1. $|V(B)| \ge 4$.

Let $\overline{G}$ denote the graph obtained from G by contracting the edge bx into a single vertex. Let $\bar{b}$ denote the new vertex of $\overline{G}$ arising from bx through its contraction, and let $\overline{H}$ denote the subgraph of $\overline{G}$ arising from H (so $\overline{H} \cong B$). Note that $V(C) \cap N_G(H) = V(C) \cap N_{\overline{G}}(\overline{H})$. Note also that from the assumption that C is locally longest with respect to H in

G, it follows that C is locally longest with respect to $\overline{H}$ in $\overline{G}$. Let $T \subset V(C) \cap N_{\overline{G}}(\overline{H})$ be a maximum strong attachment of $\overline{H}$ to C in $\overline{G}$, and set $S = (V(C) \cap N_{\overline{G}}(\overline{H})) - T$. Let $t = |T|$, $s = |S|$. By Lemma 2.2, $\overline{H}$ is locally 4-connected to C in $\overline{G}$. Hence $t \geq 4$ by Lemma 2.1. By Lemma 2.4, we may assume T satisfies (1). Define subsets P_0, P_1 of $V(C) \cap N_{\overline{G}}(\overline{H})$ by

$$P_0 := N_G(b) \cap N_G(x) \cap T \quad \text{and} \quad P_1 := N_G(b) \cap N_G(x) \cap S,$$

and let $p_0 := |P_0|$ and $p_1 := |P_1|$. Since $P_0 \subset T$ and $P_1 \subset S$, we have $p_0 \leq t$ and $p_1 \leq s$.

Write $V(C) \cap N_{\overline{G}}(\overline{H}) = \{c_1, c_2, \ldots, c_\lambda\}$ so that $c_1, c_2, \ldots, c_\lambda$ occur on C in this order (we take $c_{\lambda+1} = c_1$).

Claim 1. (I) *Let $1 \leq i \leq \lambda$, and suppose that $c_i \in T$. Then the following hold.*

(i) $|E(C[c_i, c_{i+1}])| \geq 4.$

(ii) $$|E(C[c_i, c_{i+1}])| \geq \frac{r(|V(\overline{H})| + 1) - (p_0 + p_1 + 2)}{|V(\overline{H})|} + 2 - t - \frac{s}{|V(\overline{H})|}.$$

(iii) *If equality holds in* (ii), *then*

$$|V(\overline{H})| = \frac{r(|V(\overline{H})| + 1) - (p_0 + p_1 + 2)}{|V(\overline{H})|} + 1 - t - \frac{s}{|V(\overline{H})|},$$

$\overline{H} \cong K_{|V(\overline{H})|}$, *and* $V(\overline{H}) \subset N_{\overline{G}}(c_i) \cap N_{\overline{G}}(c_{i+1})$.

(II) *There exists an index i with $c_i \in T$ for which strict inequality holds in* (I) (ii).

Proof. (I) (i) Since $c_i \in T$ and T satisfies (1), there exist $y \in V(\overline{H}) \cap N_{\overline{G}}(c_i)$ and $z \in V(\overline{H}) \cap N_{\overline{G}}(c_{i+1})$ such that $y \neq z$. Since $|V(\overline{H})| \geq 4$ and $\overline{H}$ is nonseparable, it follows that $d^*_{\overline{H}}(y, z) \geq 2$. Hence by Lemma 2.6, $|E(C[c_i, c_{i+1}])| \geq d^*_{\overline{H}}(y, z) + 2 \geq 4$.

(ii) Let H' denote the graph obtained from the subgraph induced by $V(\overline{H}) \cup \{c_i, c_{i+1}\}$ in $\overline{G}$ by joining c_i and c_{i+1} (if c_i and c_{i+1} are not joined in $\overline{G}$). Let

$$r' := \frac{r(|V(\overline{H})| + 1) - (p_0 + p_1 + 2)}{|V(\overline{H})|} + 2 - t - \frac{s}{|V(\overline{H})|}.$$

We first estimate $(\sum_{z \in V(\overline{H})} \deg_{H'}(z))/|V(\overline{H})|$. Since $N_G(b) \cap N_G(x) \subset V(C)$, we have $\deg_{\overline{G}}(\bar{b}) = \deg_G(b) + \deg_G(x) - |N_G(b) \cap N_G(x)| - 2 = \deg_G(b) + \deg_G(x) - (p_0 + p_1 + 2)$, and hence

$$\begin{aligned} \sum_{z \in V(\overline{H})} \deg_{\overline{G}}(z) &= \Big(\sum_{z \in V(H)} \deg_G(z) \Big) - (\deg_G(b) + \deg_G(x) - \deg_{\overline{G}}(\bar{b})) \\ &= r(|V(\overline{H})| + 1) - (p_0 + p_1 + 2). \end{aligned} \tag{4}$$

Write $T = \{u_1, u_2, \ldots, u_t\}$ with $u_1 = c_i$ so that $u_1, u_2, \ldots, u_t$ occur on C in this order. Set $T' := (T - \{u_2\}) \cup \{c_{i+1}\}$ and $S' := (V(C) \cap N_{\overline{G}}(\overline{H})) - T'$. Since T satisfies (1), T' is also a maximum strong attachment of $\overline{H}$ to C, and hence

$$|E_{\overline{G}}(S', V(\overline{H}))| \leq s \tag{5}$$

by Lemma 2.3(i). Also we clearly have

$$|E_{\overline{G}}(T' - \{c_i, c_{i+1}\}, V(\overline{H}))| \leq (t-2)|V(\overline{H})|. \tag{6}$$

Combining (4), (5) and (6), we obtain $(\sum_{z \in V(\overline{H})} \deg_{H'}(z))/|V(\overline{H})| \geq r'$. Consequently,

$$d^*_{H'}(c_i, c_{i+1}) \geq \frac{1}{|V(\overline{H})|} \sum_{z \in V(\overline{H})} \deg_{H'}(z) \geq r' \tag{7}$$

by Lemma 2.5(i), and hence $|E(C[c_i, c_{i+1}])| \geq r'$ by the fact that C is locally longest with respect to $\overline{H}$.

(iii) Suppose that equality holds in (ii). Then equality holds in (7). Since $H' - \{c_i, c_{i+1}\} = \overline{H}$ is connected, this together with Lemma 2.5(ii) implies that $\overline{H} \cong K_{r'-1}$ and $V(\overline{H}) \subset N_{H'}(c_i) \cap N_{H'}(c_{i+1}) \subset N_{\overline{G}}(c_i) \cap N_{\overline{G}}(c_{i+1})$.

(II) Suppose that eqality holds in (I)(ii) for all i with $c_i \in T$. Let r' be as in the proof of (I). Then by (I)(iii),

$$\overline{H} \cong K_{r'-1}, \tag{8}$$

and

$$V(\overline{H}) \subset N_{\overline{G}}(c_{i+1}) \tag{9}$$

for each i with $c_i \in T$. Note that by Lemma 2.3(i), (9) implies $c_{i+1} \in T$. Thus $c_{i+1} \in T$ for each i with $c_i \in T$. Since $T \neq \emptyset$, this forces $T = V(C) \cap N_G(\overline{H})$ which, in turn, implies that (9) holds for all $1 \leq i \leq \lambda$. Now since H is locally 4-connected to C in G, $\deg_G(x) \geq 4$, and hence $V(C) \cap N_G(x) \neq \emptyset$; that is to say, there exists c_j such that $x \in N_G(c_j)$. Take $y \in V(B - \{b\})$. Since (9) holds for all i, we have $y \in N_G(c_{j+1})$. Since $d^*_B(b, y) = r' - 2$ by (8), we now obtain $|E(C[c_j, c_{j+1}])| \geq d^*_H(x, y) + 2 = (1 + d^*_B(b, y)) + 2 = r' + 1$ by Lemma 2.6. Since $T = V(C) \cap N_G(\overline{H})$, this contradicts the assumption that equality holds in (I) (ii) for all i with $c_i \in T$.

We return to the proof of Proposition B. If $t > r - 2$, it immediately follows from Claim1(I) (i) that $c > 4(r-2)$. Thus we may assume $t \leq$

$r-2$. Since C is locally longest with respect to $\overline{H}$, $|E(C[c_i,c_{i+1}])| \geq 2$ for each $1 \leq i \leq \lambda$ with $c_i \in S$, and hence $\sum_{c_i \in S} |E(C[c_i,c_{i+1}])| \geq 2s$. Consequently,

$$c \geq \sum_{i=1}^{\lambda} |E(C[c_i,c_{i+1}])| = \sum_{c_i \in T} |E(C[c_i,c_{i+1}])| + 2s. \tag{10}$$

With (I) (i), (I) (ii) and (II) of Claim1 in mind, we substitute

$$\frac{r(|V(\overline{H})|+1)-(p_0+p_1+2)}{|V(\overline{H})|} + 2 - t - \frac{s}{|V(\overline{H})|}$$

for four of the terms $|E(C[c_i,c_{i+1}])|$ in the right-hand side of (10), including a term for which strict inequality holds in Claim 1(I) (ii), and substitute 4 for the other $t-4$ terms. Then we obtain

$$\begin{aligned} c &> 4\left\{\frac{r\left(|V(\overline{H})|+1\right)-(p_0+p_1+2)}{|V(\overline{H})|} + 2 - t - \frac{s}{|V(\overline{H})|}\right\} \\ &\quad + 4(t-4) + 2s \\ &= 4r - 8 + \left(s - \frac{4p_1}{|V(\overline{H})|}\right) + \frac{4}{|V(\overline{H})|}\left((r-2)-p_0\right) \\ &\quad + s\left(1 - \frac{4}{|V(\overline{H})|}\right). \end{aligned}$$

Since $|V(\overline{H})|(=|V(B)|) \geq 4$ and $p_0 \leq t \leq r-2$ and $p_1 \leq s$, this implies $c > 4(r-2)$, as desired.

Case 2. $|V(B)| = 3$ or 2.

Let T be a maximum strong attachment of $H-\{b\}$ to C in G, and set $S = (V(C) \cap N_G(H-\{b\})) - T$. Let $t = |T|$, $s = |S|$. Since H is locally 4-connected to C in G and $|V(H)| \leq 4$, it follows from Lemma 2.1 that there exist $|V(H)|$ independent edges joining H and C, and hence

there exist $|V(H)|-1$ independent edges joining $H-\{b\}$ and C. (11)

In particular, $t \geq 2$. By Lemma 2.4, we may assume T satisfies (1). Let

$$P_0 = N_G(b) \cap T, P_1 = N_G(b) \cap S, P_2 = N_G(b) \cap (V(C)-T-S);$$
$$p_0 = |P_0|, p_1 = |P_1|, p_2 = |P_2|.$$

Since $P_0 \subset T$ and $P_1 \subset S$, we have $p_0 \leq t$ and $p_1 \leq s$.

Claim 2.

(i) $(|V(H)|-1)\,t+s+p_0+p_1+p_2 \ge |V(H)|\,(r-2)$.

(ii) *If equality holds in* (i), *then* $|V(H)|=4$.

Proof. If $|V(H)| = 4$, then $B \cong K_3$, and hence $\sum_{z\in V(H)} \deg_H(z) = 8 = 2|V(H)|$; if $|V(H)| = 3$, then $B \cong K_2$, and hence $\sum_{z\in V(H)} \deg_H(z) = 4 < 2|V(H)|$. Thus

$$\sum_{z\in V(H)} \deg_H(z) \le 2|V(H)|. \tag{12}$$

On the other hand, since we clearly have $|E_G(V(H)-\{b\},T)| \le (|V(H)|-1)t$, and since $|E_G(V(H)-\{b\},S)| \le s$ by Lemma 2.3(i), it follows that $|E_G(V(H)-\{b\},V(C))| \le (|V(H)|-1)t+s$. Since $|E_G(\{b\},V(C))| = p_0+p_1+p_2$, this implies

$$(|V(H)|-1)\,t+s+p_0+p_1+p_2 \ge |E_G(H,C)|. \tag{13}$$

Since $\sum_{z\in V(H)} \deg_H(z) + |E_G(H,C)| = \sum_{z\in V(H)} \deg_G(z) = r|V(H)|$, (i) now follows from (12) and (13). Further, if equality holds in (i), then equality holds in (12), and hence it follows from the proof of (12) that $|V(H)|=4$, which proves (ii).

Write $V(C)\cap N_G(H-\{b\}) = \{c_1, c_2, \dots, c_\lambda\}$ so that $c_1, c_2, \dots, c_\lambda$ occur on C in this order (we take $c_{\lambda+1}=c_1$).

Claim 3. *Let* $1 \le i \le \lambda$, *and suppose that* $c_i \in S$. *Then the following hold.*

(i) $|E(C[c_i,c_{i+1}])| \ge 2+2|P_2 \cap V(C(c_i,c_{i+1}))|$.

(ii) *If* $c_i \in P_1, |E(C[c_i,c_{i+1}])| \ge 3+2|P_2\cap V(C(c_i,c_{i+1}))|$.

Proof. By Lemma 2.3(i) and (1), we can write $V(H-\{b\})\cap N_G(c_i)\,(= V(H-\{b\})\cap N_G(c_{i+1})) = \{y\}$. Let $q = |P_2 \cap V(C(c_i,c_{i+1}))|$, and write $V(C[c_i, c_{i+1}])\cap N_G(H) = \{a_0,a_1,\dots,a_{q+1}\}$ with $a_0 = c_i$ and $a_{q+1} = c_{i+1}$ so that $a_0,a_1,\dots,a_{q+1}$ occur on $C[c_i,c_{i+1}]$ in this order. Then $|E(C[a_j,a_{j+1}])| \ge 2$ for each $0\le j\le q$, and hence $|E(C[c_i,c_{i+1}])| \ge \sum_{0\le j\le q} |E(C[a_j,a_{j+1}])| \ge 2(q+1)$. This proves (i). To prove (ii), suppose that $c_i \in P_1$. If $q=0$, then since $b\in N_G(c_i)$ by the definition of P_1, we get $|E(C[c_i,c_{i+1}])| \ge d^*_H(y,b)+2 \ge 3 = 3+2q$ by Lemma 2.6. If $q\ge 1$, then since $b \in N_G(a_1)$ by the definition of P_2, we get $|E(C[a_0,a_1])| \ge d^*_H(y,b)+2 \ge 3$ by Lemma 2.6, and hence $|E(C[c_i,c_{i+1}])| = \sum_{0\le j\le q} |E(C[a_j,a_{j+1}])| \ge 3+2q$. Thus (ii) is proved.

Claim 4.

(i) *Let* $1 \le i \le \lambda$, *and suppose that* $c_i \in T$. *Then* $|E(C[c_i, c_{i+1}])| \ge 4 + 2|P_2 \cap V(C(c_i, c_{i+1}))|$.

(ii) *If* $|V(H)| = 4$, *then there exists an index* i *with* $c_i \in T$ *such that* $|E(C[c_i, c_{i+1}])| \ge 5 + 2|P_2 \cap V(C(c_i, c_{i+1}))|$.

Proof. (i) Take $y \in V(H - \{b\}) \cap N_G(c_i)$ and $z \in V(H - \{b\}) \cap N_G(c_{i+1})$ so that $y \ne z$. Let $q = |P_2 \cap V(C(c_i, c_{i+1}))|$, and write $V(C[c_i, c_{i+1}]) \cap N_G(H) = \{a_0, a_1, \ldots, a_{q+1}\}$ with $a_0 = c_i$ and $a_{q+1} = c_{i+1}$ so that $a_0, a_1, \ldots, a_{q+1}$ occur on $C[c_i, c_{i+1}]$ in this order. Assume first that $q = 0$. Since $y \ne z$ and $b \notin \{y, z\}$, ybz is a yz-path in H, and hence $d^*_H(y, z) \ge 2$. Consequently, it follows from Lemma 2.6 that $|E(C[c_i, c_{i+1}])| \ge d^*_H(y, z) + 2 \ge 4 = 4 + q$. Assume now that $q \ge 1$. Then $b \in N_G(a_1) \cap N_G(a_q)$, and hence $|E(C[a_0, a_1])| \ge d^*_H(y, b) + 2 \ge 3$ and $|E(C[a_q, a_{q+1}])| \ge d^*_H(b, z) + 2 \ge 3$ by Lemma 2.6. Consequently,

$$|E(C[c_i, c_{i+1}])| = \sum_{0 \le j \le q} |E(C[a_j, a_{j+1}])| \ge 3 + 2(q-1) + 3 = 4 + 2q.$$

(ii) Suppose that $|V(H)| = 4$. By (11), there exist $u, v \in V(C)$ with $u \ne v$ such that $x \in V(H) \cap N_G(u)$ and $V(B - \{b\}) \cap N_G(v) \ne \emptyset$. Hence it follows from Lemma 2.3(i) and (1) that there exists an index i with $c_i \in T$ such that

$$x \in N_G(c_i) \quad \text{and} \quad V(B - \{b\}) \cap N_G(c_{i+1}) \ne \emptyset.$$

For this i, let q, a_0, a_1, $\ldots$, a_{q+1} be as in the proof of (i). Take $y \in V(B - \{b\}) \cap N_G(c_{i+1})$ and write $V(B - \{b\}) = \{y, z\}$. Assume first that $q = 0$. Since $xbzy$ is an xy-path, $d^*_H(x, y) \ge 3$. Consequently, it follows from Lemma 2.6 that $|E(C[c_i, c_{i+1}])| \ge d^*_H(x, y) + 2 \ge 5 = 5 + 2q$. Assume now that $q \ge 1$. Since bzy is a by-path, $d^*_H(b, y) \ge 2$, and hence $|E(C[a_q, a_{q+1}])| \ge d^*_H(b, y) + 2 \ge 4$ by Lemma 2.6. Also $|E(C[a_0, a_1])| \ge d^*(x, b) + 2 \ge 3$ by Lemma 2.6. Consecuently,

$$|E(C[c_i, c_{i+1}])| = \sum_{0 \le j \le q} |E(C[a_j, a_{j+1}])| \ge 3 + 2(q-1) + 4 = 5 + 2q.$$

Thus (ii) is proved.

We return to the proof of Proposition B. By Claim 3, $\sum_{c_i \in S} |E(C[c_i, c_{i+1}])| \ge 2(s - p_1) + 3p_1 + 2\sum_{c_i \in S} |P_2 \cap V(C(c_i, c_{i+1}))| = 2s + p_1 + 2\sum_{c_i \in S} |P_2 \cap V(C(c_i, c_{i+1}))|$. By Claim 4(i), $\sum_{c_i \in T} |E(C[c_i, c_{i+1}])| \ge 4t +$

$2\sum_{c_i\in T}|P_2\cap V(C(c_i,c_{i+1}))|$. Consequently,

$$\begin{aligned} c &\geq \sum_{1\leq i\leq \lambda} |E(C[c_i,c_{i+1}])| \\ &\geq 4t+2s+p_1+2\sum_{1\leq i\leq \lambda}|P_2\cap V(C(c_i,c_{i+1}))| \\ &= 4t+2s+p_1+2p_2. \end{aligned} \tag{14}$$

Assume for the moment that $|V(B)|=3$. Then by Claim 2(i), $3t+s+p_0+p_1+p_2\geq 4(r-2)$. Since $p_0\leq t$, $0\leq s$ and $0\leq p_2$, this implies

$$4t+2s+p_1+2p_2\geq 4(r-2). \tag{15}$$

Assume now that $|V(B)|=2$. Then multiplying both sides of Claim 2 (i) by 4/3, we obtain

$$\frac{8t}{3}+\frac{4s}{3}+\frac{4p_0}{3}+\frac{4p_1}{3}+\frac{4p_2}{3}\geq 4(r-2).$$

Since $p_0\leq t$, $0\leq p_1\leq s$ and $0\leq p_2$, this again implies (15). Thus (15) holds in either case. Now combining (14) and (15), we obtain $c\geq 4(r-2)$.

Suppose that $c=4(r-2)$. Then equality holds in both (14) and (15). The equality in (14) implies that in Claim 4(i), equality holds for all i with $c_i\in T$, and hence $|V(H)|=3$ by Claim 4(ii). On the other hand, the equality in (15) implies that equality holds in Claim 2(i), and hence $|V(H)|=4$ by Claim 2(ii). This is a contradiction, and this contradiction shows that we have $c>4(r-2)$, completing the proof of Proposition B.

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References

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