Two-Dimensional Bandwidth of Graphs *

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Abstract

The two dimensional bandwidth problem is to determine an embedding of graph G in a grid graph in the plane such that the longest edges are as short as possible. In this paper we study the problem under the distance of L_∞-norm.
Keywords: graph labeling, two-dimensional bandwidth, L_∞-norm.

AMS: 05C78

1 Introduction

The bandwidth minimization problem for graphs has a wide range of applications, including sparse matrix computations, error-correcting code designs, data structures and the circuit layout of VLSI designs.

Many years ago, stimulated by the rectilinear network layout designs, the case of two-dimensional grid graphs has been studied in the literature ([1,2]). Here, the (two-dimensional) grid graph is a product of two paths $P_n \times P_n$ (where n is considered to be large enough, at least $n \geq |V(G)|$). In other words, the grid graph has vertex set

$$\{(i,j)|i,j \in Z, 1 \leq i,j \leq n\},$$

*The Project Supported by Zhejiang Provincial Natural Science Foundation of China(102055)

and (i,j) is adjacent to (i',j') if

$$|i-i'|+|j-j'|=1.$$

Note that the distance between two vertices (i,j) and (i',j') in grid graph H is

$$d_H((i,j),(i',j'))=|i-i'|+|j-j'|.$$

This is called the rectangular distance, i.e., Manhattan distance, as opposed to the normal Euclidean distance, in the plane. The bandwidth relative to this host graph is called the two-dimensional bandwidth, denoted by $B_2(G)$.

In [3], we studied the problem under the distance of L_∞-norm. Namely, the distance between two points $(i,j),(i',j')$ in H is defined by

$$\partial_H((i,j),(i',j'))=\max\{|i-i'|,|j-j'|\}.$$

Then, the two-dimensional bandwidth is defined by

$$\beta_2(G,f)=\max_{uv\in E(G)}\partial_H(f(u),f(v)),$$

where f is a one–to–one mapping from $V(G)$ to $V(H)$, i.e., an injection $f:V(G)\to V(H)$, which can be viewed as an embedding of G into H, and

$$\beta_2(G)=\min_f\beta_2(G,f)$$

In this paper, we continue the study of the two-dimensional bandwidth under distance of L_∞-norm. For example, the following is an example of an embedding f of C_3.

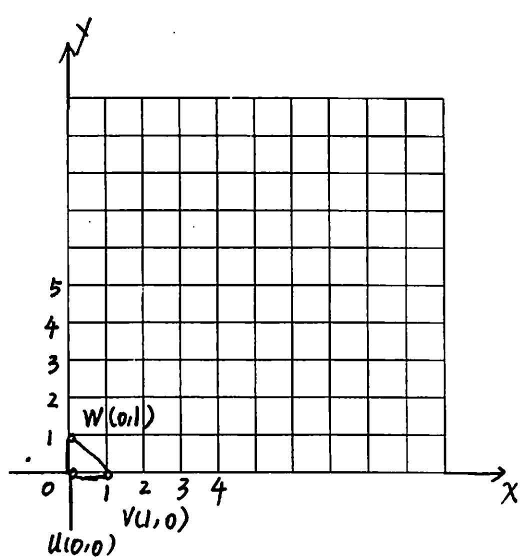

Figure 1. The embedding of C_3

By the definitions of $d_H((i,j),(i',j'))$ and $\partial_H((i,j),(i',j'))$, we obtain

$$d_H(vu) = |0-1| + |0-0| = 1.$$

$$\partial_H(vu) = \max\{|0-1|, |0-0|\} = 1.$$

Similarly,

$$d_H(uw) = 1, \partial_H(uw) = 1.$$

$$d_H(vw) = 2, \partial_H(vw) = 1.$$

Thus, by the definitions of $B_2(G)$ and $\beta_2(G)$, we obtain

$$B_2(C_3, f) = 2,$$

$$\beta_2(C_3, f) = 1.$$

The organization of the paper is as follows. In section 2, we restate some preliminaries, we present some results in section 3 and some remarks in section 4.

2 Preliminaries

For convenience, we restate some definitions and lemmas as follows (see [4] for further details).

Definition 2.1 *In grid plane H, we define rectangle $H(\alpha,\beta)$ as a set of intersection points: we take α columns continuously and then take β rows continuously. It is easy to see that*

$$|H(\alpha,\beta)| = \alpha\beta,$$

$$\partial_H(H(\alpha,\beta)) = \max\{\alpha - 1, \beta - 1\},$$

where

$$\partial_H(H(\alpha,\beta)) = \max_{x,y\in H(\alpha,\beta)} \partial_H(xy).$$

Definition 2.2 [4] *The product of simple graphs G and H is the simple graph $G \times H$ with vertex set $V(G) \times V(H)$, in which (u,v) is adjacent to (u',v') if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.*

Lemma 2.3 [3] *For a complete graph K_n of n vertices.*

$$\beta_2(K_n) = \lceil \sqrt{n} - 1 \rceil.$$

3 On β_2 of Product Graphs

Theorem 3.1 *Let $|V(G_1)| = m$, $|V(G_2)| = n$, $3 \le m \le n$, then*

$$\beta_2(G_1 \times G_2) \le (m-1)\left\lceil \frac{-1+\sqrt{4n(m-1)+1}}{2(m-1)} \right\rceil .$$

Proof: At first, we embed G_2 in an $H(\alpha,\beta)$, then, we move it α columns, $\cdots$, $(m-1)\alpha$ columns rightly to obtain embeddings of m copies of G_2. In order to let $H(\alpha,\beta)$ contain G_2, we must let $|H(\alpha,\beta)| \ge n$. That is

$$\alpha\beta \ge n \qquad (1)$$

The maximum distance among the corresponding points of the m copies of G_2 is $(m-1)\alpha$, the distance of $H(\alpha,\beta)$ under $L_\infty-$ norm is

$$\max\{\beta-1, \alpha-1\}.$$

We set the two numbers equal. That is, in this situation it is clear that

$$(m-1)\alpha = \max\{\beta-1,\alpha-1\} = \beta-1 \qquad (2)$$

Solving (2), we obtain

$$\beta = (m-1)\alpha+1.$$

Substituting into (1) we obtain

$$(m-1)\alpha^2+\alpha-n \ge 0.$$

From this inequality, we get the minimum value of α:

$$\alpha = \left\lceil \frac{-1+\sqrt{4n(m-1)+1}}{2(m-1)} \right\rceil .$$

For this embedding f, we have

$$\beta_2(G_1\times G_2, f) = (m-1)\alpha.$$

The theorem follows. □

Theorem 3.2 *Let G_1 and G_2 be two connected graphs, $|V(G_1)| \ge 2$, $|V(G_2)| \ge 2$, we have*

$$\max\{\beta_2(G_1\times K_2), \beta_2(G_2\times K_2)\} \le \beta_2(G_1\times G_2)$$
$$\le \max\{B(G_1), B(G_2)\},$$

where $B(G_i)$ ($i = 1, 2$) is the standard bandwidth as given in [4], page 248.

Proof: Since $G_1 \times K_2$ and $G_2 \times K_2$ are subgraphs of $G_1 \times G_2$, we obtain

$$\max\{\beta_2(G_2 \times K_2), \beta_2(G_2 \times K_2)\}$$
$$\leq \beta_2(G_1 \times G_2).$$

Let $V(G_1) = \{v_1, v_2, \cdots, v_n\}$, $V(G_2) = \{u_1, u_2, \cdots, u_m\}$. Let f be the optimal embedding of G_1 in a line with bandwidth $B(G_1)$. Without loss of generality, let $f(v_i) = i$, where $i = 1, 2, \cdots, n$. Similarly, let g be the optimal embedding of G_2 in a line with bandwidth $B(G_2)$ and $g(u_j) = j$, where $j = 1, 2, \cdots, m$. In a rectangular coordinate system, we construct an embedding π for $G_1 \times G_2$ as follows:

$$\pi((v_i, u_j)) = (j, i),$$

where $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$. Thus,

$$\beta_2(G_1 \times G_2) \leq \beta_2(G_1 \times G_2, \pi)$$
$$= \max\{B(G_1), B(C_2)\}.$$

The theorem follows. □

Theorem 3.3 *Let $m, n \geq 2$, denote $d = \lceil \sqrt{n} - 1 \rceil$.*
(1) If $d^2 < n \leq d(d+2) = (d+1)^2 - 1$, we have $\beta_2(K_n \times P_m) = d$;
(2) If $n = (d+1)^2$, we have $\beta_2(K_n \times P_m) = d + 1$.

Proof: **Case 1** When $d^2 < n \leq d(d+2)$, by Lemma 2.3, we have

$$\beta_2(K_n) = \lceil \sqrt{n} - 1 \rceil = d.$$

Because $K_n \subseteq K_n \times P_m$, we have

$$\beta_2(K_n \times P_m) \geq \beta_2(K_n) = d.$$

On the other hand, we can embed K_n in a $(d+1) \times (d+1)$ square of maximum distance d. Place the points of K_n as far right as possible. In the leftmost column containing points, let the points be as far up as possible. Let $K_n^{(i)}$ be a copy of K_n, $i = 1, 2, \cdots, m$. If the leftmost column of the square contains none of the vertices of $K_n^{(i)}$, we embed the copy of $K_n^{(2)}$ just left one column of $K_n^{(1)}$, and so on (For example, $K_5 \times P_3$ in Figure 2), we get the embeddings of $K_n^{(i)}$, $i = 1, 2, \ldots, m$. Connect the points between the corresponding points of $K_n^{(i)}$ and $K_n^{(i+1)}$, $i = 1, 2, \ldots, m-1$. Thus, we get an embedding of $K_n \times P_m$. It is easy to see that

$$\beta_2(K_n \times P_m) \leq d.$$

If there is no column unoccupied, say x points being occupied in the left-most column, where $x < \lceil \sqrt{n} - 1 \rceil + 1 = d + 1$, we embed $K_n^{(2)}$ in the leftmost column of $K_n^{(1)}$ and let $K_n^{(2)}$ be under the x points, (For example, $K_7 \times P_3$ in Figure 2). In the same way, we get the embeddings of $K_n^{(3)}$, ..., $K_n^{(m)}$. Connect the points between the corresponding points of $K_n^{(i)}$ and $K_n^{(i+1)}$, $i = 1, 2, \ldots, m-1$. Thus, we get an embedding of $K_n \times P_m$, and we obtain, if $n \le d(d+2)$,

$$\beta_2(K_n \times P_m) \le d.$$

In Figure 2, for clarity, the lines among the copies of K_i ($i = 5, 7$) are omitted.

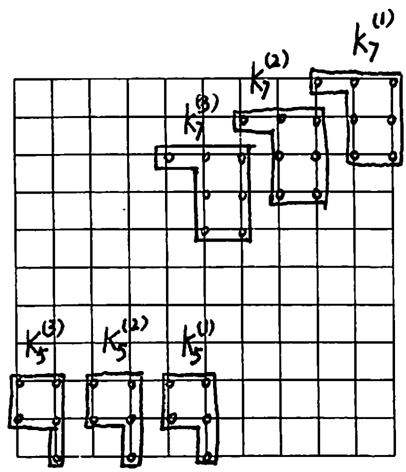

Figure 2. The embeddings of $K_5 \times P_3$ and $K_7 \times P_3$

Case 2 If $\beta_2(K_n \times P_m) = d$, $K_n^{(1)}$ must be contained in a $(d+1) \times (d+1)$ square, using all the points of the square. But then $K_n^{(i)}$, for $i > 1$, must use other points, a contradiction. It follows that $\beta_2(K_n \times P_m) \ge d + 1$. On the other hand, we embed K_n in a square, whose diameter is d, then, we move K_n parallelly $d+1$ columns, $\cdots$, $(m-1)(d+1)$ columns. In this way, we get m copies of K_n. At last, we draw lines among the corresponding points, which is the embedding of $K_n \times P_m$. Thus,

$$\beta_2(K_n \times P_m) \le \lceil \sqrt{n} - 1 \rceil + 1.$$

The theorem follows. □

Theorem 3.4 *Let $n, m \geq 3$, denote $d = \lceil \sqrt{n} - 1 \rceil$.*

(1) If $n = d^2 + 1$, we have $\beta_2(K_{d^2+1} \times C_3) = d$.

If $d^2 + 2 \leq n \leq (d+1)^2$, we have $\beta_2(K_n \times C_3) = d + 1$.

(2) If $m \geq 4$, we have $d \leq \beta_2(K_n \times C_m) \leq d + 1$.

Proof: case 1 Let $m = 3$.

Case 1.1 Let $m = 3$ and $n = d^2 + 1$. Since $K_n \subseteq K_n \times C_3$, by Lemma 2.3, we have

$$\beta_2(K_n \times C_3) \geq \beta_2(K_n) = \lceil \sqrt{n} - 1 \rceil = d.$$

On the other hand, we can embed $K_n \times C_3$ as follows: in a rectangular coordinate system, we embed the n points of $K_n^{(1)}$ at coordinates: $(d-1, d)$; $(0, d+1)$, $(1, d+1)$, $\cdots$, $(d-1, d+1)$; $(0, d+2)$, $(1, d+2)$, $\cdots$, $(d-1, d+2)$; $\cdots$; $(0, 2d)$, $(1, 2d)$, $\cdots$, $(d-1, 2d)$. Similarly, we embed the n points of $K_n^{(2)}$ at coordinates: $(2d-1, d)$; $(d, d+1)$, $(d+1, d+1)$, $\cdots$, $(2d-1, d+1)$; $(d, d+2)$, $(d+1, d+2)$, $\cdots$, $(2d-1, d+2)$; $\cdots$; $(d, 2d)$, $(d+1, 2d)$, $\cdots$, $(2d-1, 2d)$. We embed the n points of $K_n^{(3)}$ at coordinates: $(d-2, d)$, (d, d), $(d+1, d)$, $\cdots$, $(2d-2, d)$; $(d-1, d-1)$, $(d, d-1)$, $\cdots$, $(2d-2, d-1)$; $\cdots$; $(d-1, 1)$, $(d, 1)$, $\cdots$, $(2d-2, 1)$; $(2d-2, 0)$. We use $K_5 \times C_3$ and $K_{10} \times C_3$ as an example. (For clarity, the lines among the copies of K_i are omitted, where $i = 5, 10$.)

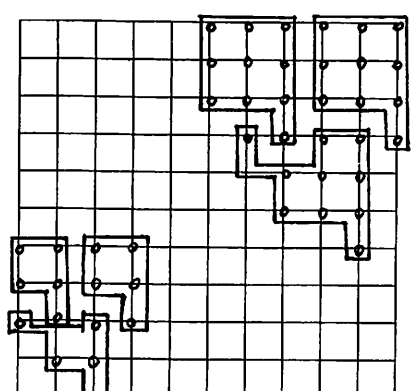

Figure 3 The embeddings of $K_5 \times C_3$ and $K_{10} \times C_3$

Thus,

$$\beta_2(K_n \times C_3) \leq d.$$

Therefore, $\beta_2(K_n \times C_3) = d$.

Case 1.2 Where $m = 3$ and $d^2 + 2 \leq n \leq (d+1)^2$. At first, we make three square regions S_1, S_2 and S_3 as follows: in a rectangular coordinate system, let

$$S_1 = \{(x,y)|y \leq 2d+1\} \cap \{(x,y)|y \geq d+1\}$$
$$\cap\{(x,y)|x \geq 0\} \cap \{(x,y)|x \leq d\}.$$
$$S_2 = \{(x,y)|y \leq 2d+1\} \cap \{(x,y)|y \geq d+1\}$$
$$\cap\{(x,y)|x \geq d+1\} \cap \{(x,y)|x \leq 2d+1\}.$$
$$S_3 = \{(x,y)|y \geq 0\} \cap \{(x,y)|y \leq d\}$$
$$\cap\{(x,y)|x \geq 1\} \cap \{(x,y)|x \leq d+1\}.$$

Then, we place the points of $K_n^{(i)}$ in S_i, and as before let the points of $K_n^{(i)}$ be as far right as possible and, in the leftmost column, as far up as possible, where $i = 1,2,3$. At last, we connect the corresponding points $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, where $x^{(1)} \in K_n^{(1)}$, $x^{(2)} \in K_n^{(2)}$, $x^{(3)} \in K_n^{(3)}$. From the embedding above, we obtain

$$\beta_2(x^{(1)}x^{(2)}) \leq d+1, \beta_2(x^{(1)}x^{(3)}) \leq d+1,$$
$$\beta_2(x^{(2)}x^{(3)}) \leq d+1.$$

Clearly, for $xy \in E(K_n^{(i)})$, we have

$$\beta_2(xy) \leq d,$$

where $i = 1,2,3$. Thus,

$$\beta_2(K_n \times C_3) \leq d+1.$$

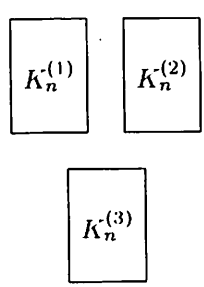

Figure 4. The embedding of $K_n \times C_3$.

In the following, we want to prove that

$$\beta_2(K_n \times C_3) \geq d + 1.$$

Because $K_n \times C_3 \supseteq K_{d^2+2} \times C_3$, we only need prove that

$$\beta_2(K_{d^2+2} \times C_3) \geq d + 1.$$

By way of contradiction, suppose that

$$\beta_2(K_{d^2+2} \times C_3) \leq d.$$

By Lemma 2.3, we have

$$\beta_2(K_{d^2+2}) = d.$$

Since K_{d^2+2} is a subgraph of $K_{d^2+2} \times C_3$, we obtain, under our supposition

$$\beta_2(K_{d^2+2} \times C_3) = d.$$

Since any three corresponding points from $K_n^{(1)}$, $K_n^{(2)}$ and $K_n^{(3)}$ form a triangle, we also want to keep the corresponding points between $K_n^{(1)}$ and $K_n^{(2)}$ at most distance d. Thus we must have one column of the square which contains $K_n^{(1)}$ to contain a point of $K_n^{(2)}$, without loss of generality, say, the leftmost column of $K_n^{(1)}$. Similarly, we must embed the first row of the square which contains $K_n^{(3)}$ into the last rows of the squares which contain $K_n^{(1)}$ and $K_n^{(2)}$. Thus at least two additional points are embedded among the first row of $K_n^{(3)}$, thus, the distance among the first row of $K_n^{(3)}$ is $d + 1$, which is a contradiction. In the following, we use $K_{11} \times C_3$ as an example.

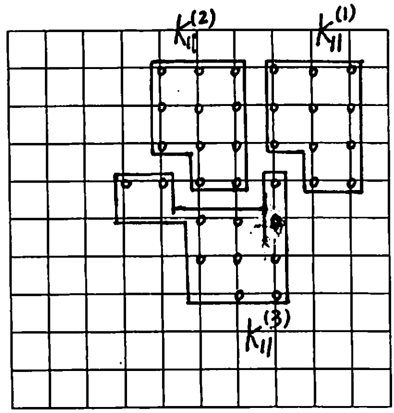

Figure 5

Case 2 Let us assume that $m \geq 4$.

In the following, we want to prove that

$$\beta_2(K_n \times C_m) \leq d + 1.$$

Since $K_n \times C_m \subseteq K_{(d+1)^2} \times C_m$, we only need prove that

$$\beta_2(K_{(d+1)^2} \times C_m) \leq d + 1.$$

Suppose that m is even, say $m = 2p$, where p a is natural number. In a rectangular coordinate system, we make $2p$ square regions $R_1, R_2, \cdots, R_{2p}$ as follows:

$$R_1 = \{(x,y)|x \geq 0\} \cap \{(x,y)|x \leq d\}$$
$$\cap\{(x,y)|y \geq 0\} \cap \{(x,y)|y \leq d\},$$
$$R_2 = \{(x,y)|x \geq d+1\} \cap \{(x,y)|x \leq 2d+1\}$$
$$\cap\{(x,y)|y \geq 0\} \cap \{(x,y)|y \leq d\},$$
$$\cdots,$$
$$R_p = \{(x,y)|x \geq (p-1)(d+1)\} \cap \{(x,y)|x \leq p(d+1)-1\}$$
$$\cap\{(x,y)|y \geq 0\} \cap \{(x,y)|y \leq d\},$$

$$R_{p+1} = \{(x,y)|x \geq (p-1)(d+1)\} \cap \{(x,y)|x \leq p(d+1)-1\}$$
$$\cap\{(x,y)|y \geq d+1\} \cap \{(x,y)|y \leq 2d+1\}.$$
$$\cdots,$$
$$R_{2p-1} = \{(x,y)|x \geq d+1\} \cap \{(x,y)|x \leq 2d+1\}$$
$$\cap\{(x,y)|y \geq d+1\} \cap \{(x,y)|y \leq 2d+1\}.$$
$$R_{2p} = \{(x,y)|x \geq 0\} \cap \{(x,y)|x \leq d\}$$
$$\cap\{(x,y)|y \geq d+1\} \cap \{(x,y)|y \leq 2d+1\}.$$

We place the $(d+1)^2$ points of $K^{(i)}_{(d+1)^2}$ at the grid points of R_i, where $i = 1, 2, \cdots, 2p$, in the same manner. We embed $K_9 \times C_6$ as an example. For clarity, the lines among the copies of K_9 are omitted.

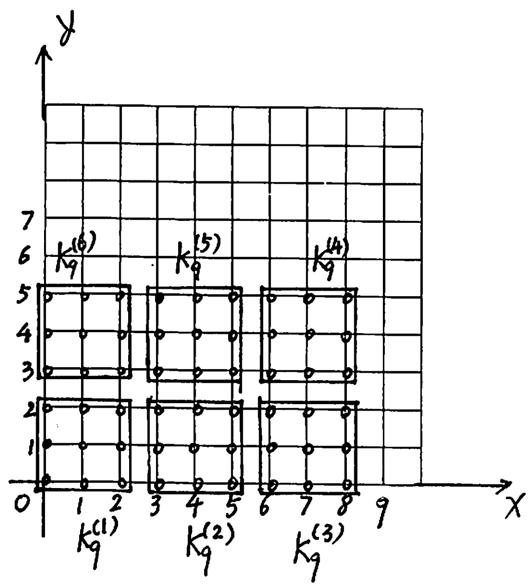

Figure 6. The embedding of $K_9 \times C_6$

From the embedding above, it is easy to see that

$$\beta_2(K_{(d+1)^2} \times C_{2p}) \leq d+1.$$

When m is odd, we can prove similarly.

Since $K_n \subseteq K_n \times C_m$, by Lemma 2.3, we have

$$\beta_2(K_n \times C_m) \geq \beta_2(K_n) = d.$$

The theorem follows. □

4 Concluding Remarks

Although they are special cases, product graphs are important in practical applications. In engineering, most special graphs are lattices, especially product graphs, such as $P_m \times P_n$. Therefore, many papers concerning bandwidth study product graphs, see [5], [6]. We hope that one can determine $\beta_2(K_n \times K_m)$.

Acknowledgement. The author would like to thank the referees for their many useful suggestions.

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