Domination Graphs of Tournaments and Other Digraphs

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Abstract

Domination graphs of directed graphs have been defined and studied in a series of papers by Fisher, Lundgren, Guichard, Merz, and Reid. A tie in a tournament may be represented as a double arc in the tournament. In this paper we examine domination graphs of tournaments, tournaments with double arcs, and more general digraphs.

Keywords: domination graph, tournament, underlying graph

1 Introduction

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. If $(x, y) \in A(D)$, we say that x beats y or write $x \to y$. Vertices x and y are said to dominate a digraph if for all vertices $z \neq x, y$, either $x \to z$ or $y \to z$. If D is a digraph, the *domination graph* of D, denoted $dom(D)$, is the graph G with vertex set $V(D)$ and an edge between each pair of vertices that dominate D. A directed graph with exactly one arc between each pair of vertices in D is a *tournament*. Fisher, Lundgren *et al* have completely characterized domination graphs of tournaments [9], [5], [6]. They have also obtained results on the domination graphs of arbitrary digraphs [8].

In section 2 of this paper we characterize those digraphs D whose domination graph is isomorphic to the (undirected) graph underlying D. We also characterize those connected graphs that are the domination graph of a "proper oriented graph" (defined below.) In section 3, we look at domination graphs of tournaments which may have double arcs. In particular, we prove that if T is a tournament with double arcs and $dom(T)$ is isomorphic to K_n, then T must have at least $\binom{n}{2} - n$ double arcs. We also show that every n-cycle with n even is an induced subgraph of a domination graph of a tournament with double arcs.

The domination graph of a tournament was introduced by Fisher, Lundgren, Merz, and Reid [9] who extended the concept to the domination graph of a digraph [8] and continued the work in additional papers with Guichard ([5] and [6].) Their work was originally motivated by the observation that the domination graph of a tournament is the complement of the competition graph of its reversal. The competition graph of a digraph D is the graph with vertex set $V(D)$ and an edge between vertices x and y if there is a vertex $z \neq x, y$ such that both x and y beat z in D. Competition graphs arise naturally when a digraph models a food web, where an arc from species x to species y means x preys on y, so that an edge between x and y in the competition graph means that x and y compete for the same prey. Results on domination graphs thus increase the understanding of competition graphs. See the survey article by Lundgren [11] for more on competition graphs.

2 Domination graphs of proper oriented graphs

Following the terminology of Fisher,et al [8], a directed graph with no loops or multiple edges will be called an *oriented graph*. An oriented graph on n vertices with fewer than $\binom{n}{2}$ edges will be called a *proper oriented graph*. We first characterize those oriented graphs whose underlying graph is isomorphic to the domination graph.

Notation 1 *D will represent an oriented graph, $UG(D)$, the underlying undirected graph, and $dom(D)$ the domination graph of D. When $UG(D)$ is isomorphic to $dom(D)$, then $\mathcal{D}(dom(D))$ will represent the domination graph with the orientation inherited from D. A directed star is a directed graph on n vertices and n edges in which one vertex beats all others. For a vertex $v \in D$, $O(v)$ will represent the set of all vertices x such that $v \rightarrow x$ and $I(v)$ the set of all vertices x such that $x \rightarrow v$.*

Example 2 *Let D have the vertices a, b, c with $a \rightarrow b$ and c isolated. Then the domination graph of D has an edge between a and c and b is isolated. Clearly, the domination graph is isomorphic to $UG(D)$. To simplify notation below, we will label the vertices of $dom(D)$ as A, B, C, where the isomorphism takes a to A, b to B, and c to C and the orientation in $\mathcal{D}(dom(D))$ is $A \rightarrow B$.*

Lemma 3 *If $UG(D)$ is isomorphic to $dom(D)$ then in D we cannot have vertices a, b,and c with $a \rightarrow b$ and $c \rightarrow b$.*

Proof. Since $UG(D)$ is isomorphic to $dom(D)$, D is isomorphic to $\mathcal{D}(dom(D))$. If a and c both beat b, then there are corresponding edges $\{A, B\}$ and $\{C, B\}$ in the domination graph, and so $A \rightarrow B$ and $C \rightarrow B$ in $\mathcal{D}(dom(D))$ and both pairs A, B and C, B dominate $\mathcal{D}(dom(D))$. But since A and B dominate $\mathcal{D}(dom(D))$ and $C \rightarrow B$, it follows that $A \rightarrow C$. Similarly we show that $C \rightarrow A$. Thus, it is not possible that both a and c beat b in D. ∎

Lemma 4 *If D is an oriented graph with $n \geq 4$ vertices and $UG(D)$ is isomorphic to $dom(D)$, then D has no isolated vertices or n isolated vertices.*

Proof. If D has exactly one isolated vertex v, then the only pair that can possibly dominate D is a pair $\{x, v\}$, where $x \to z$ for all other vertices z. But then there can be at most one edge in $dom(D)$, so at most one edge in D. This is not possible since $n \geq 4$. If D has two or more isolated vertices, then there can be no edges in $dom(D)$, so none in D. ■

Theorem 5 *Let D be an oriented graph on $n \geq 3$ vertices. Then $UG(D)$ is isomorphic to $dom(D)$ iff ($n = 3$ and D does not contain one vertex which is beaten by the other two) or ($n \geq 4$ and D is a directed star or has n isolated vertices.)*

Proof. If $n = 3$, then there are five non-isomorphic directed graphs on three vertices which are not the forbidden type; each one has its underlying graph isomorphic to its domination graph. The two non-isomorphic forbidden directed graphs do not have underlying graphs isomorphic to the domination graph. If $n \geq 4$, it is straight-forward to show that the two specified types of directed graphs have underlying graphs isomorphic to the domination graph. For the converse, assume D is an oriented graph with at least four vertices and that $dom(D)$ is graph isomorphic to $UG(D)$. It follows from Lemma 4, that D has all isolated vertices or no isolated vertices. Assume D has no isolated vertices and k is the number of weak components of D, each of which has at least one edge. Then since $dom(D)$ must have at least k edges and no isolated vertices, k must be less than or equal to two. If $k = 2$, then $dom(D)$ has at most one edge while D has at least two edges. Thus, there is exactly one component in D and so in $dom(D)$. Pick a vertex v in D with $|O(v)| \geq 1$. By Lemma 3, $|I(v)| \leq 1$. If $|I(v)| = 1$, then there are vertices x and y with $x \to v \to y$. If $|I(v)| = 0$, then either $v \to x$ for every vertex $x \in D$, or there are vertices x and y in D with $v \to x \to y$. Thus, in either case, D is either a star or has a path, which we indicate as $v \to x \to y$. Assume D is not a star. Since $UG(D)$ is isomorphic to $dom(D)$, in $domD$ there is a corresponding path $V - X - Y$ which can be directed $V \to X \to Y$. Since $\{X, Y\}$ must be a dominating edge, we must have have $y \to v$ in D. Since $n \geq 4$ and we have only one component, there must be another vertex z in D with a directed edge connecting it to this 3-cycle. By Lemma 3, it must be directed outward, so assume $x \to z$. But then by the same argument we used above, we must also have $z \to v$. But this contradicts lemma 3 since we also have $y \to v$. It follows that if D has no isolated vertices, it must be a directed star. ■

We will now classify all connected graphs which can occur as the domination graph of a proper oriented graph. The next lemma is presented without proof.

Lemma 6 *Let D be a directed graph. Then $O(v)$ is an independent set in $dom(D)$ for every vertex v.*

Definition 7 *If n is odd, U_n is the tournament on the vertices $0, ..., n-1$ with directed edges (i, j) iff $j - i$ is positive and odd or negative and even.*

Lemma 8 *[9] Let T be an n-tournament, where n is odd. Then C_n is a subgraph of dom(T) iff T is isomorphic to U_n.*

Lemma 9 *[8] If D is an oriented graph, then dom(D) is either an odd cycle with or without isolated and/or pendant vertices, or a forest of caterpillars.*

Theorem 10 *If H is a connected graph, then H is the domination graph of some proper oriented graph iff H is a caterpillar or a spiked odd cycle with at least two spikes on some vertex.*

Proof. To prove necessity, notice that Fisher and Lundgren et al. [8] have shown that the domination graph of an oriented graph is a caterpillar or a spiked odd cycle. It remains to show that a spiked odd cycle with at most one pendant vertex at each vertex in the odd cycle cannot be the domination graph of a proper oriented graph, although it must be the domination graph of a tournament.

If C_n is an odd cycle and is equal to *dom*(D) for some oriented graph D, then by adding edges to D arbitrarily we find a tournament T such that, C_n = *dom*(T). But then $T = U_n$ by Lemma 8, and since no subset of U_n will generate C_n, it follows that $D = U_n$. Then if C_n is a subset of *dom*(D) for an oriented graph D, $U_n \subset D$. Now assume $H =$ *dom*(D) contains C_n, where n is odd, and has at most one spike on each vertex of C_n. We will show D must be a tournament. Label the vertices of C_n by $0, 1, ..., n-1$. Fix vertex i. If there is a spike at vertex i, we label it x_i. Since $i-1 \rightarrow i$ in U_n, and $\{i, x_i\}$ dominates, we cannot have $i-1 \rightarrow x_i$. But then, since $\{i-1, i\}$ dominates, we must have $i \rightarrow x_i$. Let k be another vertex in C_n. If $k \rightarrow i$ then since $\{i, x_i\}$ dominate, $x_i \rightarrow k$. If $i \rightarrow k$, then $k-1$ (mod n) must beat k (by the definition of U_n), and since $\{k-1, k\}$ dominate, $k-1 \rightarrow i$. But then since $\{k, k-1\}$ dominate, we must have $k \rightarrow x_i$. That is, there must be directed edges in D between x_i and every vertex k in C_n.

Now say vertices i and j in C_n each have spikes x_i and x_j. Assume $i \rightarrow j$.Then since j, x_j dominate, we must have $x_j \rightarrow i$, but then since i, x_i also dominate, $x_i \rightarrow x_j$. That is, there are edges in D between each pair of spikes. It follows that D must be a tournament.

For the sufficiency, let H be a caterpillar. By Theorem 5.2 of [8, Theorem 5.2], H is the domination graph of an oriented graph. The proof of that theorem shows that H is in fact the domination graph of a proper oriented graph. Now let H be a spiked odd cycle with at at least two spikes on at least one vertex of the cycle. Pick a subgraph H' of H which consists of the cycle and at most one of the spikes (x_i) at each vertex i. Then H' is the domination graph of a tournament T. Now we add the other vertices to T. If y_i is another spike at vertex i then, for each other vertex v in C_n define $y_i \rightarrow v$ in T iff $x_i \rightarrow v$ and for v a spike at $j \neq i$, define $y_i \rightarrow v$ iff $x_i \rightarrow x_j$. We do not add edges between spikes on the same vertex. Then the resulting oriented graph is proper and has H as its domination graph. ■

Example 11 *A triangle is the domination graph of a tournament but not the domination graph of a proper oriented graph. A graph with one edge and one* *isolated vertex is the domination graph of a proper oriented graph but not the domination graph of any tournament. A four cycle is the domination graph neither of a tournament nor of a proper oriented graph. A caterpillar with exactly two pendant vertices on one end is the domination graph of a proper oriented graph but not the domination graph of any tournament. (This last result follows from Theorem 5 of [7].)*

3 Domination graphs of tournaments with double arcs

If a tournament, T, on n vertices has all double arcs, then $dom(T) = K_n$. We pose the following problem.

For each positive integer n, find $d(n)$, the smallest number of double edges in any tournament, T, on n vertices for which $dom(T) = K_n$, the complete graph on n vertices. $d(n)$ is characterized in the theorem below.

The authors learned in April 2002 that Theorem 12 was obtained independently by Factor and Factor. Their result has now appeared in [4]. Related work has also appeared in [3] and [12].

Theorem 12 *For* $n \geq 3$, $d(n) = \binom{n}{2} - n$

Proof. Given the positive integer n, construct a tournament T as follows. Label the vertices $0, 1, ..., n-1$. Direct $i \to i+1 (mod\, n)$. for each i. Construct double arcs between each other pair. (Of course, for $n = 3$ there are no other pairs.) Then there are $\binom{n}{2} - n$ double arcs. If i, j is any pair of vertices, consider the vertex k. If $k = i + 1 (\text{mod } n)$ or $k = j + 1 (\text{mod } n)$ then $i \to k$ or $j \to k$. Otherwise, at least one of $\{i, k\}$ or $\{j, k\}$ is a double arc and so either $i \to k$ or $j \to k$. We conclude that $\{i, j\}$ dominates T. Since this is true for every pair, $dom(T) = K_n$. Thus, $d(n) \leq \binom{n}{2} - n$.

To show that this number is actually the minimum, let T be a tournament on n vertices with $dom(T) = K_n$ and assume T has *at least* n single edges. We show that T has *exactly* n single arcs, and $\binom{n}{2} - n$ double arcs. That is, for any tournament whose domination graph is K_n the number of single arcs is at most n so that the number of double arcs is at least $\binom{n}{2} - n$ and hence, $d(n) \geq \binom{n}{2} - n$.

Consider the subgraph S of T consisting of n of the single arcs in T. Observe that we cannot have vertices a, b, c with $a \to b$ and $a \to c$ in S. (Since every pair dominates T, b, c must dominate, but neither beats a.) Since there are n vertices in T, each of which can have outdegree in S of at most one, and the sum of these outdegrees must equal n (the number of edges in S), it follows that each of the n vertices of T is the tail of exactly one arc from S.

Let v and w be vertices in T and assume that $\{v, w\}$ is not an edge in S. We claim that $\{v, w\}$ is a double arc in T. Since v and w are tails of single arcs in S, there are vertices v^* and w^* (perhaps equal) with $v \to v^*$ and $w \to w^*$. Since every pair in T must dominate T, $\{v, w^*\}$ dominates, so that $v \to w$ in T. Similarly, $\{w, v^*\}$ dominates so that $w \to v$ in T, and hence, $\{v, w\}$ is a double arc. It follows that the number of single arcs is exactly n. ∎

Example 13 *Given the positive integer n, the tournament which has the minimal number of double arcs need not be unique. Further, the components of the subgraph of single arcs may not consist of disjoint cycles.*

Let T_1 be the tournament on $n = 7$ vertices which has single arcs on a seven cycle and double arcs everywhere else. We define a second tournament, T_2, on seven vertices as follows. We have two components. The first has a directed three-cycle. The second has a directed three cycle plus one arc pendant at a vertex of the three cycle. That arc is oriented toward the cycle. Make all other arcs in T_2 double. Then it can be verified that $dom(T_1) = dom(T_2) = K_7$.

Remark 14 *The main open problem in this area is to completely characterize those graphs which are the domination graphs of tournaments with double arcs. We have only partial results and a conjecture in this direction. We begin with n-cycles. Every n-cycle with n-odd is the domination graph of a tournament (with single arcs) (Lemma 8) and so is the domination graph of a tournament which may have double arcs. However, an even n-cycle cannot be a subgraph of a domination graph of tournament ([9], Lemma 2.1). In contrast, for tournaments with double arcs we have Corollary 16 below.*

Theorem 15 *Let G be a graph with n vertices, where is n odd. If G contains an n-cycle and exactly n+1 edges, G is a domination graph of a tournament with double arcs.*

Proof. Assume the cycle is labeled $0, 1, ..., n-1$ and let $\{0, j\}$ be the extra edge in G. We can assume $j \leq (n+1)/2$ (or relabel the cycle.) We construct a tournament T so that $G = dom(T)$. First, put the single arcs of U_n in T, so that the $n-$cycle of G is in $dom(T)$ (Lemma 8).

Case 1. $j \leq (n+1)/2$, j *odd*

Add arcs $0 \to 4, ..., j-1$ to T (making these arcs double) and also $j \to 2$.Then $0 \to 4, ..., j-1,\ 1, 3, ..., n-2$ and $j \to 2,\ j+1, j+3, ..., n-1$ (from U_n.) Thus, the edge $\{0, j\}$ is in $dom(T)$. We claim there are no extra edges in the domination graph. Since we only have added arcs originating at 0, we need only check edges containing 0, so consider $\{0, v\}$, where $v \neq j, 1, n-1$. If v is odd, then $\{0, v\}$ does not dominate 2, so $\{0, v\}$ is not in $dom(T)$. Similarly, if v is even $\{0, v\}$ does not dominate $n-1$, so $\{0, v\}$ is not in $dom(T)$.Thus, $G = dom(T)$.

Case 2. $j \leq (n+1)/2$, j *even*

Add arcs $0 \to j+2, j+4, ..., n-3$ to Also add arc $j \to n-1$. Then $0 \to, 1, 3, 5, n-2, j+2, j+4, ..., n-3$ and $j \to 2, 4, ..., j-2, j+1, j+3, ..., n-2$. Thus, $\{0, j\}$ dominates G. We claim we have not added any other edges to the domination graph. It is sufficient to check edges of the form $\{0, v\}$ for $v \neq 1,\ j,$ *or* $n-1$ and $\{j, v\}$ for $v \neq 0,\ j-1,$ *or* $j+1$. Consider $\{0, v\}$ for $v \neq j,$ or $n-1$, v *even*. Then $\{0, v\}$ does not dominate $n-1$. Now consider $\{0, v\}$ for $v \neq j,\ v \neq 1,\ v$ *odd*. Then for $v \neq j, v$ does not dominate 2. Now we consider edges of the form $\{j, v\}$ for $v \neq 0,\ j-1,$ *or* $j+1$, where v is odd. Then, if $j < v$, $\{j, v\}$ does not dominate $v-1$, while if $v < j$, $\{j, v\}$ does not dominate

$j-1$. Finally, we consider edges of the form $\{j, v\}$ for $v \neq 0,\ j-1,$ *or* $j+1$ where v is even. Then $\{j, v\}$ does not dominate 1.

It follows that $G = dom(T)$. ∎

Corollary 16 *If n is even, C_n is an induced subgraph of a domination graph of a tournament with double arcs on $n+1$ vertices.*

The following is an attractive conjecture. We have been able to prove it only for the case $n = 5$.

Conjecture 17 *Let n be an odd integer. If G is a graph on n vertices and contains an n-cycle, then G is the domination graph of a tournament which may have double arcs.*

We end with a statement of a theorem characterizing those graphs on 5 vertices which are the domination graphs of tournaments with double arcs. We omit the proof.

Theorem 18 *A connected graph on 5 vertices is the domination graph of a tournament with double arcs iff it satisfies one of the following:*

(a) contains a 5-cycle;

(b) is a spiked 4 cycle with one or two diagonals;

(c) is 3 cycle with two spikes;

(d) is a star.

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