Catalan Numbers, Lucas numbers, and Circuits

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Abstract

Shapiro [8] asked what simple family of circuits will have resistances C_{2n}/C_{2n-1} (or something similar) where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the mth *Catalan* number. In this paper we give a construction of such circuits; we also discuss some related problems.

1. Introduction

Consider the following circuits

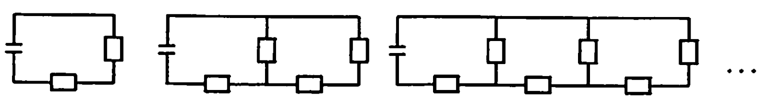

Figure 1

where all resistors are one ohm. It is easy to see that these circuits have resistances

$$2/1, 5/2, 13/8, 34/21, \cdots, F_{2n}/F_{2n-1},$$

where $\{F_n\}_{n\geq 0} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots\}$ is the Fibonacci sequence.

In [8], Shapiro posed some open questions, the first of which is the following problem.

Question. What simple family of circuits will have resistances C_{2n}/C_{2n-1} (or something similar) where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the mth *Catalan* number ?

In this paper, we answer this question by giving a construction of the circuits satisfying the required condition.

2. Answer to the Question

Theoretically there exist simple circuits with $f(2n)/f(2n-1)$ for any positive-valued rational function$f(n)$, because a circuit with resistances m/n, a positive rational number, can be constructed in this way: first connect n resistors in parallel (this circuit has resistances $1/n$), then connect m such circuits in series. (Note that all resistors must be one ohm.)

However, the above construction may not be the best one. For example, a circuit with resistances $\frac{2n+1}{n+1}(= 1 + \frac{n\times 1}{n+1})$ can be constructed in a simpler way: first, connect n resistors in series; then, connect it with one resistor in parallel; finally, connect it with one resistor in series. Note that this simpler construction needs only $n+2$ resistors, whereas the above construction needs $(2n+1)(n+1)$ resistors.

So, in what follows we are only interested in the function $f(n)$, which satisfies the following condition: the circuits with resistances $f(n)$ can be constructed according to the value of n or $f(k)$ $(0 < k < n)$ (In the later case we can construct the circuits recursively). We call such a function $f(n)$ *simple.* We also require the resistors used as few as possible.

Theorem 1. Let a, b, c, d and e be nonnegative integers. Then both $\dfrac{cn+d}{an+b}$ and $\dfrac{cn^2+dn+e}{an+b}$ are simple.

Proof. We need only to prove that $\dfrac{cn+d}{an+b}$ is simple. Note that the resistances of a circuit constructed by connecting two resistors in parallel with resistances R_1 and R_2, respectively, are $\frac{1}{1/R_1+1/R_2} = \frac{R_1R_2}{R_1+R_2}$. Since

$$\begin{aligned}\frac{cn+d}{an+b} &= \frac{cn}{an+b} + \frac{d}{an+b} \\ &= \frac{1}{\frac{a}{c}+\frac{b}{cn}} + \frac{d}{an+b},\end{aligned}$$

we need at most $ac + bcn + d(an+b) = ac + bd + (ad+bc)n$ resistors to construct the required circuit.

Corollary 1. Let C_n be the nth Catalan number. Then both C_{2n}/C_{2n-1} and C_n/C_{n-1} are simple.

Proof. The conclusion follows from Theorem 1 and the following computation:

$$\begin{aligned}
\frac{C_{2n}}{C_{2n-1}} &= \frac{1}{2n+1}\cdot\binom{4n}{2n}\Big/\frac{1}{2n}\cdot\binom{4n-2}{2n-1}\\
&= \frac{1}{2n+1}\cdot\frac{(4n)!}{(2n)!(2n)!}\Big/\frac{1}{2n}\cdot\frac{(4n-2)!}{(2n-1)!(2n-1)!}\\
&= \frac{2(4n-1)}{2n+1}\\
&= 2\times\frac{2n\times 1}{2n+1}+\frac{(2n-1)\times 2}{(2n-1)+2}.
\end{aligned}$$

and

$$\begin{aligned}
\frac{C_n}{C_{n-1}} &= \frac{1}{n+1}\cdot\binom{2n}{n}\Big/\frac{1}{n}\cdot\binom{2n-2}{n-1}\\
&= \frac{4n-2}{n+1}\\
&= 2\times\frac{n\times 1}{n+1}+\frac{(n-1)\times 2}{(n-1)+2}.
\end{aligned}$$

The corresponding circuits are shown in Figures 2-3.

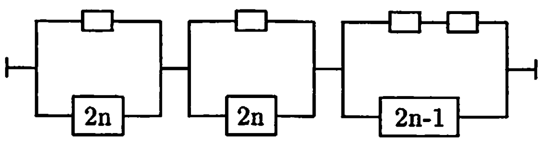

Figure 2 Circuit with resistances C_{2n}/C_{2n-1}

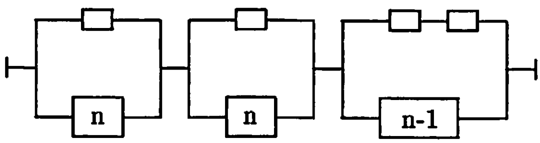

Figure 3

Note: [n] denotes the circuit constructed by connecting n resistors in series.

Corollary 2. Let $f(n)=\binom{2n}{n}$ be the central binomial coefficient. Then both $f(2n)/f(2n-1)$ and $f(n)/f(n-1)$ are simple.

Proof. Since $f(n)=\binom{2n}{n}$, the conclusion follows from Theorem 1 and the following computation:

$$\begin{aligned}
\frac{f(2n)}{f(2n-1)} &= \binom{4n}{2n}\Big/\binom{4n-2}{2n-1}\\
&= \frac{4n-1}{n}\\
&= 3+\frac{(n-1)\times 1}{(n-1)+1}
\end{aligned}$$

and

$$\begin{aligned} \frac{f(n)}{f(n-1)} &= \frac{4n-2}{n} \\ &= 2+2\times\frac{(n-1)\times 1}{(n-1)+1}. \end{aligned}$$

The corresponding circuits are as in Figures 4-5.

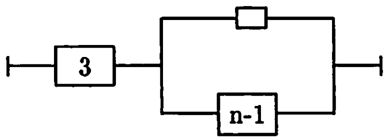

Figure 4

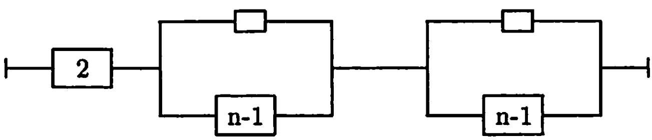

Figure 5

3. Lucas Numbers and Circuits

Let $\{u_n\}_{n\geq 0}$ be the Lucas sequence [7] defined by $u_0 = 1$, $u_1 = A$, and $u_{n+1} = Au_n + Bu_{n-1}(n \geq 1)$, where A and B are positive integers. (For other definitions about Lucas numbers see [3], [5], [6], [9].) In the case $A = B = 1$, $\{u_n\}_{n\geq 0}$ is simply the Fibonacci sequence; in the case $A = 2$ and $B = 1$, $\{u_n\}_{n\geq 0}$ gives the Pell sequence.

Theorem 2. If $A = kB + 1$ where k is a nonnegative integer, then both u_{2n}/u_{2n-1} and u_n/u_{n-1} are simple.
Proof. Since $A = kB + 1$, we have

$$\begin{aligned} \frac{u_{2n}}{u_{2n-1}} &= \frac{(kB+1)u_{2n-1}+Bu_{2n-2}}{u_{2n-1}} \\ &= kB+1+\frac{Bu_{2n-2}}{(kB+1)u_{2n-2}+Bu_{2n-3}} \\ &= kB+1+\frac{B\cdot u_{2n-2}/u_{2n-3}}{B+(kB+1)u_{2n-2}/u_{2n-3}} \\ &= kB+1+\frac{1}{\frac{1}{B}+\frac{1}{u_{2n-2}/u_{2n-3}}+\underbrace{\frac{1}{1}+\cdots+\frac{1}{1}}_{k}}. \end{aligned}$$

Note that the resistances of a circuit constructed by connecting n resistors in parallel with resistances $R_1, R_2, \cdots, R_n$, respectively, are

$$\frac{1}{1/R_1 + 1/R_2 + \cdots + 1/R_n}.$$

It is easy to see that the circuits in Figure 6 have resistances

$$u_2/u_1, u_4/u_3, \cdots, u_{2n}/u_{2n-1}.$$

For u_n/u_{n-1}, the proof is similar.

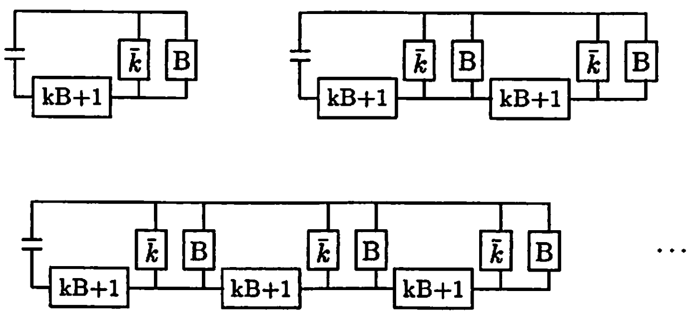

Figure 6

Note: $\boxed{\bar{k}}$ denotes the circuit constructed by connecting k resistors in parallel. When $k = 0$, $\boxed{\bar{0}}$ denotes empty circuit.

Remark 1. In the case $k = 0$ and $B = 1$, Figure 6 reduces to Figure 1, as expected.

Remark 2. For the applications of Lucas numbers in ladder networks and electric line theory we refer the reader to [2] and [4], and the literature they cited.

At the end of this paper, we pose a open question: What simple family of circuits will have resistances M_n/M_{n-1} (or something similar) where M_n is the *Motzkin* number(cousin of *Catalan* number)?

Note: *Motzkin* numbers [1] are defined by

$$M_0 = 1, \ M_{n+1} = M_n + \sum_{k=0}^{n-1} M_k M_{n-1-k} \ (n \geq 0),$$

the recurrence of which is similar to that of Catalan numbers.

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