k-factor-covered Regular Graphs*

Jia Shen Heping Zhang†

Department of Mathematics, Lanzhou University, Lanzhou
Gansu 730000, P. R. China

Abstract

A graph G is called f-factor-covered if every edge of G is contained in some f-factor. G is called f-factor-deleted if $G - e$ contains a f-factor for every edge e. Bäbler proved that every r-regular, $(r-1)$-edge-connected graph of even order has a 1-factor. In the present article, we prove that every $2r$-regular graph of odd order is both $2m$-factor-covered and $2m$-factor-deleted for all integers m, $1 \leq m \leq r-1$, and every r-regular, $(r-1)$-edge-connected graph of even order is both m-factor-covered and m-factor-deleted for all integers m, $1 \leq m \leq \lfloor \frac{r}{2} \rfloor$.

Keywords Regular graph; f-factor; k-factor; k-factor-covered graph; k-factor-deleted graph.

1 Introduction

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. For a vertex $v \in V(G)$, $E_G(v)$ denotes the set of edges of G incident to v, and the degree of v is $d_G(v) := |E_G(v)|$. Let $f\colon V(G) \to \mathbb{Z}_+$ be a function, where $\mathbb{Z}_+$ denotes the set of nonnegative integers. An f-factor of G is a spanning subgraph F such that $d_F(v) = f(v)$ for all $v \in V(G)$. An f-factor is said to be a k-factor if $f(v) = k$ for all $v \in V(G)$, where $k \in \mathbb{Z}_+$ is a constant. A necessary and sufficient condition for a graph to have an f-factor was given by Tutte [11].

An f-factor-covered graph G is a graph such that every edge of G is contained in some f-factor. An f-factor-deleted graph G is a graph such that $G - e$ contains an f-factor for every edge $e \in E(G)$.

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†Corresponding author. E-mail address: zhanghp@lzu.edu.cn(H. Zhang)

Petersen [9] showed that every cubic graph with no more than two cut-edges has a 1-factor. In 1938, Bäbler [1] generalized Petersen's result as follows: every r-regular and $(r-1)$-edge-connected graph of even order has a 1-factor. Further, Plesnik [10] obtained the following stronger result.

Theorem 1.1 (Plesnik, [10]) *Let G be an r-regular and $(r-1)$-edge-connected graph of even order. Then G has a 1-factor that contains no any given $r-1$ edges.*

In 1972, Little [4] introduced the concept of 1-factor-covered graph. By virtue of Plesnik's theorem, we can easily derive the following result.

Theorem 1.2 *If G is an r-regular, $(r-1)$-edge-connected graph of even order, $r \geq 2$, then G is both 1-factor-covered and 1-factor-deleted.*

On the other hand, Katerinis proved the following result, which examines the existence of a k-factor in a vertex-deleted subgraph of a regular graph of odd order.

Theorem 1.3 (Katerinis, [3]) *Let G be a $2r$-regular, $2r$-edge-connected graph of odd order and m be an integer such that $1 \leq m \leq r$. Then for every $u \in V(G)$, the graph $G-u$ has an m-factor.*

This article presents further generalizations of Theorems 1.2 and 1.3. First, we consider r-regular graphs G of odd order. Obviously r must be even; if G has an m-factor, m must be even too. The following result is proved in Section 2.

Theorem 1.4 *If G is a $2r$-regular graph of odd order, then G is both $2m$-factor-covered and $2m$-factor-deleted for every integer m, $1 \leq m \leq r-1$.*

For r-regular graphs of even order, we obtain the following main result of this article, which is a further generalization of Bäbler's theorem.

Theorem 1.5 *Let G be an r-regular and $(r-1)$-edge-connected graph of even order. Then G is both m-factor-covered and m-factor-deleted for every integer m, $1 \leq m \leq \lfloor \frac{r}{2} \rfloor$.*

This main theorem can be proved by Kano's theorem [2]. In Section 3 we give a direct proof by Tutte's f-factor theorem [11] and Liu's characterization theorem [6] of both f-factor-covered and f-factor deleted graphs.

2 Proof of Theorem 1.4

The following classical theorem is useful for the proof of theorem 1.4.

Theorem 2.1 (Petersen, [9]) *(1) Every r-regular bipartite graph is the union of edge-disjoint 1-factors.*
(2) Every 2r-regular graph is the union of edge-disjoint 2-factors.

Proof of Theorem 1.4. By Theorem 2.1, we know that every $2r$-regular graph is the union of r edge-disjoint 2-factors, denoted by $H_1, H_2, \dots, H_r$. For an edge $e \in E(G)$, assume without loss of generality that $e \in H_1$. Then $e \in \cup_{i=1}^{m} H_i$, and $e \notin \cup_{i=2}^{m+1} H_i$, $1 \le m < r$, which are $2m$-factors of G. □

3 Proof of Theorem 1.5

For $X, Y \subseteq V(G)$, let $f(X) := \sum_{v \in X} f(v)$ and $d_G(X) := \sum_{v \in X} d_G(v)$. Let $E_G(X,Y)$ denote the set of edges with one end-vertex in X and the other in Y, and $e_G(X,Y) := |E_G(X,Y)|$.

To prove Theorem 1.5, we need Tutte's f-factor theorem. We first recall some definitions. For any given disjoint sets $S, T \subseteq V(G)$, a component C of $G-(S \cup T)$ is called an odd or even component of $G-(S \cup T)$ according to whether $e_G(V(C),T) + f(V(C))$ is odd or even. Let $q_G(S,T;f)$ denote the number of odd components of $G-(S \cup T)$.

Theorem 3.1 (Tutte, [11]) *Let G be a graph and $f\colon V(G) \to \mathbb{Z}_+$ a function. G has an f-factor if and only if*

$$\delta_G(S,T;f) := f(S) - f(T) + d_{G-S}(T) - q_G(S,T;f) \ge 0$$

for all disjoint sets $S, T \subseteq V(G)$. Moreover, $\delta_G(S,T;f) \equiv f(V(G)) \pmod 2$ for all disjoint sets $S, T \subseteq V(G)$.

The following result is a characterization of graphs which are both f-factor-covered and f-factor-deleted.

Theorem 3.2 (Liu, [6]) *Let G be a graph and $f\colon V(G) \to \mathbb{Z}_+$ a function. Then G is both f-factor-covered and f-factor-deleted if and only if G is 2-edge-connected and for all disjoint sets $S, T \subseteq V(G)$,*

$$\delta_G(S,T;f) := f(S) - f(T) + d_{G-S}(T) - q_G(S,T;f) \ge \varepsilon(S,T),$$

where $\varepsilon(S,T) = 2$ *if* S *or* T *is not independent or* $S \cup T \neq \emptyset$ *and* $G-(S\cup T)$ *has an even component;* $\varepsilon(S,T) = 0$*, otherwise.*

Proof of Theorem 1.5. If r=2 or 3, Theorem 1.5 holds by Theorem 1.2. Therefore we assume that $r \geq 4$ from now on. Suppose for a contradiction that for some integer m, $1 \leq m \leq \lfloor \frac{r}{2} \rfloor$, G is not m-factor-covered and m-factor-deleted. We know from Theorem 3.2 that there exist disjoint sets $S, T \subseteq V(G)$ such that

$$\delta_G(S,T;m) < \varepsilon(S,T) \leq 2.$$

Since by Theorem 3.1 $\delta_G(S,T;m) \equiv m|V| \equiv 0 \pmod 2$, we have

$$\delta_G(S,T;m) \leq 0.$$

That is

$$m|T| - d_{G-S}(T) + q_G(S,T;m) \geq m|S|. \tag{1}$$

Then $S \cup T \neq \emptyset$, for otherwise $\varepsilon(S,T) = 0$ and $\delta_G(S,T;m) = 0$, contradicting $\varepsilon(S,T) > \delta_G(S,T;m)$.

Let $W := G - (S \cup T)$ and let $\omega(W)$ be the number of components of W. There are two cases to consider.

Case 1. $\omega(W) = 0$. Since $\omega(W) \geq q_G(S,T;m)$, (1) implies

$$m|T| - d_{G-S}(T) \geq m|S|, \tag{2}$$

and

$$|T| \geq |S|.$$

Since G is r-regular, G itself is an r-factor. By Theorem 3.1 we have

$$r|T| - d_{G-S}(T) \leq r|S|.$$

Subtracting (2) from the above inequality, we have that

$$(r-m)|T| \leq (r-m)|S|,$$

so that

$$|T| \leq |S|.$$

Therefore $|T| = |S|$, which together with (2) implies that $d_{G-S}(T) = 0$, and T is thus an independent set. Then we have

$$e(S,T) = d_G(T) = r|T| = r|S| = d_G(S),$$

and

$$d_{G-T}(S) = 0,$$

that is, S is an independent set. Hence G is an r-regular bipartite graph. We know from Theorem 2.1 that G is the union of r edge-disjoint 1-factors. Since the union of m edge-disjoint 1-factors is an m-factor, by arguments analogous to the proof of Theorem 1.4 we can see that G is both m-factor-covered and m-factor-deleted, contradicting our supposition.

Case 2. $\omega(W) \geq 1$. Note that

$$e_G(S,T) = d_G(T) - d_{G-S}(T) = r|T| - d_{G-S}(T),$$

and

$$\begin{aligned} e_G(S,W) &= e_G(S \cup T, W) - e_G(T,W) \\ &\geq (r-1)\omega(W) - e_G(T,W), \end{aligned}$$

since G is $(r-1)$-edge-connected. Then we have

$$\begin{aligned} d_G(S) = r|S| &\geq e_G(S,T) + e_G(S,W) \\ &\geq (r-1)\omega(W) - e_G(T,W) + r|T| - d_{G-S}(T). \end{aligned}$$

Since

$$e_G(T,W) = d_{G-S}(T) - 2e_G(T,T),$$

we have

$$\begin{aligned} r|S| &\geq (r-1)\omega(W) + 2e_G(T,T) - 2d_{G-S}(T) + r|T| \\ &= (r-3)\omega(W) + 2e_G(T,T) + 2[\omega(W) - d_{G-S}(T) + m|T|] \\ &\quad +(r-2m)|T| \\ &\geq (r-3)\omega(W) + 2e_G(T,T) + 2m|S| + (r-2m)|T| \text{ (by (1))}. \end{aligned}$$

Since $r \geq 4$ by our assumption, $e_G(T,T) \geq 0$ and $\omega(W) \geq 1$, the above inequalities imply that

$$(r-2m)|S| \geq (r-3) + (r-2m)|T|,$$

which implies that $2m < r$ and

$$|S| > |T|. \qquad (3)$$

We now consider two subcases as follows.

Subcase 2.1. m is even. For every odd component C of W we have

$$m|V(C)| + e_G(V(C), T) \equiv 1 \pmod 2.$$

Since m is even, $e_G(V(C), T)$ must be odd. Hence

$$e_G(V(C), T) \geq 1,$$

so that

$$d_{G-S}(T) \geq q_G(S, T; m).$$

Combining the above inequality with (1), we have that

$$m|T| \geq m|S|,$$

contradicting (3).

Subcase 2.2. m is odd. For every odd component C of W, we have

$$m|V(C)| + e_G(V(C), T) \equiv 1 \pmod 2,$$

which is equivalent to

$$|V(C)| + e_G(V(C), T) \equiv 1 \pmod 2.$$

Thus a component C of $G - (S \cup T)$ is an odd component with respect to some m-factor in G if and only if C is an odd component with respect to some 1-factor in G. Then we have

$$q_G(S, T; m) = q_G(S, T; 1).$$

By Bäbler's theorem, G has a 1-factor. By Theorem 3.1, we have that

$$q_G(S, T; 1) + |T| - d_{G-S}(T) \leq |S|,$$

that is,

$$q_G(S, T; m) + |T| - d_{G-S}(T) \leq |S|. \tag{4}$$

Subtracting (4) from (1) we have that

$$(m-1)|T| \geq (m-1)|S|.$$

If $m = 1$, we know from Theorem 1.2 that G is 1-factor-covered and 1-factor-deleted, contradicting our supposition. Then we have $m > 1$. Thus we have

$$|T| \geq |S|,$$

contradicting (3). This completes the proof of Theorem 1.5. □

References

[1] F. Bäbler, Über die Zerlegung regulärer Streckenkomplexe ungerader Ordnung, *Comment. Math. Helverici* **10** (1938) 275-287.

[2] M. Kano, Sufficient conditions for a graph to have factors, *Discrete Math.* **80** (1990) 159-165.

[3] P. Katerinis, Regular factors in vertex-delete subgraphs of regular graphs, *Discrete Math.* **131** (1992) 357-361.

[4] C.H.C. Little, A theorem on connected graphs in which every edge belongs to a 1-factor, *J. Austra. Math. Soc.* **18** (1974) 450-452.

[5] G. Liu, On (g, f)-covered graphs, *Acta Math. Sci.* **8** (1988) 181-184.

[6] G. Liu, On (g, f)-uniform graphs, *Adv. Math.* (China) **29** (2000) 285-287.

[7] L. Lovász, Subgraphs with prescribed valencies, *J. Combin. Theory* **8** (1970) 391-416.

[8] L. Lovász and M.D. Plummer, Matching Theory, Ann. Discrete Math. 29, North-Holland, Amsterdam, 1986.

[9] J. Petersen, Die Theorie der regularen Graph, *Acta Math.* **15** (1891) 193-220.

[10] J. Plesnik, Connectivity of regular graphs and the existence of 1-factors, *Mathematicky Časopis* **22** (1972) 310-318.

[11] W. T. Tutte, The factors of graphs, *Canad. J. Math.* **4** (1952), 314-328.

[12] W. T. Tutte, A short proof of the factor theorem for finite graphs, *Canad. J. Math.* **6** (1954) 347-352.

[13] W. T. Tutte, Graph factors, *Combinatorica* **1** (1981) 79-97.