On the nonexistence of perfect codes in $J(2w + p^2, w)$

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Abstract

We consider the nonexistence of e-perfect codes in the Johnson scheme $J(n, w)$. It is proved that for each $J(2w + 3p, w)$ for p prime and $p \neq 2, 5$, $J(2w + 5p, w)$ for p prme and $p \neq 3$ and $J(2w + p^2, w)$ for p prime, it does not contain non trivial e-perfect codes.

1 Introduction

Let N be a finite set with n elements. Let $\binom{N}{w} := \{u \subset N \mid |u| = w\}$ be the vertex set. We define the Johnson distance $\rho(u, v) = w - |u \cap v|$. For two subsets $u, v \in \binom{N}{w}$, we define relations $R_i = \{(u, v) \mid \rho(u, v) = i\}$. Then $\mathfrak{X} = (\binom{N}{w}, \{R_i\}_{0 \le i \le w})$ forms an association scheme. $\mathfrak{X} = J(n, w)$ is well known as the Johnson association scheme or Johnson scheme[3].
$C(\subset \binom{N}{w})$ is called e-perfect code in $J(n, w)$ if for $v, u (v \neq u) \in C$,

$$\binom{N}{w} = \sum_{c \in C} S_e(c);$$
$$S_e(u) \cap S_e(v) = \emptyset,$$

where $S_e(c) := \{v \in \binom{N}{w} \mid \rho(v, c) \le e\}$.
There exist some trivial perfect codes in $J(n, w)$. Let us give the following examples: (i) $C = \binom{N}{w}$ is a 0-perfect code in $J(n, w)$. (ii) $C = \{c\}$, any $c \in \binom{N}{w}$ is a w-perfect code in $J(n, w)$. (iii) $C = \{c, N \backslash c\}$, any $c \in \binom{N}{w}$ is a $\frac{w-1}{2}$-perfect code in $J(2w, w)$, where w is odd.
Delsarte conjectured the nonexistence of e-perfect codes in $J(n, w)$ (except for some

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trivialities)[3]. The known results are the following: E.Bannai[2] showed the nonexistence for $J(2w+1, w)$, for $e \geq 2$. P.Hammond[6] showed the nonexistence for $J(2w+1, w), J(2w+2, w)$. C.Roos[7] found the bound for n, w, e. T.Etzion[4, 5] found the relationship between e-perfect code in $J(n, w)$ and Steiner system design and showed the nonexistence for $J(2w+p, w)$, $J(2w+2p, w)$ $p \neq 3$, $J(2w+3p, w)$ $p \neq 2, 3, 5$, p prime, $J(2w \pm r, w)$ for $1 \leq r \leq 14, r \neq 6, 9, 12$. Up to now no non trivial e-perfect code in $J(n, w)$ has been found. In this paper we show the nonexistence of e-perfect codes in $J(2w+3p, w)$ $p \neq 2, 5$, $J(2w+5p, w)$ $p \neq 3$ and $J(2w+p^2, w)$.

2 Perfect codes and Steiner systems

C.Roos [7] showed the following important result. This result is the existence bound on e-perfect codes in $J(n, w)$. For each given w, e, if we let n be larger than the bound, an e-perfect code in $J(n, w)$ does not exist. T.Etzion [5] improved this bound.

Proposition 2.1 ([7, 4]). *If an e-perfect code in $J(n, w)$ exists then $n \leq \frac{(w-1)(2e+1)}{e}$.*

Johnson scheme is related with design theory[3], and so does the existence of e-perfect codes in $J(n, w)$. Specially, in T.Etzion[4], he found the relationship between the existence of a Steiner system and the existence of e-perfect codes. The following theorem is well known as the existence condition of a Steiner system design, for example we can find it in [1].

Proposition 2.2. *If $S(l, m, n)$ exists, so does $S(l-u, m-u, n-u)$ for each $0 \leq u \leq l-1$. This result gives yet more necessary conditions on l, m, n since $\frac{\binom{n-u}{l-u}}{\binom{m-u}{l-u}}$ for each $0 \leq u \leq l-1$ must be an integer.*

We imply the following result from $J(n, w) \simeq J(n, n-w)$

Lemma 2.3. *The complement of an e-perfect code in $J(n, w)$ is an e-perfect code in $J(n, n-w)$.*

Finally, from the following results, we can show effectively the non existence of non trivial e-perfect codes in $J(n, w)$.

Proposition 2.4 ([4]). *If an e-perfect code in $J(n, w)$ exists then $S(e+1, 2e+1, w)$ and $S(e+1, 2e+1, n-w)$ exist.*

Proposition 2.5 ([4]). *There are no e-perfect codes in $J(2w+e+1, w)$.*

Proposition 2.6 ([5]). *If an e-perfect code in $J(n, w)$ exists and $n < \frac{(w-1)(2e+1)}{e}$ then $S(2, e+2, n-w+2)$ exists.*

Proposition 2.7 ([5]). *If an e-perfect code in $J(n, w)$ exists and $w \leq n-w$ then $S(2, e+2, w+2)$ exists.*

3 Main Results

Proposition 3.1 **([4, 5]).** *If an e-perfect code in $J(n, w)$ exists, then $w \equiv e \pmod{e+1}$ and $n - w \equiv e \pmod{e+1}$*

Proof. We immediately infer this result from Proposition 2.4 and Proposition 2.2. □

The next Lemma is a subtle extension of [4].

Lemma 3.2. *There are no nontrivial e-perfect codes in $J(2w + 3p, w)$, p prime and $p \neq 2, 5$.*

Proof. Let C be an e-perfect code in $J(2w + 3p, w)$, p be a prime and $p \neq 2, 5$ then, by Proposition 3.1, $e + 1$ is either $1, 3p, 3$ or p.

(i). If $e + 1 = 1$ then C is a trivial perfect code.

(ii). If $e + 1 = 3p$ then the existence of C contradicts Proposition 2.5.

(iii). If $e + 1 = 3$, by Proposition 2.1, $n \leq \frac{5}{2}(w - 1)$. First, let $n = \frac{5}{2}(w - 1)$. We have w=$6p + 5$. There must be $S(3, 5, 6p + 5)$ by Proposition 2.4 and there must be $S(2, 4, 6p + 7)$ by Proposition 2.7. By Proposition 2.2, $\binom{6p+7}{2}/\binom{4}{2}$ and $\binom{6p+4}{2}/\binom{4}{2}$ must be integers. So

$$\frac{\binom{6p+7}{2}}{\binom{4}{2}} = \frac{\binom{6p+4}{2}}{\binom{4}{2}} + \frac{6p+5}{2},$$

which means that $\frac{6p+5}{2}$ must be an integer. It is not an integer. Hence, a contradiction. Next, let $n < \frac{5}{2}(w - 1)$. By Proposition 2.4, $S(3, 5, w)$ and $S(3, 5, n - w)$ exist. $S(2, 4, w + 2)$ exists by Proposition 2.7. $S(2, 4, n - w + 2)$ exists by Proposition 2.6. By Proposition 2.2, we get $n - w$ $(= w + 3p), w \equiv 2, 26, 50 \pmod{60}$. But there does not exist prime that can satisfy these conditions.

(iv). If $e + 1 = p$ by Proposition 2.1, $n \leq \frac{(w-1)(2p-1)}{p-1}$. First, let $n = \frac{(w-1)(2p-1)}{p-1}$. We have $w = 3p^2 - p - 1$. By Proposition 2.4, $S(p, 2p - 1, w)$ exists, so do $S(2, p + 1, w - p + 2)$. There exists $S(2, p + 1, w + 2)$ by Proposition 2.7. By Proposition 2.2, $\binom{w-p+2}{2}/\binom{p+1}{2}$ and $\binom{w+2}{2}/\binom{p+1}{2}$ must be integers. We get

$$\frac{\binom{3p^2-p+1}{2}}{\binom{p+1}{2}} = \frac{\binom{3p^2-2p+1}{2}}{\binom{p+1}{2}} + 6p - 9 + \frac{10}{p+1}.$$

So $p + 1$ must divide 10, there are no primes which satisfy the above conditions. Hence, a contradiction.
Next, let $n < \frac{(w-1)(2p-1)}{p-1}$. There must be $S(p, 2p-1, w)$ and $S(p, 2p-1, n-w)$ by Proposition 2.4, so do $S(2, p+1, w-(p-2))$ and $S(2, p+1, n-w-(p-2))$.

$S(2,p+1,w+2)$ exists by Proposition 2.7. By Proposition 2.2 for these designs,

$$(w-p+2)(w-p+1)/(p+1)p, \tag{1}$$
$$(w+2p+2)(w+2p+1)/(p+1)p, \tag{2}$$
$$(w+2)(w+1)/(p+1)p, \tag{3}$$
$$(w+1)/p \tag{4}$$

must be integers. By (4), let w be $ap-1$ where a is an integer. Since (2)−(3)= $(4p(a+1)+2)/(p+1)$ and (3)−(1)= $(2ap+1-p)/(p+1)$ are integers, $(4p(a+1)+2)/(p+1) - 2(2ap+1-p)/(p+1) = 6p/(p+1)$ must be an integer. The prime numbers which satisy these conditions are 2 and 5. Hence, a contradiction.

□

Lemma 3.3. *There are no nontrivial e-perfect codes in $J(2w+5p,w)$, p prime and $p \neq 3$.*

Proof. Let C be an e-perfect code in $J(2w+5p,w)$, p be a prime and $p \neq 3$, then by Proposition 3.1, $e+1$ is either $1, 5p, 5$ or p.

(i). If $e+1=1$ then C is a trivial perfect code.

(ii). If $e+1=5p$ then the existence of C contradicts Proposition 2.5.

(iii). If $e+1=5$, by Proposition 2.1, $n \leq \frac{9}{4}(w-1)$. First, let $n=\frac{9}{4}(w-1)$, we have $w=20p+9$. By Proposition 2.4, $S(5,9,20p+9)$ and $S(5,9,25p+9)$ exist, so do $S(2,6,20p+6)$ and $S(2,6,25p+6)$. By Proposition 2.2, $\binom{20p+6}{2}/\binom{6}{2}$ and $\binom{25p+6}{2}/\binom{6}{2}$ must be integers. We get the following equation from these integers:

$$\frac{\binom{25p+6}{2}}{\binom{6}{2}} = \frac{\binom{20p+6}{2}}{\binom{6}{2}} + \frac{p(45p+11)}{6}.$$

We know that $\frac{p(45p+11)}{6}$ must be an integer. The prime numbers which satisy this condition are 2 and 3. If $p=2$ then $\frac{p(45p+11)}{6} = \frac{101}{3}$ is not an integer. Next, let $n < \frac{9}{4}(w-1)$. By Proposition 2.4, $S(5,9,w)$ and $S(5,9,n-w)$ exist. By Proposition 2.6, $S(2,6,n-w+2)$ exists. By Proposition 2.7, $S(2,6,w+2)$ exists. By Proposition 2.2 for $S(5,9,w)$ and $S(2,6,w+2)$, we obtain $w \equiv 4, 79, 199, 94, 184, 109 \pmod{210}$. In a similar way, we get $n-w \equiv 4, 79, 199, 94, 184, 109 \pmod{210}$ from $S(5,9,n-w)$ and $S(2,6,n-w+2)$. But there are no primes which satisfy these conditions.

(iv). If $e+1=p$ by Proposition 2.1, $n \leq \frac{(w-1)(2p-1)}{p-1}$. First, let $n=\frac{(w-1)(2p-1)}{p-1}$, we get $w=5p^2-3p-1$. By Proposition 2.4, $S(p,2p-1,w)$ exists, so do $S(2,p+1,w-p+2)$. By Proposition 2.7, $S(2,p+1,w+2)$ exist. By Proposition 2.2, $\binom{w-p+2}{2}/\binom{p+1}{2}$, $\binom{w+2}{2}/\binom{p+1}{2}$ must be integers. Since

$$\frac{\binom{w-p+2}{2}}{\binom{p+1}{2}} = \frac{\binom{w+2}{2}}{\binom{p+1}{2}} + 50p - 65 + \frac{70}{p+1},$$

$p+1$ must divide 70. Such prime is only 13. We have $w = 805$ for $p = 13$ and know that $\binom{w+2}{2}/\binom{p+1}{2}$ is not in integers. Therefore, it is a contradiction. Next, let $n < \frac{(w-1)(2p-1)}{p-1}$. By Proposition 2.4, $S(p, 2p-1, w)$ and $S(p, 2p-1, n-w)$ exist, so do $S(2, p+1, w-(p-2))$ and $S(2, p+1, n-w-(p-2))$. By Proposition 2.6, $S(2, p+1, n-w+2)$ exist. By Proposition 2.7, $S(2, p+1, w+2)$ exists. By Proposition 2.2 for these designs,

$$(w-p+2)(w-p+1)/(p+1)p, \tag{5}$$
$$(w+4p+2)(w+4p+1)/(p+1)p, \tag{6}$$
$$(w+5p+2)(w+5p+1)/(p+1)p, \tag{7}$$
$$(w+2)(w+1)/(p+1)p, \tag{8}$$
$$(w+1)/p \tag{9}$$

must be integers. By (9), let w be $ap-1$ where a is an integer. Since (5)−(8)= $-(2ap+1-p)/(p+1)$ and (7)−(8)= $5(2ap+1+5p)/(p+1)$ are integers, $-5(2ap+1-p)/(p+1)+5(2ap+1+5p)/(p+1) = 30p/(p+1)$ must be an integer. The prime numbers which satisy these conditions are $2, 5$ and 29. Since (6)−(8)+4((5)−(8)) $= 20p/(p+1)$ must be an integer, the above three prime numbers are not suitable. Hence, a contradiction.

□

Theorem 3.4. *There are no nontrivial e-perfect codes in $J(2w+p^2, w)$, p prime.*

Proof. Let C be an e-perfect code in $J(2w+p^2, w)$, p be a prime, then by Proposition 3.1, $e+1$ is either $1, p^2$ or p.

(i). If $e+1 = 1$ then C is a trivial perfect code.

(ii). If $e+1 = p^2$ then the existence of C contradicts Proposition 2.5.

(iii). If $e+1 = p$, by Proposition 2.1, $n \le \frac{(w-1)(2p-1)}{p-1}$. First, let $n = \frac{(w-1)(2p-1)}{p-1}$. We have $w = p^3-p^2+2p-1$. By Proposition 2.4, $S(p, 2p-1, p^3-p^2+2p-1)$ and $S(p, 2p-1, p^3+2p-1)$ exist, So do $S(2, p+1, p^3-p^2+p+1)$ and $S(2, p+1, p^3+p+1)$.

$$\frac{\binom{p^3+p+1}{2}}{\binom{p+1}{2}} = \frac{\binom{p^3-p^2+p+1}{2}}{\binom{p+1}{2}} + 2p^3 - 3p^2 + 5p - 4 + \frac{4}{p+1}$$

$p+1$ must divide 4, there is only $p = 3$. By Lemma 3.2, such a C does not exist. Next, let $n < \frac{(w-1)(2p-1)}{p-1}$. By Proposition 2.4, $S(p, 2p-1, w)$ and $S(p, 2p-1, n-w)$ exist, so do $S(2, p+1, w-(p-2))$ and $S(2, p+1, n-w-(p-2))$. By Proposition 2.7, $S(2, p+1, w+2)$ exists. By Proposition 2.2 for these designs,

$$(w-p+2)(w-p+1)/(p+1)p, \tag{10}$$
$$(w+p^2-p+2)(w+p^2-p+1)/(p+1)p, \tag{11}$$
$$(w+2)(w+1)/(p+1)p, \tag{12}$$
$$(w+1)/p \tag{13}$$

must be integers. By (13), let w be $ap-1$ where a is an integer. Since (10)−(12)= $-(2ap+1-p)/(p+1)$ and (11)−(12)= $(p-1)(1-p+p^2+2ap)/(p+1)$ are integers, $(p-1)(1-p+p^2+2ap)/(p+1)-(p-1)(2ap+1-p)/(p+1)=(p-1)p^2/(p+1)$ must be an integer. There are no primes which satisfy this condition. Hence, a contradiction.

□

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