On 3-coloring of plane triangulations

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Abstract

For a 3-vertex coloring, a face of a triangulation whose vertices receive all three colors is called a *vivid* face with respect to it. In this paper, we show that for any triangulation G with n faces, there exists a coloring of G with at least $\frac{1}{2}n$ faces and construct an infinite series of plane triangulations such that any 3-coloring admits at most $\frac{1}{5}(3n-2)$ vivid faces.

Keywords: triangulation, plane triangulation, 3-coloring

1 Introduction

A *plane triangulation* G is a simple graph embedded in the plane such that each face is bounded by three edges and that any two faces share at most one edge. (Hence we don't regard K_3 as a plane triangulation.) The unbounded face of G is said to be *outer* and other finite faces are *inner*. Let $\lambda : V(G) \to \{1,2,3\}$ be a color-assignment, which we refer to as a *3-coloring* of G here. (Note that in our 3-colorings, two vertices with the same color might be adjacent.) A face f of G is said to be *vivid* with respect to λ if the three vertices of f receive three distinct colors.

The following is the well-known result for 3-colorings of plane triangulations:

Theorem 1 (Sperner's Lemma [5]) *Let G be any plane triangulation with outer face xyz. For any 3-coloring $\lambda : V(G) \to \{1,2,3\}$ such that $\lambda(x) = 1$, $\lambda(y) = 2$ and $\lambda(z) = 3$, there exists a vivid inner face of G with respect to λ.*

Theorem 1 can combinatorially be proved by using the Odd Point Theorem, but it surprisingly has an application to prove the planar version of Brouwer's fixed point theorem (see [4]). By Sperner's lemma, for a plane triangulation G with outer face xyz, every surjective 3-coloring $\lambda : V(G) \to \{1,2,3\}$ with $\lambda(x) = 1$, $\lambda(y) = 2$ and $\lambda(z) = 3$ yields a vivid inner face. However, for any plane triangulation G with at least 5 vertices, there exists a surjective 3-coloring λ of G with no vivid face (Since G is not complete, G has two non-adjacent vertices p and q. Thus, we have a 3-coloring λ such that $\lambda(p) = 1$, $\lambda(q) = 2$ and $\lambda(v) = 3$ for any vertex $v \in V(G) - \{p, q\}$.) Thus, we can define the minimum number k, called the *looseness* $\xi(G)$ of G, such that for any surjection $c : V(G) \to \{1, 2, \ldots, 3 + k\}$, there exists a vivid face [3, 6]. (This notion is first established to distinguish two embeddings of large complete graphs into surfaces with high genera [1, 2].)

In this paper, we consider the following problem: Conversely to the direction of Sperner's lemma, for a given plane triangulation G, how many faces of G can we make vivid by specifying some specific 3-coloring to G? We formalize our problem, as follows:

For a graph G, let $\mathcal{C}(G)$ denote the set of 3-colorings of G. Let $\mathcal{G}(n)$ denote the set of plane triangulations with n faces. For $G \in \mathcal{G}(n)$ and $\lambda \in \mathcal{C}(G)$, let $h_\lambda(G)$ denote the number of vivid faces of G with respect to λ. Define

$$h(G) := \max\{h_\lambda(G) \mid \lambda \in \mathcal{C}(G)\}, \quad \text{and}$$

$$h(n) := \min\{h(G) \mid G \in \mathcal{G}(n)\}$$

Then the following is our result:

Theorem 2 *For any even n, $h(n) \geq \frac{1}{2}n$. And for infintely many n, $h(n) \leq \frac{1}{5}(3n - 2)$.*

2 Lemmas

First we prepare some notations to prove Theorem 2. Let H be the octahedron shown in Figure 1, where H consists of three hilighted faces ACD, BCE, EDF (called *blue* faces), four faces ABC, ADF, BFE, CED (called *white* faces) and the infinite face.

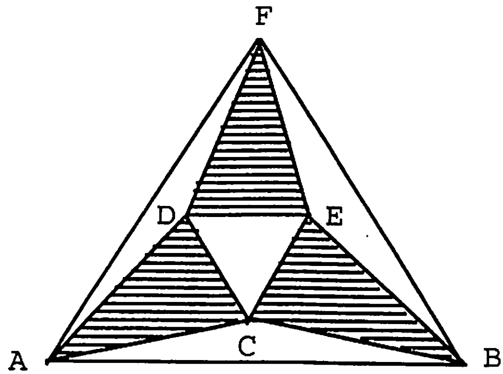

Figure 1: Three blue faces and four white faces.

For a 3-coloring λ of H, a blue face R is said to be *good* if $|\{\lambda(v) \mid v \in R\}| = 2$. We define $t(\lambda)$ to be the sum of the number of good blue faces and the number of vivid white faces in H.

We prepare two lemmas to prove the main theorem.

Lemma 3 *Let H be the octahedron as shown in Figure 1. Then $t(\lambda) \leq 5$ for any 3-coloring $\lambda \in \mathcal{C}(H)$. Moreover, the equality $t(\lambda) = 5$ holds only if $|\lambda(\{A, B, F\})| = 2$.*

Proof. Let λ be a 3-coloring of H with $t(\lambda) \geq 5$. Let $\mathcal{B}$ be the set of good blue faces, and let $\mathcal{W}$ be the set of vivid white faces in H. Since $t(\lambda) = |\mathcal{B}| + |\mathcal{W}| \geq 5$, we have $|\mathcal{B}| \geq 1$ and $|\mathcal{W}| \geq 2$. Note that for each face in $\mathcal{B}$, one of the faces incident with it cannot belong to $\mathcal{W}$. Since $\mathcal{B} \neq \emptyset$, we have $|\mathcal{W}| \leq 3$, and hence $|\mathcal{B}| \geq 2$.

Suppose, in particular, that $|\mathcal{B}| + |\mathcal{W}| \geq 6$. Then we have $|\mathcal{B}| = |\mathcal{W}| = 3$. In this case, the unique white face not in $\mathcal{W}$ must be the central face CDE, because each blue face must be incident with it. We may assume that $\lambda(C) = \lambda(D) = \lambda(E) = 1$, and $\lambda(A), \lambda(B), \lambda(F) \in \{2, 3\}$. But then, one of the white faces ABC, ADF and BEF cannot be vivid. This is a contradiction.

Thus we have $|\mathcal{B}| + |\mathcal{W}| = 5$. (Here, we have proved that $t(\lambda) \leq 5$.) Since $|\mathcal{B}| \geq 2$ and $|\mathcal{W}| \geq 2$, we may assume that $ABC \in \mathcal{W}$ and $ACD \in \mathcal{B}$. We may also assume that $\lambda(A) = 1$, $\lambda(B) = 2$ and $\lambda(C) = 3$. By way of contradiction, we assume that $\lambda(F) = 3$.

Since $ACD \in \mathcal{B}$, we have $\lambda(D) \in \{1, 3\}$. If $\lambda(D) = 3$, then the white faces CDE and ADF are not in $\mathcal{W}$. Since $|\mathcal{W}| \geq 2$, the white face BEF must be vivid. Thus we have $\lambda(E) = 1$. Then, $|\mathcal{B}| = |\mathcal{W}| = 2$, which contradicts our assumption $|\mathcal{B}| + |\mathcal{W}| = 5$. If $\lambda(D) = 1$, then ADF is not vivid, and one of CDE and BEF cannot be vivid. Thus $|\mathcal{W}| = 2$, and hence $|\mathcal{B}| = 3$. The unique way to make both blue faces BCE and EDF good is to set $\lambda(E) = 3$. but then we have $|\mathcal{W}| = 1$, a contradiction. □

Now we shall define a sequence of plane triangulations G_k as follows. Let G_0 be the tetrahedron. For $k \geq 1$, G_k is defined to be the plane triangulation obtained from the octahedron H in Figure 1 by replacing each blue face with G_{k-1} so that the boundary of the outer face of G_{k-1} coincides with the boundary of the blue face (see Figure 2).

For a plane triangulation G and $\lambda \in \mathcal{C}(G)$, let $h'_\lambda(G)$ denote the number of finite vivid faces in G with respect to λ, and define $h'(G) = \max\{h'_\lambda(G) \mid \lambda \in \mathcal{C}(G)\}$.

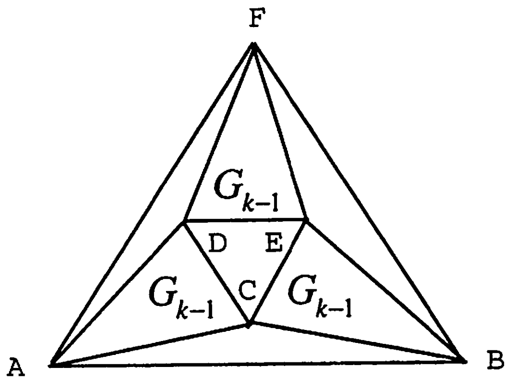

Figure 2: G_k.

Lemma 4 *$h'(G_k) = 3h'(G_{k-1}) + 2$ for any $k \geq 1$. Moreover, if $h'_\lambda(G_k) = h'(G_k)$ with $k \geq 0$, then λ assigns exactly two colors to the vertices on the outer boundary of G_k. As a consequence, it holds $h(G_k) = h'(G_k)$ for any $k \geq 0$.*

Proof. For the case $k = 0$, we can easily check the second assertion. (Note that $h'(G_0) = 2$.)

Let $k \geq 1$, and let $\lambda \in \mathcal{C}(G_k)$ be a 3-coloring with $h'_\lambda(G_k) = h'(G_k)$. Let H be the octahedron in G_k from which we obtain G_k by replacing each blue face with G_{k-1}. Consider the 3-coloring $\lambda|_H \in \mathcal{C}(H)$, and define $\mathcal{B}$ to be the set of good blue faces of H and $\mathcal{W}$ to be the set of vivid white faces of H.

Each finite vivid face of G_k with respect to λ is either a finite vivid face of one of G_{k-1}'s or a white vivid face of H. By the induction hypothesis, for each blue face not in $\mathcal{B}$, the G_{k-1} replaced with it contains at most $h'(G_{k-1})-1$ finite vivid faces. Thus, by using Lemma 3, we have

$$\begin{aligned} h'(G_k)=h'_\lambda(G_k) &\le 3h'(G_{k-1})-(3-|\mathcal{B}|)+|\mathcal{W}| \\ &= 3h'(G_{k-1})-3+t(\lambda|_H) \\ &\le 3h'(G_{k-1})+2. \end{aligned}$$

On the other hand, by setting $\mu(A)=\mu(B)=1$, $\mu(F)=2$, and $\mu(C)=\mu(D)=\mu(E)=3$, we can find a 3-coloring μ of G_k with $h'_\mu(G_k)=3h'(G_{k-1})+2$. Thus the equality must hold in the above inequalities. In particular, we have $t(\lambda|_H)=5$. Thus by Lemma 3, we have $|\lambda(\{A,B,F\})|=2$. This completes the proof of Lemma 4. □

Now we shall prove 2.

Proof of Theorem 2. First we prove that $h(n)\ge\frac{1}{2}n$. For any $G\in\mathcal{G}(n)$, if we put $p:=|V(G)|$, then by Euler's formula, we have $|E(G)|=3p-6$ and $n=2p-4$.

By the four color theorem, there exists a proper coloring $c:V(G)\to\{1,2,3,4\}$. For $i,j\in\{1,2,3,4\}$ with $i\ne j$, let q_{ij} denote the number of edges $xy\in E(G)$ such that $\{c(x),c(y)\}=\{i,j\}$. We may assume that q_{34} is smallest among all q_{ij}. Since $\sum_{i,j}q_{ij}=|E(G)|$, we have $q_{34}\le\frac{1}{6}|E(G)|=\frac{p}{2}-1$.

Now, we define a 3-coloring λ of G by $\lambda(x):=\min\{c(x),3\}$ for each $x\in V(G)$. Then, the number of non-vivid faces with respect to λ is at most $2q_{34}\le p-2$. Hence, we have $h_\lambda(G)\ge 2p-4-(p-2)\ge p-2=\frac{1}{2}n$.

In order to obtain an upperbound for $h(n)$, we consider the plane triangulation G_k defined in the previous section. It is not difficult to deduce that the number of faces in G_k is $5\cdot 3^k-1=:n$. By Lemma 4 with the fact that $h(G_0)=h'(G_0)=2$, we obtain $h(G_k)=h'(G_k)=3^{k+1}-1=3\cdot\frac{n+1}{5}-1=\frac{1}{5}(3n-2)$. □

3 Conclusions

In this paper we proved that $\frac{1}{2}n\le h(n)\le\frac{1}{5}(3n-2)$. We would like to pose a conjecture about this subject.

Conjecture *There exists a constant c such that for any even n*

$$h(n)\ge\frac{3}{5}n-c.$$

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