The total number of maximal k-independent sets in the generalized lexicographical product of graphs

by
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ABSTRACT: In this paper we determine the number of all maximal k-independent sets in the generalized lexicographical product of graphs. We construct a polynomial which calculate this number using the concept of Fibonacci polynomial and generalized Fibonacci polynomial. Also for special graphs we give the recurrence formula.

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1. Introduction

For general concepts we refer the reader to [1] and [2]. By a graph G we mean a finite, undirected, connected, simple graph. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. By a P_n we mean graph with the vertex set $V(P_n) = \{t_1, ..., t_n\}$ and the edge set $E(P_n) = \{\{t_i, t_{i+1}\}; i = 1, ..., n-1\}$, $n \geqslant 1$. Let K_x denotes complete graph on x vertices, $x \geqslant 1$. Let G be a graph on $V(G) = \{t_1, ..., t_n\}, n \geqslant 2$, and $H_i, i = 1, ..., n$, are graphs on $V(H_i) = V = \{y_1, ..., y_x\}, x \geqslant 1$. By generalized lexicographical product of G and $H_1, ..., H_n$ we mean a graph $G[H_1, ..., H_n] = \{\{(t_i, y_p), (t_j, y_q)\}; (t_i = t_j \text{ and } \{y_p, y_q\} \in E(H_i)) \text{ or } \{t_i, t_j\} \in E(G)\}$. If $H_i = H, i = 1, ..., n$, then $G[H_1, ..., H_n] = G[H]$, where $G[H]$ is a lexicographical product of two graphs. By $d_G(x, y)$ we denote the length of the shortest path joining vertices x and y in G.

In [7] it has been proved:

Theorem 1.1 [7]. *Let* $(t_i, y_p), (t_j, y_q) \in V(G[H_1, ..., H_n])$. *Then*

$$d_{G[H_1,...,H_n]}((t_i, y_p), (t_j, y_q)) = \begin{cases} d_G(t_i, t_j) & \text{for} \quad i \neq j, \\ 1 & \text{for} \quad i = j \quad \text{and} \quad d_{H_i}(y_p, y_q) = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Let k be a fixed integer, $k \geqslant 2$. A subset $S \subseteq V(G)$ is said to be a k-independent set of G (also named as k-stable set of G) if for two distinct vertices $x, y \in S, d_G(x, y) \geqslant k$. In addition the empty set and a subset containing only one vertex also are meant as k-independent sets of G. Note that for $k = 2$ the definition reduces to the definition of an independent set of the graph G. The total number of independent sets of graph G was named in [5] by H.Prodinger and R.F.Tichy as Fibonacci number of a graph

G. They denote it by $F(G)$. Let $|V(G)| = n$. If $f_G(n,p)$ denotes the number of all p-elements independent sets of G, then $F(G) = \sum_{p\geqslant 0} f_G(n,p)$.

In [4] it was defined more general concept, namely generalized Fibonacci number of graph G. It was denoted by $F_k(G)$ and defined as the total number of k-independent sets of a graph G. If $f_G(k,n,p)$ denotes the number of all p-elements k-independent sets of G then $F_k(G) = \sum_{p\geqslant 0} f_G(k,n,p)$.

It is interesting to know that $F(P_n) = \sum_{p\geqslant 0} \binom{n-p+1}{p}$ and $F(C_n) = 1 + \sum_{p\geqslant 1} \binom{n-p-1}{p-1}\frac{n}{p}$, so they are equal to the Fibonacci number and the Lucas number, respectively, see [1],[5]. For $k \geqslant 2$ it was proved in [4] that $F_k(P_n) = \sum_{p\geqslant 0} \binom{n-p-(p-1)(k-2)+1}{p}$ and $F_k(C_n) = 1 + \sum_{p\geqslant 1} \frac{n}{p}\binom{n-p(k-1)-1}{p-1}$. In [3] G.Hopkins and W.Staton defined the Fibonacci polynomial $F_G(x)$ of a graph G by $F_G(x) = F(G[K_x])$, for $x \geqslant 1$. They proved:

Theorem 1.2 [3]. *For an arbitrary graph G on n vertices*

$$F_G(x) = \sum_{p\geqslant 0} f_G(n,p)x^p.$$

In [8] it was determined the generalized Fibonacci number of graph $G[H_1, ..., H_n]$ using the concept of a generalized Fibonacci polynomial. It has been proved:

Theorem 1.3 [8]. *Let $k \geqslant 3$, $n \geqslant 2$, $x \geqslant 1$. Then for a given graph G on n vertices and for an arbitrary seguence of n graphs $H_1, ..., H_n$ on x vertices*

$$F_k(G[H_1, ..., H_n]) = \sum_{p\geqslant 0} f_G(k,n,p)x^p.$$

A k-independent set $S \subseteq V(G)$ is a maximal if for each vertex $t \in (V(G)\backslash S)$ the set $S \cup \{t\}$ is not a k-independent set of G. Evidently every maximal k-independent set of G has at least one vertex. It is known that every maximal k-independent set of undirected graph is a $(k, k-1)$-kernel of G. Let $j_G(k,n,p)$ denotes the number of all maximal p-elements k-independent sets of the graph G. If by $J_k(G)$ we denote the number of all maximal k-independent sets of the graph G, then it is clear that $J_k(G) = \sum_{p\geqslant 1} j_G(k,n,p)$.

For $k = 2$ we put $J_2(G) = J(G)$ and $j_G(2,n,p) = j_G(n,p)$.

The total number of k-independent sets and maximal k-independent sets in different classes of graphs were determined in [3], [4], [5], [6], [8]. In this paper we calculate the total number of maximal k-independent sets in the lexicographical product of graphs using the concept of the Fibonacci polynomial and generalized Fibonacci polynomial.

2. Main results

Theorem 2.1. *Let $k \geqslant 3$, $n \geqslant 2, x \geqslant 1$. Then for a given graph G on n vertices and for an arbitrary sequence of n graphs $H_1, ..., H_n$ on x vertices*

$J_k(G[H_1,...,H_n]) = \sum_{p\geqslant 1} j_G(k,n,p)x^p.$

Proof: Let G be a given graph on n vertices, $n \geqslant 2$. We shall show that if $k \geqslant 3$, then for an arbitrary sequence of graphs $H_1,...,H_n$ the number $J_k(G[H_1,...,H_n])$ is equal to $\sum_{p\geqslant 1} j_G(k,n,p)x^p$. It suffices to calculate the number $J_k(G[H_1,...,H_n])$. From the definition of the graph $G[H_1,...,H_n]$ and by Theorem 1 we deduce that to obtain a p-elements, $p \geqslant 1$, maximal k-independent set of $G[H_1,...,H_n]$ first we have to choose a p-elements maximal k-independent set of the graph G. Evidently we can do it on $j_G(k,n,p)$ ways. Next we have to choose one of the x vertices in each of the p choosen copies of H_i, $i = 1,...,n$. Clearly, by Theorem 1.1 and $k \geqslant 3$, we deduce that for an arbitrary graph H_i, $i = 1,...,n$ only one vertex from its copy can be choosen to a maximal k-independent set. Because every vertex of p-copies can be choosen on x ways, so we have $j_G(k,n,p)x^p$ maximal k-independent sets having exactly p-elements in $G[H_1,...,H_n]$. Hence $J_k(G[H_1,...,H_n]) = \sum_{p\geqslant 1} j_G(k,n,p)x^p$. ∎

In the similar way we can prove:

Theorem 2.2. *Let $n \geqslant 2$, $x \geqslant 1$. Then for a given graph G on n vertices* $J(G[K_x]) = \sum_{p\geqslant 1} j(n,p)x^p$. ∎

Corollary 2.3. *If $x = 1$, then $J_k(G[H_1,...,H_n]) = J_k(G)$ and $J(G[K_x]) = J(G)$.*

Evidently to study numbers $J_k(G[H_1,...,H_n])$ and $J(G[K_x])$ it suffices to study the coefficients $j_G(k,n,p)$. It is clear that the constant coefficient is 0, because the empty set is not a maximal k-independent set of graph. Moreover if $diam(G) \leqslant k-1$, then every vertex is a maximal k-independent set of graph G and there is not exist a maximal k-independent set of G having more than one vertex. Consequently $J_k(G[H_1,...,H_n]) = nx$.

Now we consider the graph P_n, $n \geqslant 2$, instead of G and we present numbers $J_k(P_n[H_1,...,H_n])$ and $J(P_n[K_x])$. Firstly we calculate the number $j_{P_n}(k,n,p)$, for convinience we denote this number by $j(k,n,p)$.

Theorem 2.4. *Let $k \geqslant 2, n \geqslant 1, p \geqslant 2$. If $n < (p-1)k+1$ or $n > p(2k-1)$, then $j(k,n,p) = 0$.*

Proof: Assume that k,n,p are as in the statement of theorem. Then it is clear that to construct a maximal k-independent set S of P_n having p elements we need at least $(p-1)k+1$ vertices, then $S = \{t_1, t_{k+1}, ..., t_{(p-1)k+1}\}$, so if $n < (p-1)k+1$ then $j(k,n,p) = 0$. Moreover let $S = \{t_k, t_{k+2k-1}, t_{k+2(2k-1)}, ..., t_{k+(p-1)(2k-1)}\}$. Evidently S is a maximal p-elements k-independent set of P_n and maximum number of vertices of P_n is $k+(p-1)(2k-1)+k-1 = p(2k-1)$. So if $n > p(2k-1)$ then there is not exist a maximal p-elements k-independent set of P_n, hence $j(k,n,p) = 0$. ∎

Theorem 2.5. *Let $k \geqslant 2, n \geqslant 1, p \geqslant 1$. Then the number $j(k,n,p)$ satisfy*

the following recurrence relations:
$j(k,n,1) = n$ *if* $1 \leqslant n \leqslant k$,
$j(k,n,1) = n - 2r$ *if* $k+1 \leqslant n = k + r \leqslant 2k-1$, $1 \leqslant r \leqslant k-1$,
$j(k,n,1) = 0$ *if* $n \geqslant 2k$,
and for $p \geqslant 2$ *we have*
$j(k,n,p) = 0$ *if* $n \leqslant (p-1)k$
$j(k,n,p) = j(k,n-k,p-1) + j(k,n-1,p)$ *for* $(p-1)k+1 \leqslant n \leqslant 2k$
$j(k,n,p) = j(k,n-k,p-1) + j(k,n-1,p) - j(k,n-2k,p-1)$
for $2k+1 \leqslant n \leqslant p(2k-1)$
$j(k,n,p) = 0$ *if* $n \geqslant p(2k-1)+1$.

Proof: Let k, n, p be as in the statement of theorem. First let $p = 1$. If $1 \leqslant n \leqslant k$, then there exists only the maximal k-independent sets having exactly one vertex. Because every vertex of $V(P_n)$ is a maximal k-independent set of the graph P_n, so $j(k,n,1) = n$. If $k+1 \leqslant n = k+r \leqslant 2k-1$, $1 \leqslant r \leqslant k-1$, then it is clear that every vertex $\{t_j\}, r+1 \leqslant j \leqslant n-r$ is a maximal k-independent set of P_n, so we have exactly $n-2r$ such sets, what gives $j(k,n,1) = n-2r$. If $n \geqslant 2k$, then it is obviously that there is not exist a maximal k-independent sets of P_n having exactly one vertex, so $j(k,n,1) = 0$. Let now $p \geqslant 2$. By Theorem 2.4 we have that $j(k,n,p) = 0$ if $n \leqslant (p-1)k$. Suppose that $(p-1)k+1 \leqslant n \leqslant p(2k-1)$ and let S be an arbitrary maximal k-independent set of P_n. We consider two casses.

Case 1. $t_n \in S$.

In this case, for $i = n-1, n-2, ..., n-(k-1), t_i \notin S$. Furthermore if S^* is an arbitrary $(p-1)$-elements maximal k-independent set of $P_n - \bigcup_{i=0}^{k-1} t_{n-i}$, then $S^* \cup \{t_n\}$ is a k-independent set of P_n. We shall show that $S^* \cup \{t_n\}$ is maximal. By an easy observation it follows that among the vertices of S^* there must be a vertex t_j such that $n-k-(k-1) \leqslant j \leqslant n-k$. Otherwise we could add the vertex t_{n-k} to S^*. Consequently to prove that $S^* \cup \{t_n\}$ is maximal it sufficies to estimate the distance between vertices t_j and t_n in P_n. By an simple calculations we obtain that $d_{P_n}(t_n, t_j) \leqslant n-(n-2k+1) = 2k-1 < 2k$. This means that it is not possible to add to $S^* \cup \{t_n\}$ any vertex of the succesive vertices $t_j, t_{j+1}, ..., t_{n-1}$. This shows that $S^* \cup \{t_n\}$ is maximal and $S = S^* \cup \{t_n\}$. Because we have $j(k,n-k,p-1)$ sets S^*, this implies that the number of p-elements maximal k-independent sets of P_n containing the vertex t_n is equal to $j(k,n-k,p-1)$.

Case 2. $t_n \notin S$.

Then all maximal p-elements k-independent sets of $P_n - t_n$ are p-elements k-independent sets of P_n. Suppose that S^* is a p-elements maximal k-independent set of $P_n - t_n$. Evidently we have $j(k,n-1,p)$ such sets. It should be noted that, if $t_{n-k} \in S^*$, then S^* could not be a maximal k-independent set of P_n, since then $d_{P_n}(t_n, S^*) = k$. Observe, that if $t_{n-k} \notin S$, then there must be a vertex $t_j, n-k < j \leqslant n-1$, which belongs to S^* by the maximality of S^* in the graph $P_n - t_n$. This means that to calculate the total number of maximal p-elements k-independent sets of P_n not containing the vertex t_n it sufficies to substract the number of all subsets S^* which contain the vertex t_{n-k} from the number $j(k, n-1, p)$. Let r denotes the number of all maximal p-elements k-independent sets of $P_n - t_n$ containing the vertex t_{n-k}.

Consider two possibilities:

Subcase 2.1. $(p-1)k+1 \leqslant n \leqslant 2k$.

Since $n-k \leqslant k$, so for any $p \geqslant 2$ there is not exist a p-elements maximal k-independent set containing the vertex t_{n-k}. Consequently $r=0$.

Subcase 2.2. $2k+1 \leqslant n \leqslant p(2k-1)$

Consider the graph $P_n - \bigcup_{i=0}^{k-1} t_{n-i}$ isomorphic to P_{n-k}. Since r denotes the number of all maximal p-elements k-independent sets S^* of the graph $P_n - t_n$ containing the vertex t_{n-k} it follows from case 1 and preceding observations that $r = j(k, n-k-k, p-1) = j(k, n-2k, p-1)$. All this togeather gives the result for $p \geqslant 2$

$j(k,n,p) = j(k, n-k, p-1) + j(k, n-1, p)$ if $(p-1)k+1 \leqslant n \leqslant 2k$,

$j(k,n,p) = j(k, n-k, p-1) + j(k, n-1, p) - j(k, n-2k, p-1)$ if $2k+1 \leqslant n \leqslant p(2k-1)$.

If $n \geqslant p(2k-1)+1$, then by Theorem 2.4 we have $j(k,n,p) = 0$. ∎

Corrolary 2.6. *Let $k \geqslant 2$, $n \geqslant 2$. Then* $J_k(P_n) = \sum_{p \geqslant 1} j(k,n,p)$.

Corrolary 2.7. *Let $k \geqslant 3$, $n \geqslant 2$, $x \geqslant 1$. Then for an arbitrary seqence of n graphs $H_1, ..., H_n$ on x vertices* $J_k(P_n[H_1, ..., H_n]) = \sum_{p \geqslant 1} j(k,n,p)x^p$.

Corrolary 2.8. *Let $n \geqslant 2$, $x \geqslant 1$. Then* $J(P_n[K_x]) = \sum_{p \geqslant 1} j(2,n,p)x^p$.

At the end we present numbers $J_k(P_n[H_1, ..., H_n])$ and $J(P_n[K_x])$ by the linear recurrence relations.

Theorem 2.9. *Let $k \geqslant 3$, $n \geqslant 2, x \geqslant 1$. Then for an arbitrary sequence of n graphs $H_1, ..., H_n$ on x vertices, the number $J_k(P_n[H_1, ..., H_n])$ satisfy the following reccurence relations:*

$J_k(P_n[H_1, ..., H_n]) = nx$, $n = 2, ..., k$,

$J_k(P_{k+1}[H_1, ..., H_{k+1}]) = x^2 + (k-1)x$,

$J_k(P_n[H_1, ..., H_n]) = J_k(P_{n-1}[H_1, ..., H_{n-1}]) + x(J_k(P_{n-k}[H_1, ..., H_{n-k}]) - 1)$, *for* $k+2 \leqslant n \leqslant 2k$,

$J(P_{2k+1}[H_1, ..., H_{2k+1}]) = J_k(P_{2k}[H_1, ..., H_{2k}]) + x(J_k(P_{k+1}[H_1, ..., H_{k+1}]) - x)$,

$J_k(P_n[H_1, ..., H_n]) = J_k(P_{n-1}[H_1, ..., H_{n-1}]) + x(J_k(P_{n-k}[H_1, ..., H_{n-k}]) - J_k(P_{n-2k}[H_1, ..., H_{n-2k}]))$, *for* $n \geqslant 2k+2$

Proof: Let k, n, x be as it was mentioned in the statement of the theorem.If $n = 2, ..., k$, then every vertex of $V(P_n[H_1, ..., H_n])$ is a maximal k-independent set of the graph $P_n[H_1, ..., H_n]$. Moreover there is not exist a maximal

k-independent set of $P_n[H_1, ..., H_n]$ having at least two elements. This implies that $J_k(P_n[H_1, ..., H_n]) = nx$.
If $n = k+1$, then every set on the form $\{(x_i, y_j)\}$, $2 \leqslant i \leqslant k$, $1 \leqslant j \leqslant x$ is a maximal k-independent set of $P_{k+1}[H_1, ..., H_{k+1}]$ so we have $(k-1)x$ such sets. Moreover in this case we have also maximal k-independent sets having exactly two elements. Every two elements maximal k-independent set has the form $\{(t_1, y_j), (t_{k+1}, y_q)\}$, where $1 \leqslant j \leqslant x$ and $1 \leqslant q \leqslant x$. Then we have x^2 such subsets and consequently $J_k(P_{k+1}[H_1, ..., H_{k+1}]) = x^2 + (k-1)x$. Now suppose that $n \geqslant k+2$ and let S be an arbitrary maximal k-independent set of $P_n[H_1, ..., H_n]$. Because at most one vertex from each copy of H_i, $i = 1, ..., n$, can belong to the maximal k-independent set of $P_n[H_1, ..., H_n]$, by Theorem 1.1 and $k \geqslant 3$, so two cases can occur now:
Case 1. There exists $1 \leqslant j \leqslant x$ such that $(t_n, y_j) \in S$. From the definition of $P_n[H_1, ..., H_n]$ we have that $(t_{n-i}, y_j) \notin S$, for $i = 1, ..., (k-1)$, $j = 1, ..., x$. Furthermore if S^* is an arbitrary maximal k-independent set of $P_n[H_1, ..., H_n] \setminus \bigcup_{i=0}^{k-1} \bigcup_{j=1}^{x} \{(t_{n-i}, y_j)\}$, then $S^* \cup \{(t_n, y_j)\}$ is a k-independent set of P_n. We shall show that $S^* \cup \{(t_n, y_j)\}$ is maximal. By an easy observation it follows that among of vertices of S^* there must be a vertex (t_p, y_j), $1 \leqslant j \leqslant x$, such that $n - k - (k-1) \leqslant p \leqslant n - k$. Otherwise it could be add the vertex (t_{n-k}, y_j) to S^*, but it contradicts the maximality of S^*. Consequently, to prove that $S^* \cup \{(t_n, y_j)\}$ is maximal it suffices to estimate the distance between vertices (t_p, y_j),(t_n, y_j) in $P_n[H_1, ..., H_n]$. By simple calculations and by Theorem 1.1 we obtain that $d_{P_n[H_1,...,H_n]}((t_p, y_j), (t_n, y_j)) = d_{P_n}(t_n, t_p) \leqslant n - (n - 2k + 1) < 2k$. This means that it is possible to add to $S^* \cup \{(t_n, y_j)\}$ no vertex among of succesive vertices $(t_p, y_j), (t_{p+1}, y_j), ..., (t_{n-1}, y_j)$, $j = 1, ..., x$, which shows that $S^* \cup \{(t_n, y_j)\}$ is maximal and $S = S^* \cup \{(t_n, y_j)\}$. Because the vertex (t_n, y_j) we can choose on x ways this implies, that the total number of maximal k-independent sets of $P_n[H_1, ..., H_n]$ containing the vertex (t_n, y_j) is equal to $xJ_k(P_{n-k}[H_1, ..., H_{n-k}])$.
Case 2. For each $j = 1, ..., x$ holds $(t_n, y_j) \notin S$. Then maximal k-independent sets of $P_n[H_1, ..., H_n] \setminus \bigcup_{j=1}^{x} \{(t_n, y_j)\}$ are k-independent sets of P_n. Suppose that S^* is a maximal k-independent set of $P_n[H_1, ..., H_n] \setminus \bigcup_{j=1}^{x} \{(t_n, y_j)\}$, which is isomorphic to $P_{n-1}[H_1, ..., H_{n-1}]$. It should be noted that if there exists $1 \leqslant j \leqslant x$, such that $(t_{n-k}, y_j) \in S^*$, then S^* could not be maximal, since it would be $d_{P_n[H_1,...,H_n]}((t_{n-k}, y_j), (t_n, y_j)) = d_{P_n}(t_{n-k}, t_n) = k$. Observe that if $(t_{n-k}, y_j) \notin S^*$, then there must be a vertex (t_p, y_q), such that $n - k < p \leqslant n - 1$, $1 \leqslant q \leqslant x$ which belongs to S^*, by maximality of S^*

in the graph $P_n[H_1, ..., H_n] \setminus \bigcup_{j=1}^{x} \{(t_n, y_j)\}$. (Thus we can conclude that S^* is a maximal k-independent set of $P_n[H_1, ..., H_n]$ in this possibility.) This means that to calculate the total number of maximal k-independent sets of $P_n[H_1, ..., H_n]$ not containing the vertex (t_n, y_j) it sufficies to substract the number of all subsets S^* which contain one of the vertices from set $\{(t_{n-k}, y_j),\ j = 1, ..., x\}$ from the number $J_k(P_{n-1}[H_1, ..., H_{n-1}])$.
Let r denotes the number of all maximal k-independent sets of $P_n[H_1, ..., H_n] \setminus \bigcup_{j=1}^{x} \{(t_n, y_j)\}$ containing one vertex from set $\{(t_{n-k}, y_j),\ j = 1, ..., x\}$.
Consider three possibilities:
Subcase 2.1. $k + 2 \leqslant n \leqslant 2k$.
Since $n - k \leqslant k$, so there exists exactly x maximal k-independent sets S^* of the graph $P_n[H_1, ..., H_n] \setminus \bigcup_{j=1}^{x} \{(t_n, y_j)\}$ containing the vertex (t_{n-k}, y_j), namely $S^* = \{(t_{n-k}, y_j)\},\ 1 \leqslant j \leqslant x\}$. This means that $r = x$. So, in this case the number of all maximal k-independent sets is equal to $J_k(P_{n-1}[H_1, ..., H_{n-1}]) - x$.
Subcase 2.2. $n = 2k + 1$.
Because $n = 2k+1$, so $n-k = k+1$, hence every maximal k-independent set S^* of the graph $P_n[H_1, ..., H_n] \setminus \bigcup_{j=1}^{x} \{(t_n, y_j)\}$ containing the vertex (t_{n-k}, y_j) has the form $S^* = \{(t_1, y_j), (t_{k+1}, y_s)\}$. Because $1 \leqslant j \leqslant x$ and $1 \leqslant s \leqslant x$ then this gives that we have x^2 such sets, so $r = x^2$.
Subcase 2.3. $n \geqslant 2k + 2$.
Consider the graph $P_n[H_1, ..., H_n] \setminus \bigcup_{i=0}^{k-1} \bigcup_{j=1}^{x} \{(t_{n-i}, y_j)\}$ which is isomorphic to $P_{n-k}[H_1, ..., H_{n-k}]$. Since r denotes the number of all maximal k-independent sets containing one of the vertices from set $\{(t_{n-k}, y_j); j = 1, ..., x\}$, it follows from Case 1 and preceeding observations that
$r = x(J_k(P_{(n-k)-k}[H_1, ..., H_{(n-k)-k}]) = xJ_k(P_{n-2k}[H_1, ..., H_{n-2k}])$.
All this togeather gives that:
$J_k(P_n[H_1, ..., H_n]) = J_k(P_{n-1}[H_1, ..., H_{n-1}]) + x(J_k(P_{n-k}[H_1, ..., H_{n-k}]) - 1)$, for $k + 2 \leqslant n \leqslant 2k$,
$J(P_{2k+1}[H_1, ..., H_{2k+1}]) = J_k(P_{2k}[H_1, ..., H_{2k}]) + x(J_k(P_{k+1}[H_1, ..., H_{k+1}]) - x)$ and
$J_k(P_n[H_1, ..., H_n]) = J_k(P_{n-1}[H_1, ..., H_{n-1}]) + x(J_k(P_{n-k}[H_1, ..., H_{n-k}]) - J_k(P_{n-2k}[H_1, ..., H_{n-2k}]))$, for $n \geqslant 2k + 2$. ∎
Using the same method we prove:
Theorem 2.10. *Let $n \geqslant 2$, $x \geqslant 1$. Then the number $J(P_n[K_x])$ satisfy the following recurrence relations*
$J(P_2[K_x]) = 2x$,
$J(P_3[K_x]) = x^2 + x$,

$J(P_4[K_x]) = J(P_3[K_x]) + x(J(P_2[K_x]) - 1),$
$J(P_5[K_x]) = J(P_4[K_x]) + x(J(P_3[K_x]) - x),$
$J(P_n[K_x]) = J(P_{n-1}[K_x]) + x(J(P_{n-2}[K_x]) - J(P_{n-4}[K_x]))$, *for* $n \geqslant 6$. ∎

Corollary 2.11. *If* $x = 1$, *then* $J_k(P_n[H_1, ..., H_n]) = J_k(P_n)$ and $J(P_n[K_x]) = J(P_n)$.

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