Enumerations of Unlabelled Multigraphs

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Abstract

Using R. C. Read's superposition method we establish a formula for the enumeration of Euler multigraphs, with loops allowed and with given numbers of edges. In addition, applying Burnside's Lemma and our adaptation of Read's superposition method, we also derive a formula for the enumeration of Euler multigraphs without loops – via the calculation of the number of perfect matchings of the complement of complete multipartite graphs. MAPLE is employed to implement these enumerations. For one up to 13 edges, the numbers of nonisomorphic Euler multigraphs with loops allowed are 1, 3, 6, 16, 34, 90, 213, 572, 1499, 4231, 12115, 36660 and 114105, respectively and, for one up to 16 edges, the numbers of nonisomorphic Euler multigraphs without loops are 0, 1, 1, 4, 4, 15, 22, 68, 131, 376, 892, 2627, 7217, 22349, 69271 and 229553, respectively. Simplification of these methods yields the numbers of multigraphs with given numbers of edges, results which also appear to be new. Our methods also apply to multigraphs with essentially arbitrary constraints on vertex degrees.

1 Introduction

An *Euler graph* is defined to be a graph with exclusively even vertex degrees. A multigraph is one which allows multiple edges to connect the same pair of vertices. Loops are edges originating and ending on the same vertex. A graph is called *simple* if it has neither loops nor multiple edges. In [1], Robinson gave a formula enumerating the simple (unlabelled) Euler graphs on n vertices. In [2], Read provided a method for enumerating vertex-labelled Euler graphs. In [3], Chiê Nara and Shinsei Tazawa enumerated unlabelled (simple) graphs with specified degree parities. In the subsequent decades, however, the problem of enumerating non-simple Euler graphs appears to have been overlooked, that is, until it was motivated by the development of Bayesian statistics for (-1,+1)-binary n-sequences [4]. This application engenders the enumeration of unlabeled Euler multigraphs without loops and with a given number of edges: the main goal of this manuscript.

When all is said and done, the further extentions of the state-of-the-art enumerations —- which we implemented —- yield only a modest number of the leading coefficients for our enumeration, not the general terms nor the asymptotics. Indeed, the extent of the computations in our implementation appear to increase at least as fast as geometrically with the evaluation of each successive coefficient. Thus, the problems which we address herein comprise significant challenges for future combinatorial analysis.

To facilitate progress, we first considered a related problem whose solution was at hand: the enumeration of Euler multigraphs with loops allowed. R. C. Read established a method of enumerating graphs with given valencies, referred to as superpostion theory [5]. His approach is modified herein to achieve our main goal. For the enumeration of Euler multigraphs without loops, the key is to superimpose one graph upon another so that, in the construct, every edge of each graph links two different components of the other. This construction will be seen to involve the enumeration of perfect matchings of the complete multipartite graph, the complement of $G = K_{i_1} \bigcup K_{i_2} \bigcup \cdots \bigcup K_{i_s}$: the union of disjoint complete graphs. From matching theory [6], the number of perfect matchings of the complement $\overline{G}$ of G, denoted $pm(\overline{G})$ or $pm(i_1, i_2, \ldots, i_s)$, is expressible as

$$pm(\overline{G}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} \mu(G, x) dx, \tag{1}$$

where $\mu(G, x)$ is the matching polynomial of the graph G which is known for our case [6]:

$$\begin{aligned} \mu(i_1, i_2, \ldots, i_s, x) &\stackrel{def}{=} \mu(G, x) \\ &= \prod_{t=1}^{s} \left(\sum_{0 \le r \le \lfloor \frac{i_t}{2} \rfloor} (-1)^r \frac{i_t!}{r!(i_t - 2r)! 2^r} x^{i_t - 2r} \right). \end{aligned} \tag{2}$$

Sometimes we borrow notations of set theory to simplify our symbols like $pm(i_1, i_2, \ldots, i_s)$: rewriting this as $pm(i_o | o = 1, \ldots, s)$ or simply as $pm(i_o | o)$, when there is no ambiguity. For multiple subscripts we also use $pm(i_{jkl} | j, k, l)$.

2 Basics

We first recall some of the essential elements of Read's superposition theory [5].

(a) Superposition. Let $G_1, G_2, \ldots, G_k$ be k unlabelled graphs all with n vertices. A superposition of $G_1, G_2, \ldots, G_k$ is defined as a graph formed by labelling the vertices of each graph G_i with $\{1, 2, ..., n\}$ and collapsing the vertices with the same label, while retaining differentiation of edges from different graphs. Thus a superposition is, in general, a multigraph with k types of edges. Two superpositions of $G_1, G_2, \ldots, G_k$ are considered the same if one can be transformed to the other by a permutation of $\{1, 2, ..., n\}$ and the natural induced action on the k types of edges. Recall that the automorphism group of a graph is the set of permutations of the vertices' identities which, in their induced action, stablilize the (multi)set of edges. A theorem enumerates the distinct superpositions[5]:

Theorem 1 The Superposition Theorem (R.C. Read) *The number of distinct superimposed graphs that can be obtained by superimposing the graphs $G_1, G_2, \ldots, G_k$, each with n vertices, is $N(H_1, H_2, \cdots, H_k)$ where H_i is the automorphism group of G_i $(i = 1, 2, \ldots, k)$ and the function N is defined as follows, in terms of the cycle-index polynomials P_i for the respective automorphism group H_i:*

$$P_i \stackrel{def}{=} \sum_{j_1+2j_2+\cdots+nj_n=n} A^{(i)}_{j_1 j_2 \cdots j_n} f_1^{i_1} f_2^{i_2} \cdots f_n^{i_n}; i = 1, 2, \ldots, k.$$

Then, with (j) denoting this domain of summation,

$$N(H_1, H_2, \cdots, H_k) \stackrel{def}{=} \sum_{(j)} \left(\prod_{r=1}^{k} A^{(r)}_{j_1 j_2 \cdots j_n}\right)(j_1! j_2! \cdots j_n! 2^{j_2} 3^{j_3} \cdots n^{j_n})^{k-1}. \tag{3}$$

(b) Consider the rectangular array of variables

$$\begin{array}{cccc} z_{11}, & z_{12}, & \ldots, & z_{1n} \\ z_{21}, & z_{22}, & \ldots, & z_{2n} \\ \ldots, & \ldots, & \ldots, & \ldots \\ z_{m1}, & z_{m2}, & \ldots, & z_{mn} \end{array}$$

and a permutation group K of degree n and a permutation group H_j of degree m. We can permute this array of objects by, first, permuting each row with an individual permutation from K and, then, permuting the rows with any permutation from H. This plainly yields a permutation group acting upon this array, denoted $H[K]$: the wreath product. A Pólya result [7] is that the cycle index polynomial of $H[K]$ is obtained by "substituting" the cycle index of K in that of H in the following way. Let $Z(H) =$

$P(h_1, h_2, \ldots, h_m)$ be the cycle index polynomial of H, in the indeterminates $h_1, h_2, \ldots, h_n$, and $Z(K) = Q(f_1, f_2, \ldots, f_n)$ that of K, in the indeterminates $f_1, f_2, \ldots, f_n$. Then, the substitution of K into H is effected by replacing h_i in $P(h_1, h_2, \ldots, h_m)$, by $Q(f_i, f_{2i}, \ldots, f_{ni})$. (See [7], p. 178.) This composition is referred to as *plethysm* [8]. For example, we have

$$Z(S_2) = \frac{1}{2}(f_1^2 + f_2).$$

Therefore,

$$\begin{aligned} Z(S_2[S_2]) &= \tfrac{1}{2}\{[\tfrac{1}{2}(f_1^2 + f_2)]^2 + \tfrac{1}{2}(f_2^2 + f_4)\} \\ &= \tfrac{1}{8}(f_1^4 + 2f_1^2 f_2 + 3f_2^2 + 2f_4). \end{aligned}$$

(c) Enumeration of bipartite graphs with given valencies because this will facilitate the enumeration of multigraphs. A bipartite graph is a graph whose vertices can be partitioned into two independent sets A and B. We may translate the enumeration of nonisomorphic bipartite graphs – with given valencies – into that of superpositions of certain graphs. Suppose a_i is the number of vertices in A of valency i and b_j is the number of vertices in B of valency j; $i, j = 1, 2, \ldots$. Then, evidently, with e denoting the number of edges in the graph,

$$e = a_1 + 2a_2 + 3a_3 + \cdots = b_1 + 2b_2 + 3b_3 + \cdots.$$

We consider two graphs: G_1 composed of the union of a_i disjoint complete graphs of i vertices, K_i, $i = 1, 2, \ldots$, and G_2 composed of the union of b_j disjoint complete graphs of j vertices, $K_j, j = 1, 2, \ldots$, respectively.

$$G_1 = \bigcup_{i=1,2,\cdots} (\bigcup_{j=1}^{a_i} K_i), \quad G_2 = \bigcup_{i=1,2,\cdots} (\bigcup_{j=1}^{b_i} K_i).$$

The automorphism group H_1 for G_1 is, evidently,

$$H_1 = S_{a_1}[S_1] \times S_{a_2}[S_2] \times S_{a_3}[S_3] \times \cdots,$$

where $\times$ denotes the direct product. Similarly, the automorphism group of G_2 is

$$H_2 = S_{b_1}[S_1] \times S_{b_2}[S_2] \times S_{b_3}[S_3] \times \cdots.$$

Now superimpose G_1 and G_2. Then, contract each complete graph of G_1 and G_2 into a respective vertex and add as many edges between the resulting vertices from G_1 and from G_2 as the number of superimposed pairs of vertices of the respective complete graphs. Fig. 1 shows an example of this superposition and contraction. A bipartite graph with the desired valencies is thereby constructed. There is plainly a one-to-one correspondence between nonisomorphic superimposed graphs resulting from all possible labellings and the bipartite graphs we wish to enumerate.

(d) Multigraphs with given valencies. Suppose that there are to be a_i vertices of valency i. The number of edges e is given by

$$e = \frac{1}{2}(a_1 + 2a_2 + 3a_3 + \cdots).$$

Here we allow loops, multiple edges, and disconnection. If we regard the given vertex set as A and insert a vertex at the midpoint of each edge, we obtain a bipartite graph, the "B part" of the respective bipartite partition being the set of inserted vertices. All the vertices in B clearly have valency two. Conversely, given a bipartite graph with every vertex in one block of the bipartite partition having valency two, we can delete these vertices and form the given graph, having the required valencies. Thus, the correspondence between multigraphs with given valencies and the desired bipartite graphs is one-to-one.

(e) The enumeration of Euler multigraphs with loops. This is reduced to the enumeration of the graphs with given valencies, because, for each partition $[a_1, a_2, \ldots, a_e]$ of the number of edges e of the graph, with a_i i's, $i = 1, \ldots, e$, we can enumerate the graphs with a_i vertices of valency $2i$, $1 \leq i \leq e$ and then sum over all partitions of e. Thus, we have

Theorem 2 (R. C. Read) *The number of Euler multigraphs of e edges, with loops allowed, is*

$$N_e = \sum_{a_1+2a_2+\cdots+ea_e=e} N((S_{a_1}[S_2]\times S_{a_2}[S_4]\times\cdots\times S_{a_e}[S_{2e}]), S_e[S_2]).$$

We have employed MAPLE to obtain the numbers of Euler multigraphs for the number of edges 1, 2, ..., 10: 1, 3, 6, 16, 34, 90, 213, 572, 1499 and 4231, respectively.

3 Formula for the Enumeration: Loops Forbidden

Building upon the preceding, we next establish the formula for the enumeration of Euler multigraphs without loops. From the argument of (c) above, we can infer that the superposition in (e) is equivalent to the superposition of two graphs:

$$G_1 = \bigcup_{1\le i\le e} (\bigcup_{j=1}^{a_i} K_{2i})$$

and $G_2 = K_2 \bigcup K_2 \bigcup \cdots \bigcup K_2$, a perfect matching of the same number of vertices as G_1. Thus, forbidding loops amounts to not counting those superpositions in which any edges of G_2 are superimposed upon an edge of a component of G_1. Therefore, our desideratum is equivalent to enumerating the inequivalent perfect matchings of the complement graph $\overline{G}_1$ of G_1, with equivalence under the automorphism group $H_1 = Aut\ G_1$. Note, for simple graphs such as these $Aut\ G = Aut\ \overline{G}$.

To use Burnside's Lemma for this purpose, we will need to determine the number of perfect matchings fixed by each element of H_1. This motivates the following detailed analysis of the natural permutaion representation of H_1. We wish to take advantage of the cycle index of H_1. However, the cycle index only reflects the "cycle types" of the elements of H_1, while, unfortunately, elementary examples demonstrate that elements of the same cycle type don't necessarily fix the same number of perfect matchings. This motivates the following expanded characterization.

For this purpose, it suffices to consider only an arbitary direct factor of H_1 because such factors may be treated independently. Let the factor be $S_{a_i}[S_{2i}]$ and the corresponding graph be the union of a_i K_{2i}'s, labelled $A_1, A_2, \ldots, A_{a_i}$. Furthermore, for any element of S_{a_i}, that is, a permutation of $\{A_1, A_2, \ldots, A_{a_i}\}$, its independent cycles generate disjoint permutations. Let $z = (A_1, A_2, \ldots, A_s)$ be such a cycle, without loss of generality, and suppose A_t contains the vertices $\{v_{t1}, v_{t2}, \ldots, v_{tq}\}$; $t = 1, 2, \ldots, s$, and $q = 2i$.

Lemma 1 *All the permutations of $S_{a_i}[S_q]$ with row-action z are enumerated according to cycle type by $(q!)^s Q(f_s, f_{2s}, \ldots, f_{qs})$, where $Q(f_1, f_2, \ldots, f_q)$ is the cycle index of S_q.*

Proof. By induction on s. The lemma is true for s equal unity. We apply Pólya's result, from (b), for $Z_s[S_q]$ with Z_s denoting the cyclic group $\langle (A_1, A_2, \ldots, A_s) \rangle$. Note that different cycles in a permutation of Z_s generate disjoint permutations. Let $R(h_1, h_2, \ldots, h_{s-1}) + ah_s$ denote the cycle index polynomial of Z_s, where $a = \phi(s)/s$. Then all the permutations of Z_s are enumerated by $s[R(h_1, h_2, \ldots, h_{s-1}) + ah_s]$. By Pólya's result, the cycle index polynomial of $Z_s[S_q]$ is

$$R(Q(f_1, f_2, \ldots, f_q), Q(f_2, f_4, \ldots, f_{2q}), \ldots,\\ Q(f_{(s-1)}, f_{2(s-1)}, \ldots, f_{q(s-1)})) + aQ(f_s, f_{2s}, \ldots, f_{qs}).$$

Thus,

$$s(q!)^s[R(Q(f_1, f_2, \ldots, f_q), Q(f_2, f_4, \ldots, f_{2q}), \ldots,\\ Q(f_{(s-1)}, f_{2(s-1)}, \ldots, f_{q(s-1)})) + aQ(f_s, f_{2s}, \ldots, f_{qs})]$$

represents all the permutations of $Z_s[S_q]$. By the induction hypothesis,

$$s(q!)^s[R(Q(f_1, f_2, \ldots, f_q), Q(f_2, f_4, \ldots, f_{2q}), \ldots,\\ Q(f_{(s-1)}, f_{2(s-1)}, \ldots, f_{q(s-1)}))]$$

comprises all the permutations of $Z_s[S_q]$ generated by the permutations of Z_s composed of cycles less than s in length. As a result, $\phi(s)(q!)^s Q(f_s, f_{2s}, \ldots, f_{qs})$ represents all the permutations of $Z_s[S_q]$ generated by the cycles in Z_s of length s. There are altogether $\phi(s)$ s-cycles in Z_s, yielding the desired result. □

Remarks What this lemma means is that Pólya's plethysm in (b), Section 2, is true for H equal a cycle as well as for a whole group.

Note also that since the elements of $(q!)^s Q(f_s, f_{2s}, \ldots, f_{qs})$ represent permutations "built upon" z, it is plain that they are permutations composed of cycles of the elements ordered from A_1 to A_s. For example, f_{2s} represents the cycle of order $2s$ composed of two s-cycles of elements from A_1 to A_s: $(v_{1j_1} v_{2j_2} \ldots v_{sj_s} v_{1k_1} v_{2k_2} \ldots v_{sk_s})$.

Now, there are three kinds of matchings fixed by a given permutation $\sigma \in S_{a_i}[S_{2i}]$. In this respect, there is a natural parallel to [1].

The first kind involves a perfect matching within a cycle of σ, which will be denoted an *in-matching*. By the argument of Proposition 1 in [8], if a cycle has an in-matching, it must be of even order. Thus, in the case of $Q(f_s, f_{2s}, \ldots, f_{(2i)s})$ — - where $Q(f_1, f_2, \ldots, f_{2i})$ is the cycle index polynomial of K_{2i} —- in order for the cycle x corresponding to f_{ts} to have an admissible fixed in-matching s must be even and t must be odd. This is because x is composed of the cycles of the elements in the order from A_1 to A_s, and, also, the first entry of x must match the $(\frac{ts}{2} + 1)th$ entry —- but the two entries must fall into different A's in an admissible matching. For instance, the cycle $(v_{1j_1} v_{2j_2} \ldots v_{sj_s} v_{1k_1} v_{2k_2} \ldots v_{sk_s})$ has no in-matching because v_{1j_1} would match v_{1k_1}, but they are both in A_1. In summary, there is precisely one admissible fixed in-matching pertaining to f_{ts} if and only if t is odd and s is even.

The second kind of fixed matching has edges between two cycles arising from a common cycle of $\{A_1, A_2, \ldots, A_{a_i}\}$. The cycles involved must have the same size ts, by the argument in [8], and must be paired. The number of ways of pairing is the same as the number of perfect matchings if we regard the cycles in question as vertices of a complete graph, invoking the hemifactorial. Furthermore, for each pair there are exactly $t(s-1)$ fixed matchings because two points in the same A can not match each other. For example, suppose a cycle yields two cycles $(v_{1j_1}v_{2j_2}\dots v_{sj_s}v_{1k_1}v_{2k_2}\dots v_{sk_s})$ and $(v_{1p_1}v_{2p_2}\dots v_{sp_s}v_{1q_1}v_{2q_2}\dots v_{sq_s})$. In this case, v_{1j_1} can not match v_{1p_1} or v_{1q_1} since they are both in A_1.

The third kind includes the rest: that is, those matchings whose edges lie between different cycles generated by different cycles of S_{a_i}. Cycles of the same size r must be paired, and, for each pair, there are r fixed matchings; there are no constraints for these matchings since they involve different A's. The number of such matchings is plainly the same as r^w times that for a corresponding multipartite graph, where w is the number of edges of the multipartite matchings and where each part corresponds to a different cycle of S_{a_i}. By definations, cycles in $S_{a_i}[S_{2i}]$ generated by the same cycle of S_{a_i} are not allowed to match one another in these matchings. For example, if the permutation is of the form $x_1x_2y_1y_2z_1z_2$ in which x, y and z are cycles of size r generated from three different cycles of S_{a_i}, then the number of matchings fixed by this permutation is $8r^3$, because there are 8 admissible matchings of $x_1, x_2, y_1, y_2, z_1, z_2$, and each comprises 3 edges.

Suppose $P_{a_i}(h_1, h_2, \dots, h_{a_i})$ is the cycle index polynomial of S_{a_i} and $Q_{2i}(f_1, f_2, \dots, f_{2i})$ is the cycle index polynomial of S_{2i}. By the remark following Lemma 1 in the previous section we can still implement Pólya's plethysm. From the application of the Burnside lemma, however, we will also need to distinguish the cycles of S_{a_i} in the substitution, for the second kind of matchings, and we will also need to record the size of these cycles, for the first kind of matchings. This is achieved as follows:

1. Obtain $\widehat{P}_{a_i}$ from the cycle index polynomial $P_{a_i}(h_1, h_2, \dots, h_{a_i})$ by replacing $h_l^{q_l}$ with $h_l(i,1)\cdots h_l(i,q_l), i = 1, 2, \dots, e$;

2. Implement the substitution $\widehat{P}_{a_i}\{Q_{2i}\}$ of Q_{2i} into $\widehat{P}_{a_i}$ by substituting

$$Q_{2i}(f_j(i,j,k), f_{2j}(i,j,k), \ldots, f_{(2i)j}(i,j,k))$$

for $h_j(i,k)$ in $\widehat{P}_{a_i}, i = 1, 2, \ldots, e$;

3. Take the product over i, forming the final cycle index polynomial:

$$\widehat{P}_{a_1,a_2,\ldots,a_e}\{Q\} = \prod_{i=1}^{e} \widehat{P}_{a_i}\{Q_{2i}\}.$$

Now for each l, $[f_l(i,j,k)]^{r_{ijkl}}$ in

$$\prod_{i,j,k} [f_l(i,j,k)]^{r_{ijkl}},$$

$\widehat{P}_{a_1,a_2,\ldots,a_e}\{Q\}$ can have the three kinds of fixed matchings. So r_{ijkl} is partitioned into α_{ijkl}, $2\beta_{ijkl}$ and γ_{ijkl}, corresponding to the in-matchings, matchings between cycles generated by the same cycle of $\{A_1, A_2, \ldots, A_{a_i}\}$ and the rest, respectively. Recall that matches of the third kind are between different A-cycles. The number of such matchings equals

$$l^{(\frac{1}{2}\sum_{i,j,k}\gamma_{ijkl})} pm(\gamma_{ijkl}|i,j,k).$$

Hence from (1), the number of the fixed perfect matchings by a permutation of type

$$\prod_{i,j,k,l} [f_l(i,j,k)]^{r_{ijkl}}$$

in $\widehat{P}_{a_1,a_2,\ldots,a_e}\{Q\}$ equals

$$F = \prod_l \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \{\prod_{i,j,k} \sum_{\alpha_{ijkl},\beta_{ijkl},\gamma_{ijkl}} \Omega_{ijkl}\, \mu(\gamma_{ijkl}, x)\} dx, \quad (4)$$

where $\Omega_{ijkl} = \binom{r_{ijkl}}{\alpha_{ijkl}, 2\beta_{ijkl}} (2\beta_{ijkl} - 1)!!(l(j-1)/j)^{\beta_{ijkl}} l^{\frac{1}{2}\gamma_{ijkl}}$, the first factor denoting a trinomial coefficient, $(-1)!! = 1$, and where the summation over l, i, j, k, are over natural numbers with their proper ranges, where the summation over $\alpha_{ijkl}, \beta_{ijkl}, \gamma_{ijkl}$ ranges over non-negative integers satisfying the condition $\delta_{ijkl}\alpha_{ijkl} + 2\beta_{ijkl} + \gamma_{ijkl} = r_{ijkl}$, with $\delta_{ijkl} = 1$ if j is even and $\frac{l}{j}$ is odd, otherwise $\delta_{ijkl} = 0$. Then replace each term $\prod_{i,j,k,l}[f_l(i,j,k)]^{r_{ijkl}}$ in $\widehat{P}_{a_1,a_2,\ldots,a_e}\{Q\}$ with F to get the number $N(\widehat{P}_{a_1,a_2,\ldots,a_e})$.

It is not difficult to prove that $pm(i_1, i_2, \ldots, i_s) > 0$ only if

$$\max(i_1, i_2, \ldots, i_s) \leq \frac{1}{2}(i_1 + i_2 + \cdots + i_s).$$

On balance we have by Burnside's Lemma

Theorem 3 *The number of Euler multigraphs of e edges and with no loops is given by*

$$EMG(e) = \sum_{\substack{a_1 + 2a_2 + \cdots + ea_e = e \\ \max_{1\leq i\leq e}\{i, a_i > 0\} \leq \frac{e}{2}}} N(\widehat{P}_{a_1,a_2,\ldots,a_e}).$$

□

4 Further Simplification and Generalization

In implementing this enumeration by MAPLE, we have to do the repetitive calculation of polynomials with many unknowns. This entails long running times. If we can replace the calculation of polynomials by numerical caculation, as much as possible, then it will greatly reduce the operating time. Let's go back to what we have done in Section 3. The only purpose of steps 1, 2 and 3 in Section 3 is to distinguish different cycles of S_{a_i} in order to avoid matchings lying in the same A's. This only involves matchings of the first two kinds of the three mentioned in Section 3 which match nothing outside a single cycle of S_{a_i}. Hence we can compute these matchings within the single cycle first with no regard to other cycles of S_{a_i}. The consequence is that we simply substitute

$$Q_{2i}(f_j(i,j), f_{2j}(i,j), \ldots, f_{(2i)j}(i,j))$$

for any cycle of length j in S_{a_i} and work out the details for the first and second kinds of matchings.

Here we omit the index k since it is not necessary. We note also that in equation (4), the order of the multiplication and integration may be transposed (because each integral term of the product is a constant and can be put in and out of the integrals):

$$\prod_{l=1} \int_{-\infty}^{\infty} f_l(x)dx = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_l [\prod_{l=1} f_l(x_l)],$$

where $f_l(x_l)$ denotes the corresponding factor of the integral of (4). The indices of f and x suffice for our purpose. From the discussion above, we know how to streamline the algorithm:

1. First introduce an operation Φ_j on polynomials for each cycle length j:

   Let $Q(f_1, f_2, \cdots, f_n) = \sum_{i_1+2i_2+\cdots+ni_n=n} c_{i_1 i_2 \ldots i_n} \prod_{s=1}^{n} f_s^{i_s}$. Then

   $$\begin{aligned}\Phi_j(Q) = & \textstyle\sum_{i_1+2i_2+\cdots+ni_n=n} c_{i_1 i_2 \ldots i_n} \\ & \textstyle\prod_{s=1}^{n} \sum_{\delta_s\alpha_s+2\beta_s+\gamma_s=i_s, \alpha_s\ge 0, \beta_s \ge 0} \\ & \binom{i_s}{\alpha_s, 2\beta_s}(2\beta_s - 1)!![s(j-1)]^{\beta_s}(sj)^{\frac{1}{2}\gamma_s}\mu(\gamma_s, x_{sj}).\end{aligned}$$

   Here the subscript sj of x is the product of s and j, and $\delta_s = 1$ if j is even and s is odd; otherwise $\delta_s = 0$.

2. Let $a_1 + 2a_2 + 3a_3 + \cdots + ea_e = e$, where e is the number of edges, and $Z(S_{a_i})$ denotes the cycle index of S_{a_i}, $i = 1, 2, \ldots, e$:

   $$Z(S_{a_i}) = P_{a_i}(h_1, h_2, \ldots, h_{a_i}).$$

   $Q_i(f_1, f_2, \ldots, f_{2i})$ is the cycle index of S_{2i}. Then we 'substitute' Q_i into P_{a_i} in a fashion similar to Pólya's plethysm:

   $$F_{a_1 a_2 \cdots a_e} = \prod_{i=1}^{e} P_{a_i}(\Phi_1(Q_i), \Phi_2(Q_i), \ldots, \Phi_{a_i}(Q_i)).$$

3. Perform the integration, where l is the number of variables in $F_{a_1a_2\cdots a_e}$:

$$\begin{aligned} N_{a_1a_2\cdots a_e} &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-x_1{}^2}{2}}dx_1\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-x_2{}^2}{2}}dx_2\cdots \\ &\quad \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-x_l{}^2}{2}}dx_l F_{a_1a_2\cdots a_e} \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^l\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\cdots \\ &\quad \int_{-\infty}^{\infty}dx_l e^{-\frac{1}{2}\sum_{t=1}^{l}x_t{}^2}F_{a_1a_2\cdots a_e}. \end{aligned}$$

When we apply $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\frac{-x^2}{2}}x^k dx = (k-1)!!$, if k is even, or 0, otherwise, we can further simplify this formula.

4. Finally

$$EMG(e) = \sum_{\substack{a_1+2a_2+\cdots+ea_e=e \\ \max_{1\le i\le e}\{i, a_i>0\}\le\frac{e}{2}}} N_{a_1a_2\ldots a_e}.$$

Theorem 4 *The number of Euler multigraphs of e edges and with no loops is given by*

$$EMG(e) = \sum_{\substack{a_1+2a_2+\cdots+ea_e=e \\ \max_{1\le i\le e}\{i, a_i>0\}\le\frac{e}{2}}} N_{a_1a_2\ldots a_e}.$$

□

We have programmed our formulas in MAPLE, obtaining the numbers 0, 1, 1, 4, 4, 15, 22, 68, 131, 376, 892, 2627, 7217, 22349, 69271, 229553 for the respective numbers of edges from 1 to 16.

Remarks The first three steps above provides a method of enumerating multigraphs with given valencies and with loops forbidden: replacing e with $2e$ and $2i$ with i in step 2, we get

Theorem 5 *The number of multigraphs with a_i vertices of valency i, $i = 1, 2, \ldots, 2e$, $2e = a_1 + 2a_2 + \cdots$, and with no loops* *is given by*

$$N_{a_1a_2\cdots a_{2e}} = (\frac{1}{\sqrt{2\pi}})^l \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_l e^{-\frac{1}{2}\sum_{t=1}^{l} x_t^2} F_{a_1a_2\cdots a_{2e}},$$

where l is the number of variables in $F_{a_1a_2\cdots a_{2e}}$ □

As a result,

Theorem 6 *The number of multigraphs with e edges, loops forbidden, is given by*

$$\sum_{a_1+2a_2+\cdots+2ea_{2e}=2e} N_{a_1a_2\cdots a_{2e}}.$$

□

Maple has been employed to yield the numbers of multigraphs up to 10 edges: 1, 3, 8, 23, 66, 212, 686, 2389, 8682 and 33160. This result corroborates the multigraph sequence given by Vladeta Jovovic on the On-Line Encyclopedia of Integer Sequences.

In a similar fashion, we can enumerate the multigraphs with given valencies and with loops allowed ((d) in Section 2), by regarding the superposition as a kind of perfect matching. And it can also be generalized to enumerate the multigraphs, the Euler multigraphs with loops using the " perfect matching " approach. The only difference is that, with loops allowed, there are no restrictions on the perfect matchings. Hence

Theorem 7 *The number $N_{a_1a_2\cdots}$ of multigraphs with a_i vertices of valency i, $i = 1, 2, \ldots$ and with loops allowed is given by the following steps:*

1. *Construct the general cycle index polynomial*

$$\begin{aligned} F &= \prod_{i=1} Z(S_{a_i}[S_i]) \\ &= \prod_{i=1} P_{a_i}(Q_i(f_1, f_2, \ldots, f_i), Q_i(f_2, f_4, \ldots, f_{2i}), \ldots, \\ &\quad Q_i(f_{a_i}, f_{2a_i}, \ldots, f_{ia_i})). \end{aligned}$$

2. For each term $f_s^{i_s}$ in F, replace it by

$$f_s^{i_s} = \sum_{\alpha_s=1}^{[i_s/2]} \delta_{s,i_s-2\alpha_s} \binom{i_s}{2\alpha_s} (2\alpha_s - 1)!! s^{\alpha_s}$$

to get $N_{a_1 a_2 \cdots}$, where $\delta_{s,t} = 0$ if s^t is odd and, 1 otherwise.

□

And

Theorem 8 *The number of multigraphs with e edges and with loops allowed is:*

$$\sum_{a_1+2a_2+\cdots+2ea_{2e}=2e} N_{a_1 a_2 \cdots a_{2e}},$$

where $N_{a_1 a_2 \cdots a_{2e}}$ is obtained through the steps in Theorem 7.

□

MAPLE yielded the numbers: 2, 7, 23, 78, 274, 1002, 3756, 14682, 59445 and 249595 for the number of edges from 1 to 10.

Theorem 9 *The number of Euler multigraphs with e edges and with loops allowed is:*

$$\sum_{a_1+2a_2+\cdots+ea_e=e} N_{a_1 a_2 \cdots a_e},$$

where $N_{a_1 a_2 \cdots a_e}$ is obtained through the steps in Theorem 7 by replacing S_i with S_{2i}.

□

In summary, we used R.C. Read's superposition approach and (3) to enumerate the Euler multigraphs allowing loops, yielding the numbers 1,3,6,16,34,90,213,572,1499, 4231 for the numbers of edges up to 10. Then because in our case one of the superimposed graphs is effectively a perfect matching, we converted the problem to the enumeration of perfect matchings in the case in which loops are forbidden. This greatly reduced the running time for MAPLE. Furthermore, using the same perfect matching approach, we established formulas to enumerate the multigraphs with given valencies allowing and forbidding loops and to obtain the next three numbers, 12115,36660 and 114105, of the Euler multigraphs with loops allowed.

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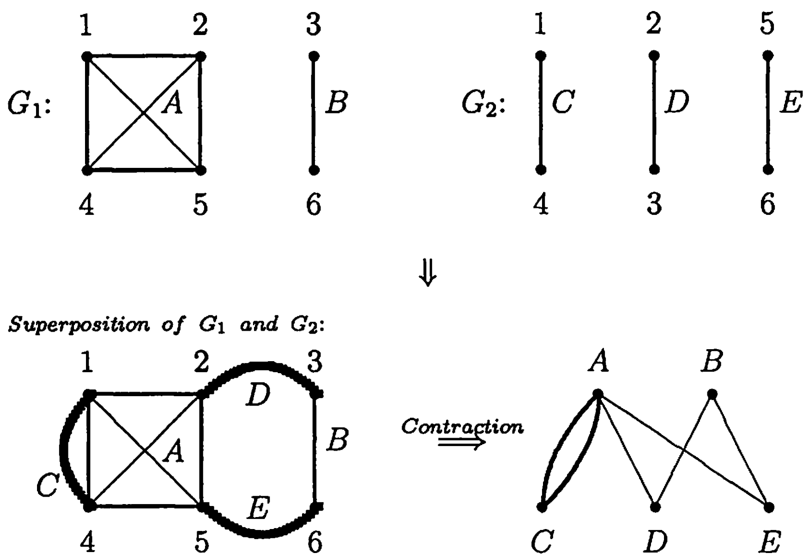

Fig. 1 An example of superposition and contraction