On wins in round robin tournaments

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In a round robin tournament between n teams each team plays each other team once. Suppose that the match between two teams is contested by just two players, one representing each team. For different matches a team may vary the player, who represents it. We ask: given a positive integer w, what is the maximum number M of players, who can achieve at least w wins? Clearly $M = 0$ if $w \geq n$, since each team only plays $n-1$ matches. For $w < n$ we shall show

Proposition: $$M = \begin{cases} 2n - 2w - 1 & \text{if } \frac{n-1}{2} < w < n \\ [\binom{n}{2}/w] & \text{if } 0 < w \leq \frac{n-1}{2} \end{cases}.$$

Here the notation $[x]$ means the largest integer not exceeding x and it is assumed that all teams have sufficiently many players.

We present two proofs of this proposition, but both rely on the recognition of an appropriate pattern of results. The first proof is constructive, whereas the second has the advantage of proving a slightly stronger result. It is also shorter, but depends on spotting a delicate algebraic inequality. The reformulation of the problem in terms of graphs is as follows: we assign orientations to the edges of the complete graph with n vertices and let $\{d_1, \ldots, d_n\}$ denote the (unordered) set of outdegrees of the vertices. The proposition gives the maximum possible value of $\sum[\frac{d_i}{w}]$. In a transportation problem one might be interested in such a maximisation. For example, the vertices might represent depots at approximately equal journey times from each other, which need to have daily contact, and a messenger can make w return trips per day and one wishes to minimise the number of messengers.

§1 Proof of Proposition in the case $\frac{n-1}{2} < w < n$.

Since each team plays only $n-1$ matches, each team can have at most one player with at least w wins. Given any set of $2r-1$

teams there are $\binom{n}{2} - \binom{n-2r+1}{2} = (n-r)(2r-1)$ matches which involve at least one of the teams. Similarly, given any set of $2r$ teams, there are $\binom{n}{2} - \binom{n-2r}{2} = 2r(n-r-\frac{1}{2})$ matches which involve at least one of these $2r$ teams. We thus see that $(n-r)$ and $[n-r-\frac{1}{2}]$ are certainly upper bounds for the number of matches that $(2r-1)$ and $2r$, respectively, teams can each win. Setting $w = n-r$ we deduce that at most $2n-2w-1$ teams can have w wins each. We leave it as an exercise for the reader to exhibit a pattern of match results (along the lines of the pattern in Case 1 in §2), that will achieve the desired upper bound. An alternative way to complete the proof is to apply the harem version of Hall's Marriage Theorem ([1], 91-97) to show that $(2r-1)$ (respectively $2r$) teams can indeed win at least w matches each if and only if $w \leq n-r$ (respectively $w < n-r-\frac{1}{2}$).

§2 Constructive Proof of Proposition in the case $n \geq 2w+1$.

An upper bound for M is $[\binom{n}{2}/w]$, since there are only $\binom{n}{2}$ matches in total and M is an integer. It remains to show that this upper bound can be achieved.

Case 1: Suppose $n = 2w+1$.

Denote the teams by $A_1, A_2, \ldots, A_w, B_1, B_2, \ldots, B_w, P$. Let A_i defeat A_j if $i < j$, let A_i defeat B_j if $w - i < j$, let B_i defeat B_j if $i > j$, let B_i defeat A_j if $j \leq w - i$, and let P defeat A_i and lose to B_i for each i. This allocation of wins is consistent and gives each of the n teams exactly w wins.

Case 2: Suppose $\frac{n-1}{3} \leq w < \frac{n-1}{2}$.

We have that $n = 2w + r$, where $1 < r \leq w+1$. Denote the teams by $A_1, A_2, \ldots, A_w, B_1, B_2, \ldots, B_w, P, C_1, \ldots, C_{r-1}$. Write $\binom{r}{2} = sw + t$, where $0 \leq t < w$ and s,t are integers. Note that $\binom{r}{2} \leq \frac{(w+1)w}{2}$ and so $s \leq \frac{w+1}{2}$. Let C_j defeat C_i if $j > i$. Consider the team sequence, which has been divided by semicolons into $w+1$ blocks of length w for clarity, $B_1, B_2, \ldots, B_w; P, B_1, \ldots, B_{w-1}; P, A_1, B_1, \ldots, B_{w-2}; P, A_1, A_2, B_1, \ldots, B_{w-3}; \ldots; P, A_1, A_2, \ldots, A_{w-1}$. For $k \geq 2$ the kth block of w entries in this sequence is obtained by prefacing $B_1, B_2, \ldots, B_{w-k+1}$ by the first $k-1$ entries of $P, A_1, \ldots, A_{w-1}$. Observe that any w consecutive teams in the sequence are distinct. Let C_1 lose to the first team in the sequence (i.e. to B_1), let C_2 lose to the next two teams in the sequence (i.e. to B_2, B_3), let C_3 lose to the next three teams in the sequence, and so on, so that for $1 \le i \le r-1$ team C_i suffers $i-1$ defeats. Apart from the losses mentioned in the previous sentence let each of $C_1, C_2, \dots, C_{r-1}$ defeat all other teams A_j, B_j and P. Take the results in the matches between teams $A_1, A_2, \dots, A_w, B_1, B_2, \dots, B_w, P$ as in §2 Case 1, but with the following exceptions. If $s > 0$ reverse the results when P plays $B_1, B_2, \dots, B_w$ (so that now P wins these matches), and if $s > 1$ also reverse the results when A_1 plays $P, B_1, B_2, \dots, B_{w-1}$ (so that A_1 now wins these matches), and in general for all $i < s$ reverse the results when A_i plays the teams in the $(i+1)$th block (so that A_i wins those matches). With this configuration of results teams $B_1, B_2, \dots, B_w, A_s, A_{s+1}, \dots, A_w$ can each contribute one player with w wins, teams $C_1, C_2, \dots, C_{r-2}$ and A_i for $i < s$ can each contribute two players with w wins and team P can contribute one such player if $s = 0$ and two such players if $s > 0$. The first t teams in the $(s+1)$th block of the sequence each have one spare win, apart from those in a block of w wins.

Remark on the case when $n = 3w+1$: When w is odd and $n = 3w+1$, we have that $r = w+1$ and hence that $s = \frac{w+1}{2}$ and $t = 0$. However when w is even and $n = 3w+1$, we have $r = w+1$ and hence $s = t = \frac{w}{2}$.

Case 3: Suppose $\frac{n}{4} \le w < \frac{n-1}{3}$.

We have that $n = 3w + r$, where $1 < r \le w$. If w is odd, write $\binom{r}{2} = sw + t$, where $0 \le t < w$ (and so $s \le \frac{w-1}{2}$). If w is even, write $\binom{r}{2} + \frac{w}{2} = sw + t$, where $0 \le t < w$ (and so $s \le \frac{w}{2}$). Let the teams be $A_1, \dots, A_w, B_1, \dots, B_w, P, C_1, \dots, C_w, D_1, \dots, D_{r-1}$. Let team D_i defeat team D_j if $i > j$ and let team C_i defeat team C_j if $i > j$. Consider the same sequence of teams as was used in Case 2. As before, let C_1 lose to the first team in the sequence (i.e. to B_1), let C_2 lose to the next two teams in the sequence (i.e. to B_2, B_3), let C_3 lose to the next three teams in the sequence, and so on until C_w has lost to w teams. Then let D_1 lose to the next team in the sequence, D_2 to the next two teams in the sequence, and so on until D_{r-1} has lost to $r-1$ teams. Note that the sequence is long enough to allow all these results, since $r \le w$ and hence $\binom{w+1}{2} + \binom{r}{2} < (w+1)w$. Apart from the losses mentioned in the previous three sentences let each of $C_1, C_2, \dots, C_w, D_1, D_2, \dots, D_{r-1}$ defeat all other teams A_j, B_j and P. Let each team D_i defeat each

C_j. Take the results in matches between teams $A_1,....,A_w,B_1,....,B_w,P$ as in §2 Case 1, except reverse the results when P plays $B_1,B_2,....,B_w$ (so that P wins these matches), when A_1 plays $P,B_1,...B_{w-1}$ (so that A_1 wins these matches) and in general, for $i \leq [\frac{w+1}{2}] + s$, when A_{i-1} plays the ith block of w matches in the sequence. With this configuration of results teams $B_1,B_2,...B_w,A_{[(w+1)/2]+s},...,A_w$ can each supply one player with at least w wins, teams $P,C_1,...,C_w,A_1,...,A_{[(w+1)/2]+s-1}$ can each supply two players and teams $D_1,...,D_{r-1}$ can each supply three players. Note that the first t teams in the $([(w+1)/2]+s+1)$th block of w teams in the sequence each have one spare win apart from these.

Case 4: Suppose $n > 4w$.

Write $n = n' + 2mw$, where $2w+1 \leq n' \leq 4w$ and n' and m are integers. If $n' \leq 3w+1$ denote the teams by $A_1^i,....,A_w^i,B_1^i,....,B_w^i,P,C_1,...,C_{r-1}$, where $1 < i \leq m+1$ and $r = n' - 2w$. Take the results of the matches between $A_1^1,....,A_w^1,B_1^1,....,B_w^1,P,C_1,...,C_{r-1}$ as in Case 2, take the results of the matches between $A_1^i,....,A_w^i,B_1^i,....,B_w^i,P$, for each fixed $i \geq 2$, as in Case 1, and let each C_j defeat $A_1^i,....,A_w^i,B_1^i,....,B_w^i$ for all $i \geq 2$. Similarly, if $3w+1 < n' \leq 4w$, denote the teams by $A_1^i,....,A_w^i,B_1^i,....,B_w^i,P,C_1,...,C_w,D_1,...,D_{r-1}$, where $1 < i \leq m+1$ and $r = n' - 3w$. Take the results of the matches between $A_1^1,....,A_w^1,B_1^1,....,B_w^1,P,C_1,...,C_w,D_1,...,D_{r-1}$ as in Case 3, take the results of the matches between $A_1^i,....,A_w^i,B_1^i,....,B_w^i,P$, for each fixed $i \geq 2$, as in Case 1, and let each C_j and D_j defeat $A_1^i,....,A_w^i,B_1^i,....,B_w^i$ for all $i \geq 2$.

§3 Alternative proof of Proposition in the case $n \geq 2w+1$.

A celebrated criterion of Landau (see [1]) says that $b_1 \leq b_2 \leq ...... \leq b_n$ are possible numbers of wins by the n teams in a round robin tournament if and only if $\sum_{i=1}^n b_i = \binom{n}{2}$ and $\sum_{i=1}^r b_i \geq \binom{r}{2}$ for each r, $1 \leq r \leq n-1$. We shall use the criterion to show that it is possible to achieve the maximum number $M = [\binom{n}{2}/w]$ of players with w wins each in such a way that these players are distributed as equally as possible over the teams, i.e. the number of such players in any team differs by at most one from the number of such players in any other team. Explicitly, set

$k = [\frac{M}{n}]$, $\varepsilon = \binom{n}{2} - Mw$ and $t = M - nk$. Then $0 \leq \varepsilon < w$, $0 \leq t < n$ and $\binom{n}{2} = nkw + tw + \varepsilon$. We shall verify that the conditions in Landau's criterion are satisfied if we take $b_i = (k+1)w$ for $i > n-t$, $b_i = kw$ for $1 \leq i < n-t$, and $b_{n-t} = kw + \epsilon$.

It is immediate that $\sum_{i=1}^{n} b_i = \binom{n}{2}$. We next show that $\sum_{i=1}^{r} b_i \geq \binom{r}{2}$ for $r \leq n-t$. Since $\sum_{i=1}^{r} b_i \geq rkw$, it suffices to show that $kw \geq \frac{r-1}{2}$, i.e. that $r \leq 2kw+1$. The trick is to rewrite the identity $\binom{n}{2} = nkw + tw + \varepsilon$ in the form $\frac{n-(2kw+1)}{n-(n-t-1)} = \frac{2w}{n} - \frac{2(w-\varepsilon)}{n(t+1)}$. Since the righthand side of this equation is strictly less than 1, we deduce that $n-t-1 < 2kw+1$, and hence $r \leq 2kw+1$. Finally to see that $\sum_{i=1}^{r} b_i \geq \binom{r}{2}$ for $r \geq n-t$ one may argue as follows. We observe that $\sum_{i=1}^{r} b_i$ is obtained from $\binom{n}{2}$ by subtracting the constant quantity $(k+1)w$ from it $(n-r)$ times. On the other hand $\binom{r}{2}$ is obtained from $\binom{n}{2}$ by subtracting successively $n-1, n-2, \ldots, r$. We note that $n-1 > (k+1)w$ (since $\binom{n}{2} > n(k+1)w$ by the definition of k). Suppose one could find an integer R such that $\sum_{i=1}^{R} b_i < \binom{R}{2}$ and $n > R \geq n-t$. It would follow from our observation that for the maximal such R we had $R < (k+1)w$ and hence that for all r in the range $n-t \leq r \leq R$ we had $r < (k+1)w$ and hence $\sum_{i=1}^{r} b_i < \binom{r}{2}$. However we have shown previously that $\sum_{i=1}^{n-t} b_i \geq \binom{n-t}{2}$, hence R cannot exist.

Remark: It is necessary that our pattern of results has taken some care over the distribution of the ε "spare" wins. For example, when $n = 23$ and $w = 10$, there are $\binom{23}{2} = 253$ matches. After distributing blocks of 10 matches as equally as possible one would have 2 teams with two such blocks and 21 teams with one such block. The 3 spare wins cannot be assigned to one of the teams who already have 20 wins, since no results can achieve this pattern of wins.

§4 Comments

The problem considered is a special case of a more general problem, which is stated in [2]. Let n teams each with P players contest a round robin tournament, where in the match between two teams each team is represented by p players and each player representing one team plays each player representing the other team. The problem is to determine the maximum number of players who play a specified number (or more) of matches and achieve at least a specified proportion of wins in their games. The above deals with the case when $p = 1$ and P is sufficiently large. In [2] the case $P = p$ was mainly discussed.

References

1. Victor Bryant, Aspects of Combinatorics (Cambridge University Press, Cambridge.
2. M.H.Eggar, A Tournament Problem, to appear in Discrete Mathematics.