On the nonexistence of Steiner t-(v,k) trades

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Abstract

We establish the nonexistence of: (i) Steiner t-(v,k) trades of volume s, for $2^t + 2^{t-1} < s < 2^t + 2^{t-1} + 2^{t-2}$; (ii) Steiner 4-$(v,k)$ trades of volume $s = 29$; (iii) Steiner t-(v,k) trades with $k > t+1$ and volume $s < (t-1)2^t + 2$.

1. Introduction

Let $0 < t < k < v$ be three positive integer and let X be a v-set. For every i, $0 \leq i \leq v$, the set of all i-subsets of X will be denoted by $P_i(X)$, and also for any $i > 1$, we will denote the set $\{x_1, x_2, \cdots, x_i\}$ by $x_1 x_2 \cdots x_i$. The elements of $P_k(X)$ are called *blocks*.

A t-(v,k) *trade* $T = \{T_1, T_2\}$ consists of two disjoint collections of blocks, T_1 and T_2, such that every element of $P_t(X)$ is contained in the same number of blocks in T_1 and T_2. For simplicity, the term t-trade is commonly used for this combinatorial object.

Let $T = \{T_1, T_2\}$ be a t-trade . Clearly, $|T_1| = |T_2|$ and $|T_1|$ is called the ***volume*** of T and is denoted by ***vol***(T) and we denote vol(T) by s. The subset of X which is covered by T_1 (and T_2) is called the *foundation* of T and is denoted by $found(T)$.

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Repeated blocks in $T_1(T_2)$ are allowed. A trade with no repeated block is called *simple*.

A t-(v,k) trade is called Steiner t-(v,k) trade if every element of $P_t(X)$ appears in at most one block of $T_1(T_2)$.

It has been shown in [4,1] that in every t-(v,k) trade, $|\text{found}(T)| \geq k + t + 1$, and $\text{vol}(T) \geq 2^t$.

A t-trade T with $\text{vol}(T) = 2^t$ and $|\text{found}(T)| = k+t+1$ is called *minimal.* A minimal t-(v,k) trade is unique, up to an isomorphism, and can be cast in the following form

$$T = (x_1 - x_2)(x_3 - x_4)\cdots(x_{2t+1} - x_{2t+2})x_{2t+3}\cdots x_{k+t+1},$$

where $x_i \in \text{found}(T)$. After a formal multiplication, the terms with plus(minus) signs are to be considered as blocks of $T_1(T_2)$.

Let $T = \{T_1, T_2\}$ be a t-(v,k) trade with $\text{vol}(T) = s$ and $x, y \in \text{found}(T)$. Then the number of blocks in $T_1(T_2)$ which contain x is denoted by r_x, and the number of blocks containing $\{x,y\}$ (for $t \geq 2$) is denoted by λ_{xy}. The set of blocks in $T_1(T_2)$ which contains $x \in \text{found}(T)$ is denoted by $T_{1x}(T_{2x})$ and the set of remaining blocks by $T'_{1x}(T'_{2x})$.

It has been shown [4] that if $r_x < s$, then $T_x := \{T_{1x}, T_{2x}\}$, is a $(t-1)$-(v,k) trade with $\text{vol}(T_x) = r_x$, and furthermore, $T'_x := \{T'_{1x}, T'_{2x}\}$ is a $(t-1)$-$(v-1,k)$ trade with $\text{vol}(T_x) = s - r_x$. If we remove x from the blocks of T_x, then the result will be a $(t-1)$-$(v-1,k-1)$ trade which is called a *derived* trade of T and is denoted by

$$D_xT = \{(D_xT)_1, (D_xT)_2\}.$$

It is easy to show that if T is a Steiner trade, then its derived trade is also a Steiner trade.

If $T = \{T_1, T_2\}$ and $T^* = \{T_1^*, T_2^*\}$ are two t-(v,k) trades, then we define $T + T^* = \{T_1 \cup T_1^*, T_2 \cup T_2^*\}$ and $T - T^* = \{T_1 \cup T_2^*, T_1^* \cup T_2\}$. Note that the blocks which appear in both sides are omitted. It is easy to see that $T \pm T^*$ are also t-(v,k) trades.

Let T be a t-(v,k) trade and $T \neq T_x + T_y$(for $x, y \in \text{found}(T)$), then $T - (T_x + T_y)$ is a $(t-1)$-(v,k) trade with volume $s - (r_x + r_y) + 2\lambda_{xy}$[4].

Hwang [4] has shown that there is no t-(v,k) trade of volume $s = 5$, and as a generalization, she has also shown that t-trades with $s = 2^t + 1$ do not

exist. Malik [6], and Mahmoodian and Soltankhah [5] have shown that there does not exist any t-(v,k) trade of volume $2^t < s < 2^t + 2^{t-1}$. We will refer to this result as Theorem MMS. In [5], the following conjecture has been stated:
Conjecture: There does not exist any trade of volume s as long as

$$2^{t+1} - 2^{t-i} < s < 2^{t+1} - 2^{t-i-1},\ i = 0, 1, \cdots, t-1.$$

In fact Theorem MMS is the answer for the conjecture for $i = 0$. Gray and Ramsay [2], have shown that no 3-$(v,4)$ trade with $s = 13$ exists, and they also prove a generalization of this: For $t \geq 3$, t-$(v, t+1)$ trades of volume $s = 2^t + 2^{t-1} + 1$ do not exist. Of course this is a partial solution for $i = 1$ of the conjecture. Prior to this proof, in [3] it is shown that there does not exist any Steiner 3-$(v,4)$ trade with $s = 13$.

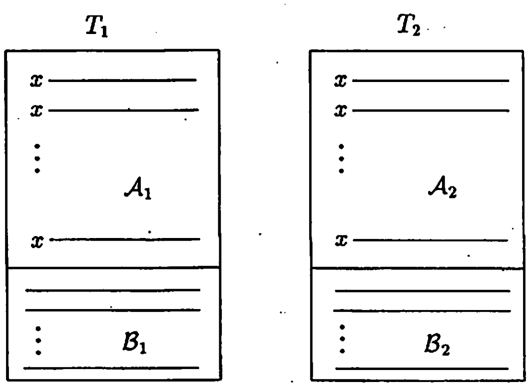
In this paper, for $i = 1$, we prove the correctness of the conjecture for Steiner trades.

2. Two useful lemmas

Lemma 1. In every Steiner t-trade $T = \{T_1, T_2\}$ with $k = t+1$ and volume s and for every $x \in$ found (T), we have

$$r_x \leq \frac{1}{2}s.$$

Proof. Based on the following figure, $\mathcal{A}_1(\mathcal{A}_2)$ consists of all blocks in $T_1(T_2)$ which contain x:



Every t-subset in $\mathcal{A}_1$ which does not contain x, appears in one of the blocks of $\mathcal{B}_2$, and every bock in $\mathcal{B}_2$ contains at most one t-subset (without x) from $\mathcal{A}_1$. For if every block of $\mathcal{B}_2$ contains at least two t-subsets of $\mathcal{A}_1$ say C_1 and C_2, then $C_1 \cup \{x\}$ and $C_2 \cup \{x\}$ would be two blocks of T_1 which have a t-subset in common, since C_1 and C_2 have a $(t-1)$-subset in common. This is contradictory with T being a Steiner trade. Therefore, $r_x \leq s - r_x$ and hence the result. □

Lemma 2. In a Steiner t-$(v, t+1)$ trade T with volume $s = 2^t + 2^{t-1}$, for every $x \in$ found (T), we have

$$r_x = 2^{t-1} \text{ or } 2^{t-1} + 2^{t-2}.$$

Proof. By Lemma 1, we have

$$r_x \leq \frac{1}{2}s = 2^{t-1} + 2^{t-2}.$$

Since T_x is a $(t-1)$ trade of volume r_x, therefore, $r_x \geq 2^{t-1}$ and by Theorem MMS,

$$2^{t-1} < r_x < 2^{t-1} + 2^{t-2},$$

do not hold. □

3. Main results

The main problem will be delt with in two parts: $k = t+1$ and $k > t+1$.

Theorem 3. For $t \geq 3$, there does not exist any Steiner t-$(v, t+1)$ trade with volume s as long as

$$2^t + 2^{t-1} < s < 2^t + 2^{t-1} + 2^{t-2}.$$

Proof. Induction on t. For $t = 3$, the statement leads to the nonexistence of Steiner 3-trade with $k = 4$ and volume $s = 13$. This was established in [2,3].

Suppose that the theorem is correct for values smaller than $t (t > 3)$ and we have to establish it for t. Suppose that the statement is not correct and there exists a Steiner t-$(v, t+1)$ trade T with volume $s = 2^t + 2^{t-1} + i$ where $0 < i < 2^{t-2}$, and $|\text{found }(T)| = f$. Then we derive a contradiction.

Consider $x \in$ found (T). Since D_xT is a Steiner $(t-1)$-trade with $k = t$ and volume $s = r_x$. Therefore, $r_x \geq 2^{t-1}$ and by Theorem MMS: $r_x = 2^{t-1}$ or $r_x \geq 2^{t-1} + 2^{t-2}$ and by Lemma 1, we have

$$r_x \leq \frac{s}{2} = 2^{t-1} + 2^{t-2} + \frac{i}{2} < 2^{t-1} + 2^{t-2} + 2^{t-3}.$$

But by induction assumption, the followings:

$$2^{t-1} + 2^{t-2} < r_x < 2^{t-1} + 2^{t-2} + 2^{t-3}$$

do not hold. Therefore, the only values remain to be checked are:

$$r_x = 2^{t-1} \quad \text{or} \quad r_x = 2^{t-1} + 2^{t-2}.$$

Case 1. There exists $x \in$ found (T) such that $r_x = 2^{t-1}$. Then

(a) Suppose that there exists $y \in$ found(T) such that $\lambda_{xy} = 0$. This leads to a contradiction. To see this, we look at $T - (T_x + T_y)$ which is a $(t-1)$-trade of volume

$$\begin{aligned} s' &= s - (r_x + r_y) = 2^t + 2^{t-1} + i - (2^{t-1} + r_y) \\ &= 2^t + i - r_y. \end{aligned}$$

Therefore, if $r_y = 2^{t-1}$, then $s' = 2^{t-1} + i$, which is $2^{t-1} < s' < 2^{t-1} + 2^{t-2}$. By, Theorem MMS, this is a contradiction. If $r_y = 2^{t-1} + 2^{t-2}$, then $s' = 2^{t-2} + i$, that is $0 < s' < 2^{t-1}$, and again we arrive at a contradiction.

(b) Suppose that for every $y \in$ found$(T), y \neq x$, we have $\lambda_{xy} \neq 0$. Then

$$|\text{found }(T_x)| = |\text{found}(T)| = f.$$

Now, since T_x is a $(t-1)$-trade with $k = t+1$ and volume $r_x = 2^{t-1}$ such that x appears in all its blocks, must be of the form

$$T_x = (y_1 - y_2) \cdots (y_{2t-1} - y_{2t})x.$$

We note that $f = 2t + 1$. This is a contradiction since in T we have

$$f \geq k + t + 1 = 2t + 2.$$

Case 2. For every $x \in \text{found}(T), r_x = 2^{t-1} + 2^{t-2}$. In this case again if there exists $y \in \text{found}(T), y \neq x$ such that $\lambda_{xy} = 0$, then the volume, s', of the $(t-1)$-trade $T - (T_x + T_y)$ is

$$s' = s - (r_x + r_y) = 2^t + 2^{t-1} + i - 2(2^{t-1} + 2^{t-2}) = i.$$

This is a contradiction since $0 < i < 2^{t-2}$. Therefore, for all $x, y \in \text{found}(T), \lambda_{xy} \neq 0$. By applying Lemma 2 to D_xT, we conclude that

$$\lambda_{xy} = 2^{t-2} \text{ or } 2^{t-2} + 2^{t-3}.$$

Now, if there exists a y such that $\lambda_{xy} = 2^{t-2}$, then

$$\begin{aligned} s' = \text{vol}(T - (T_x + T_y)) &= s - (r_x + r_y) + 2\lambda_{xy} \\ &= 2^t + 2^{t-1} + i - 2(2^{t-1} + 2^{t-2}) + 2 \times 2^{t-2} \\ &= 2^{t-1} + i. \end{aligned}$$

Therefore,

$$2^{t-1} < s' < 2^{t-1} + 2^{t-2},$$

and again by Theorem MMS, we have a contradiction.

Finally, the only remaining case is that for every $x, y \in \text{found}(T)$, we have

$$r_x = 2^{t-1} + 2^{t-2}, \quad \lambda_{xy} = 2^{t-2} + 2^{t-3}.$$

To reach a contradiction, by counting the pairs $(y, B), y \in B \in (D_xT)_1$ in two ways we obtain

$$\sum_{y \in \text{found}(D_xT)} \lambda_{xy} = t \text{ vol}(D_xT).$$

Since $|\text{found}(D_xT)| = f - 1$, therefore,

$$(f-1)(2^{t-2} + 2^{t-3}) = t(2^{t-1} + 2^{t-2}) \text{ or } f = 2t + 1,$$

and this is in contradiction with $f \geq k + t + 1 = 2t + 2$. □

Theorem 4. Let $T = \{T_1, T_2\}$ be a Steiner t-(v, k) trade with $k > t + 1$ and with volume s, then

$$s \geq (t-1)2^t + 2.$$

Note: As a result, it follows that for $t = 2$, $s \geq 6$ and for $t > 2$, we have $s \geq 2^{t+1} + 2$, and therefore the theorem establishes the truth of the Conjecture for any Steiner trade with $k > t + 1$.

Proof. Since $k > t + 1$ and T is a Steiner trade, therefore there exists a $(t + 2)$-subset $U \subseteq$ found (T) which is contained only in one of the blocks of T_1 and not in any block of T_2. Suppose that for $i = 0, \ldots, t + 2$, $a_i(b_i)$ is the number of blocks of $T_1(T_2)$ which has i intersections with U.

We note that the statement of the theorem can be deduced from

$$s = \sum_{i=0}^{t+2} a_i = \frac{1}{2}\sum_{i=0}^{t+2}(a_i + b_i) \geq \frac{1}{2}\sum_{i=0}^{t+2}|a_i - b_i|, \tag{1}$$

and we show that the sum on the right hand side of (1) can be expressed only just in terms of b_{t+1}. Since T is a Steiner trade and based on choosing U, we have

$$a_{t+2} = 1, \ b_{t+2} = 0, \ a_{t+1} = 0, \ b_{t+1} \in \{0, 1\}. \tag{2}$$

Now suppose that for $j = 0, \ldots, t$,

$$P_j(U) = \{U_1^{(j)}, \ldots, U_{\binom{t+2}{j}}^{(j)}\}$$

be the set of all j-subsets U and

$$U_1^{(j)}, \ldots, U_{\binom{t+2}{j}}^{(j)}$$

appear.

$$\lambda_1^{(j)}, \ldots, \lambda_{\binom{t+2}{j}}^{(j)}$$

times in the blocks of $T_1(T_2)$, respectively. Now by counting in two ways the number of pairs

$$(U_i^{(j)}, B), \ U_i^{(j)} \subseteq B \in T_1, \ i = 1, \ldots, \binom{t+2}{j},$$

(also for $B \in T_2$) and considering the property of balancedness, we obtain

$$\sum_{i=j}^{t+2} a_i \binom{i}{j} = \sum_{i=j}^{t+2} b_i \binom{i}{j} = \sum_{i=1}^{\binom{t+2}{j}} \lambda_i^{(j)}, \quad j = 0, \ldots, t.$$

Therefore,

$$\sum_{i=j}^{t+2} \binom{i}{j} (a_i - b_i) = 0, \quad j = 0, \dots, t. \tag{3}$$

To solve this system, we consider the following polynomial:

$$\begin{aligned} f(z) :&= \sum_{i=0}^{t+2} (a_i - b_i) z^i = \sum_{i=0}^{t+2} (a_i - b_i)((z-1)+1)^i \\ &= \sum_{i=0}^{t+2} (a_i - b_i) \sum_{j=0}^{i} \binom{i}{j} (z-1)^j \\ &= \sum_{j=0}^{t+2} \left[\sum_{i=j}^{t+2} \binom{i}{j} (a_i - b_i) \right] (z-1)^j. \end{aligned}$$

But by (3), the expression within the bracket, for $j = 0, \dots, t$, is equal to zero. Therefore

$$f(z) = \sum_{j=t+1}^{t+2} \left[\sum_{i=j}^{t+2} \binom{i}{j} (a_i - b_i) \right] (z-1)^j.$$

We note that on the right hand side of this equality, only the coefficients of $a_{t+1}, a_{t+2}, b_{t+1}, b_{t+2}$ appear. By $a_{t+1} = b_{t+2} = 0$ and $a_{t+2} = 1$, we obtain

$$f(z) = (z-1)^{t+2} + (t + 2 - b_{t+1})(z-1)^{t+1}.$$

Two cases have to be dealt with:

(i) $b_{t+1} = 0$:

$$\begin{aligned} \sum_{i=0}^{t+2} (a_i - b_i) z^i &= (z-1)^{t+2} + (t+2)(z-1)^{t+1} \\ &= z^{t+2} + \sum_{i=0}^{t+1} (-1)^{t+1-i} \left((t+2)\binom{t+1}{i} - \binom{t+2}{i} \right) z^i. \end{aligned}$$

From this it follows that

$$a_i - b_i = (-1)^{t+1-i} \left((t+2)\binom{t+1}{i} - \binom{t+2}{i} \right), \quad 0 \le i \le t+1.$$

Since for $0 \le i \le t+1$, we have

$$(t+2)\binom{t+1}{i} - \binom{t+2}{i} \ge 0,$$

therefore

$$\begin{aligned} s &\geq \frac{1}{2}\sum_{i=0}^{t+2}|a_i - b_i| = \frac{1}{2}\left\{1 + \sum_{i=0}^{t+1}\left[(t+2)\binom{t+1}{i} - \binom{t+2}{i}\right]\right\} \\ &= \frac{1}{2}\left\{1 + (t+2)2^{t+1} - (2^{t+2}-1)\right\} = t \times 2^t + 1. \end{aligned}$$

(ii) $b_{t+1} = 1$:

$$\begin{aligned} \sum_{i=0}^{t+2}(a_i - b_i)z^i &= (z-1)^{t+2} + (t+1)(z-1)^{t+1} \\ &= z^{t+2} - z^{t+1} + \sum_{i=0}^{t}(-1)^{t+1-i}\left[(t+1)\binom{t+1}{i} - \binom{t+2}{i}\right]z^i, \end{aligned}$$

and since for $0 \leq i \leq t$, we have

$$(t+1)\binom{t+1}{i} - \binom{t+2}{i} > 0,$$

therefore

$$\begin{aligned} s \geq \frac{1}{2}\sum_{i=0}^{t+2}|a_i - b_i| &= \frac{1}{2}\left\{1 + 1 + \sum_{i=0}^{t}\left[(t+1)\binom{t+1}{i} - \binom{t+2}{i}\right]\right\} \\ &= \frac{1}{2}\left\{2 + (t+1)(2^{t+1}-1) - \left(2^{t+2} - \binom{t+2}{t+1} - 1\right)\right\} \\ &= (t-1)2^t + 2. \end{aligned}$$

Since these are the only cases which occur for $f(z)$ and consequently for $\sum_i |a_i - b_i|$, therefore,

$$s \geq \min\{t2^t + 1,\ (t-1)2^t + 2\} = (t-1)2^t + 2. \qquad \square$$

Theorem 5. There does not exist any 4-(v,k) trade with volume $s = 29$.

[Note that $s = 29$ belongs to the interval $(2^{t+1} - 2^{t-2},\ 2^{t+1} - 2^{t-3})$ and $t = 4$, which is smallest.]

Proof. It suffices to prove the theorem for $k = t + 1 = 5$, since by Theorem 4, for $k \geq 5$ we have $s \geq 50$. By Lemma 1, for every $x \in \text{found}(T), r_x \leq \frac{s}{2}$, that is $r_x \leq 14$. On the other hand, D_xT is a Steiner 3-trade with $k = 4$ and volume r_x. Therefore $r_x \geq 2^3 = 8$ and $r_x \neq 13$ and by Theorem MMS, $r_x \neq 9, 10, 11$. Therefore, the possible values for r_x are

$$r_x = 8, 12, 14.$$

Now from

$$\sum_{x \in \text{found}(T)} r_x = ks = 5 \times 29,$$

we end up with a contradiction, since the sum on the left hand side is an even number. □

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