Degree Sequences of Halin Graphs, and Forcibly Cograph-graphic Sequences

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Abstract

A sequence $\pi = (d_1, \ldots, d_n)$ of nonnegative integers is *graphic* if there exists a graph G with n vertices for which $d_1, \ldots, d_n$ are the degrees of its vertices. G is referred to as a *realization* of π. Let P be a graph property. A graphic sequence π is *potentially P-graphic* if there exists a realization of π with the graph property P. Similarly, π is *forcibly P-graphic* if all realizations of π have the property P. We characterize potentially Halin graph-graphic sequences, forcibly Halin graph-graphic sequences, and forcibly cograph-graphic sequences.

1 Introduction

We consider finite simple graphs only. A sequence $\pi = (d_1, \ldots, d_n)$ of nonnegative integers is *graphic* if there exists a graph G with n vertices for which $d_1, \ldots, d_n$ are the degrees of its vertices. G is referred to as a *realization* of π and π is the *degree sequence* of G.

A graphic sequence π is *potentially P-graphic* if there exists a realization of π with the graph property P. Similarly, π is *forcibly P-graphic* if all realizations of π have the property P.

For several graph properties the forcibly P-graphic or potentially P-graphic sequences are well known. From the definitions of forcibly and potentially P-graphic sequences arises the following question:

For which graph property P is the number of forcibly P-graphic sequences finite?

We show that there are only three forcibly Halin graph-graphic sequences. Halin graph is one of the first graph property P with a finite number of forcibly P-graphic sequences.

A graph is a *cograph* if it has no induced subgraph P_4. Threshold graphs and trivially perfect graphs are subclasses of cographs and their forcibly P-graphic sequences are well known [5], [1]. We characterize forcibly cograph-graphic sequences in a very elementary way. S.R. Rao proposed a general approach to the description of forcibly hereditary P-graphic sequences [7]. S.R. Rao's criterion in case of cograph requires $O(n^7)$ operations. Our characterization requires $O(n)$ operations and it gives complete information about the structure of forcibly cograph-graphic sequences.

In general, a graphic sequence π has several realizations. We *switch* two edges of a realization of π, then we get another realization of π. Formally, assume that a, b, c, d are vertices of a graph $G = (V, E)$ such that $ab, cd \in E(G)$ and $ac, bd \notin E(G)$. The *switching* is the replacement of the edges ab and cd by the edges ac, bd. The resulting graph is $G' = G - ab - cd + ac + ad$. Clearly, switchings do not change degree of any vertex.

2 Degree Sequences of Halin Graphs

A *Halin graph* is a plane graph $H = T \cup C$, where T is a plane tree with no vertex of degree two and at least one vertex of degree three or more, and C is a cycle connecting the endvertices of T in the cyclic order determined by the embedding of T.

Lemma 1 ([6]) *A sequence $\pi = (d_1, \ldots, d_n)$ is potentially tree-graphic if and only if every $d_i > 0$ and $\sum_{i=1}^{n} d_i = 2(n-1)$.*

Theorem 1 *A sequence $\pi = (d_1, \ldots, d_n)$, $d_1 \leq \cdots \leq d_n$ is potentially Halin graph-graphic if and only if the following assertions hold:*
Let l be the number of elements $d_i = 3$ and $h := \frac{1}{2}\sum_{i=1}^{n} d_i - n + 1$.

(i) *$d_1 = 3$ and $l \geq 3$,*

(ii) *h is an integer with $3 \leq h \leq l$.*

Proof: If π is potentially Halin graph-graphic, then π has a realization $H = T \cup C$, which is a Halin graph. Let k be the number of endvertices of T. By the definition of Halin graph it is easy to see that $d_1 = 3$, $3 \leq k \leq l$ and $\sum_{i=1}^{n} d_i = 2n - 2 + 2k$. Hence $3 \leq h = k \leq l$.

Conversely, we assume that $d_1 = 3$, $l \geq 3$ and h is an integer with $3 \leq h \leq l$. We look at the sequence $\pi' = (1, \ldots, 1, d_{h+1}, \ldots, d_n)$. The sum of the elements of π' is equal to $2n-2$. By Lemma 1 π' is potentially tree-graphic. Let T' be a tree realization of π'. T' does not have a vertex with degree two and T' has at least one vertex of degree three or more. Now we add a cycle C, which connects the endvertices of T' in the cyclic order determined by the embedding of T'. $G = T' \cup C$ is a Halin graph and π is the degree sequence of G. Therefore π is potentially Halin graph-graphic. □

Lemma 2 ([4]) *Halin graphs are 3-connected graphs.*

Lemma 3 ([2]) *A graphic sequence $\pi = (d_1, \ldots, d_n)$, $d_1 \geq \cdots \geq d_n \geq 1$, $n \geq 3$ is forcibly tree-graphic if and only if $d_3 = 1$ and $d_1 + d_2 = n$.*

Theorem 2 *A graphic sequence π is forcibly Halin graph-graphic if and only if $\pi = (3,3,3,3)$, $\pi = (4,3,3,3,3)$ or $\pi = (5,3,3,3,3,3)$.*

Proof: Let $H = T \cup C$ be a Halin graph realization of π. Without loss of generality we may assume that T has at least four endvertices: otherwise $\pi = (3,3,3,3)$. In view of Lemma 2, our aim is to find a realization of π, which is not 3-connected. By Lemma 2 such a realization is not a Halin graph. We look at the degree sequence π_T of the tree T. Let G be a realization of π_T with endvertices $v_1, \ldots, v_p$ ($d_{v_j} = 1$). It is easy to see that we can always find a cycle $C' = (v_{i_1}, \ldots, v_{i_p})$ such that $G \cup C'$ is a realization of π.

We first consider the case that π_T is not forcibly tree-graphic. Then π_T has a realization G, which is disconnected. Let $S_1, \ldots, S_r$, mK_2 be the components of G, where S_i has at least three vertices and m components are K_2. Some S_i must contain a cycle, as G cannot be a forest. Each component has at least one endvertex: otherwise π has a realization, which is disconnected. Let $v_1, \ldots, v_k$ be endvertices of $S_1, \ldots, S_r$, first the endvertices of S_1, then endvertices of S_2 and so on. Let a_ib_i be the edges of K_2, for $i = 1, \ldots, m$.

If $m \geq 2$, we add the cycle $C = (v_1, \ldots, v_k, a_1, \ldots, a_m, b_1, \ldots, b_m)$. $G \cup C$ has the degree sequence π. $G \cup C$ is not a Halin graph, since $G \cup C - \{v_1, v_k\}$ is disconnected.

If $m = 0$, then necessarily $r \geq 2$. We add the cycle $C = (v_1, \ldots, v_k)$, then $G \cup C - \{v_1, v_h\}$ is disconnected, where v_1 is the first and v_h is the last endvertex of S_1.

If $m = 1$ and G has at least two S_i components, then we add the cycle $C = (v_1, \ldots, v_h, a_1, v_{h+1}, \ldots, v_k, b_1)$. $G \cup C - \{a_1, b_1\}$ is disconnected. If $r = 1$, then S_1 has only one cycle Z: otherwise π_T has another realization, which is connected and not a tree, a contradiction. Let vertex v be on the cycle Z. If it exists a path $P = (v, x, y)$ such that x and y are not on the cycle Z, then we switch a_1b_1 and vx and we get a new graph G'. For G' it holds $m = 0$. If no such a path $P = (x, y, z)$ does exist, then each vertex is either on the cycle Z or endvertex, and has a neighbor on the cycle Z. By definition of Halin graph each vertex on the cycle Z has at least one endvertex as a neighbor. If S_1 has more than four endvertices, then it is easy to find two cycles $C_1 = (v_1, a_1, v_2, b_1)$ and $C_2 = (v_2, \ldots, v_k)$ such that $G \cup C_1 \cup C_2$ is not 3-connected. If S_1 has at most four endvertices, then it exists only three degree sequences $(3,3,3,3,3,3,3,3)$, $(4,3,3,3,3,3,3,3,3)$ and $(3,3,3,3,3,3,3,3,3,3)$. Each of these sequences has a realization, which is disconnected and has K_4 as a component.

It remains to consider the case that π_T is forcibly tree-graphic. By Lemma 3 the tree T with n vertices has at least $n-2$ endvertices. If T has more then five endvertices, then we can add two disjoint cycles through these endvertices in such a way that the arising graph is not 3-connected. By Lemma 3 and the definition of a Halin graph, we get only five forcibly tree-graphic degree sequences with at most five endvertices: $\pi_{T_1} = (3,1,1,1)$, $\pi_{T_2} = (4,1,1,1,1)$, $\pi_{T_3} = (5,1,1,1,1,1)$, $\pi_{T_4} = (3,3,1,1,1,1)$, $\pi_{T_5} = (4,3,1,1,1,1,1)$. We put a cycle on the endvertices of T_i, then we get the degree sequences $\pi_1 = (3,3,3,3)$, $\pi_2 = (4,3,3,3,3)$, $\pi_3 = (5,3,3,3,3,3)$, $\pi_4 = (3,3,3,3,3,3)$, $\pi_5 = (4,3,3,3,3,3,3)$.

It is easy to see that π_1, π_2 and π_3 have unique realizations and that they are Halin graphs. π_4 has the realization $K_{3,3}$ and π_5 a non-planar realization too (see Figure 1). □

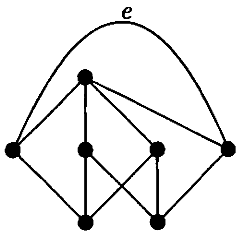

Figure 1: A non-planar graph with degree sequence $\pi = (4,3,3,3,3,3,3)$. By the contraction of edge e, $K_{3,3}$ is minor of the graph.

3 Rao's Method for a Hereditary Property

A graph property P is *hereditary* if a graph G and each induced subgraph of G have the property P. For example, cograph, planarity. S.R. Rao proposed the following general approach to characterize forcibly hereditary P-graphic sequences [7]:

> On the set of all graphic sequences the partial order $\ll$ is defined as follows: $\pi_1 \ll \pi_2$ if π_2 has a realization G and π_1 has a realization H such that H is an induced subgraph of G.
>
> Let $\mathcal{G}(P)$ be the set of graphs with property P. Then there exists the set $\mathcal{F}_0(P)$ of minimal forbidden subgraphs such that $G \in \mathcal{G}(P)$ if and only if none of the induced subgraphs of G belongs to $\mathcal{F}_0(P)$. Let $A_0(P)$ be the set of graphic sequences of elements of $\mathcal{F}_0(P)$. Let $M(P)$ be the set of minimal elements with respect to $\ll$ in $A_0(P)$.
>
> A graphic sequence $\pi = (d_1, \dots, d_n)$, $d_1 \geq \cdots \geq d_n$ is forcibly P-graphic if and only if $M(P)$ contains no elements such that $\pi' \ll \pi$.

Lemma 4 *If the property P is cograph, then $M(P) = \{(2,2,1,1)\}$.*

Proof: $\pi^* = (2,2,1,1)$ has a unique realization, the path P_4. Hence $\pi^* \in A_0(P)$. If $\pi \in A_0(P)$, then π has a realization G, which is not cograph. P_4 is an induced subgraph of G. Then $\pi^* \ll \pi$. Therefore π^* is the unique minimal element of $A_0(P)$ and $M(P) = \{(2,2,1,1)\}$. □

If $M(P)$ is known and no additional considerations are used, then time $c(\binom{n}{k} n(n-k)^2k^2 + n)$ is necessary for the verification of S.R. Rao's criterion [1]. k is the maximum length of sequences from $M(P)$, n is the length of the tested sequence and c is a constant. By Lemma 4 the direct use of S.R. Rao's criterion on a cograph requires cn^7 operations. In the next section we give a linear time characterization for forcibly cograph-graphic sequences.

4 Forcibly Cograph-graphic Sequences

We denote by G^c the *complement graph* of G. For a graphic sequence $\pi = (d_1, \ldots, d_n)$ we call $\pi^c = (n-1-d_1, \ldots, n-1-d_n)$ the *complement sequence* of π. It is easy to see that G is a realization of π if and only if G^c is a realization of π^c.

Lemma 5 ([3]) *G is a connected cograph if and only if G^c is a disconnected cograph.*

Theorem 3 *A graphic sequence $\pi = (d_1, \ldots, d_n)$, $d_1 \geq \cdots \geq d_n$, $n \geq 2$ is forcibly cograph-graphic if and only if one of the following assertions holds:*

(i) *$d_1 = n - 1$ and $\pi_1 = (d_2 - 1, \ldots, d_n - 1)$ is forcibly cograph-graphic;*

(ii) *$d_n = 0$ and $\pi_n = (d_1, \ldots, d_{n-1})$ is forcibly cograph-graphic;*

(iii) *$d_1 < n - 1$, $1 \leq d_n$ and π or π^c is equal to $(k, 1, \ldots, 1)$.*

Proof: Assertions (i) and (ii) are trivial. We prove the assertion (iii). Let $\pi = (d_1, \ldots, d_n)$ be a forcibly cograph-graphic sequence with $d_1 < n-1$ and $1 \leq d_n$. We first consider the case that π has a realization G which is disconnected. G is a cograph and it has at least two components. Each component of G has an edge, since $d_n \geq 1$. Now we show that at most one component is a $K_{1,k}$ and all other components are K_2. We assume that one of the components has a cycle. We switch an edge of a shortest cycle with an edge of another component, then we get a realization of π with an induced subgraph P_4. This is a contradiction. If two components have an induced subgraph P_3, then we again get, by switching, an induced P_4. Let S be the unique component of G with an induced P_3. S is a tree and P_3 is a longest path of S. Hence S is a $K_{1,k}$. It follows that $\pi = (k, 1, \ldots, 1)$. It remains to consider the case that π has a realization H which is connected. By Lemma 5 H^c is disconnected and cograph. Therefore $\pi^c = (k, 1, \ldots, 1)$.

Conversely, it is easy to see that the sequence $\pi = (k, 1, \ldots, 1)$ with n elements is graphic if and only if $1 \leq k \leq n-1$ and $n - k - 1$ is even. π and π^c have unique realizations G and G^c, respectively. A unique component of G is $K_{1,k}$, the remaining components of G are all K_2. G is a cograph. Therefore π and π^c are forcibly cograph-graphic. □

Corollary 1 *The forcibly cograph-graphic sequence $\pi = (d_1, \ldots, d_n)$, $d_1 \geq \cdots \geq d_n$ can be recognized in $O(n)$ elementary operations.*

Corollary 2 *If π is forcibly cograph-graphic, then π has a unique realization.*

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