The spectrum for self-converse directed BIBDs with block size four

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Abstract

A directed balanced incomplete block design ($DB(k,\lambda;v)$) $(X,\mathcal{B})$ is called self-converse if there is an isomorphic mapping f from $(X,\mathcal{B})$ to $(X,\mathcal{B}^{-1})$, where $\mathcal{B}^{-1}=\{B^{-1}:B\in\mathcal{B}\}$ and $B^{-1}=(x_k,x_{k-1},\cdots,x_2,x_1)$ for $B=(x_1,x_2,\cdots,x_{k-1},x_k)$. In this paper, we give the existence spectrum for self-converse $DB(4,\lambda;v)$ for any $\lambda\geq 1$.

Key Words: self-converse, self-converse directed balanced incomplete block design, group divisible design

AMS Classification: 05B

1 Introduction

Let v, k and λ be positive integers. A *transitive ordered k-tuple* $(a_1,a_2,\cdots,a_k)$ is defined to be the set $\{(a_i,a_j): 1\leq i<j\leq k\}$ consisting of $\binom{k}{2}$ ordered pairs. *A directed balanced incomplete block design* (directed BIBD), briefly $DB(k,\lambda;v)$, is a pair $(X,\mathcal{B})$, where X is a v-set of points and $\mathcal{B}$ is a collection of transitive ordered k-tuples of X (called *blocks*) such that every ordered pair of distinct points of X occurs in exactly λ blocks of $\mathcal{B}$. It is noted that a $DB(k,\lambda;v)$ becomes a balanced incomplete block design $B(k,2\lambda;v)$ (or $(v,k,2\lambda)$-BIBD) if the order of the blocks is ignored. The necessary conditions

for the existence of a $DB(k, \lambda; v)$ are

$$\begin{cases} 2\lambda(v-1) \equiv 0 \pmod{k-1}, \\ \lambda v(v-1) \equiv 0 \pmod{\binom{k}{2}}. \end{cases}$$

It has been shown in [7] that the necessary condition for the existence of a $DB(4, \lambda; v)$ is that $v \equiv 1 \pmod 3$ and $v \geq 4$ if $\lambda \equiv 1, 2 \pmod 3$; and any v if $\lambda \equiv 0 \pmod 3$.

Every transitive ordered k-tuple $B = (a_1, a_2, \cdots, a_k)$ has a converse $B^{-1} = (a_k, a_{k-1}, \cdots, a_1)$. So given a $DB(k, \lambda; v)$ $(X, \mathcal{B})$, one can define $\mathcal{B}^{-1} = \{B^{-1} : B \in \mathcal{B}\}$. Obviously, $(X, \mathcal{B}^{-1})$ is also a $DB(k, \lambda; v)$, which is called *the converse* of $(X, \mathcal{B})$. If there exists a permutation f on X such that $\mathcal{B}^{-1} = \{f(B) : B \in \mathcal{B}\}$, where $f(B) = (f(a_1), f(a_2), \cdots, f(a_k))$ for $B = (a_1, a_2, \cdots, a_k)$, then we say that $(X, \mathcal{B})$ and $(X, \mathcal{B}^{-1})$ are *isomorphic*. If such an isomorphism f exists, then the $DB(k, \lambda; v)$ $(X, \mathcal{B})$ is called *self-converse* and denoted by $SCDB(k, \lambda; v)$ or $(X, \mathcal{B}, f)$.

It is well known that a $DB(3, 1; v)$ exists if and only if $v \equiv 0, 1 \pmod 3$ (see [4]). In [2], it was put forward as a open problem by Colbourn and Rosa that for what orders an $SCDB(3, 1; v)$ exists. Kang, Chang and Yang gave a complete answer and proved that an $SCDB(3, 1; v)$ exists if and only if $v \equiv 0, 1 \pmod 3$ and $v \neq 6$ (see [5]). Recently, Yin gave a short new proof for Colbourn and Rosa problem (see [9]) and the authers showed that the existence spectrum of an $SCDB(4, 1; v)$ is $v \equiv 1 \pmod 3$ and $v \neq 7$ in [8]. In this paper we will establish the existence spectrum of an $SCDB(4, \lambda; v)$ for any integer $\lambda \geq 2$.

2 Preliminaries

In order to establish our construction, we need the following auxiliary designs.

A GDD with block size k and index λ, a positive integer, denoted by (k, λ)-GDD, is a triple $(X, \mathcal{G}, \mathcal{A})$ which satisfies the following properties:

1. $\mathcal{G}$ is a partition of a set X (of points) into subsets called *groups*,
2. $\mathcal{A}$ is a collection of k-subsets of X (called *blocks*) such that a group and a block contain at most one common point,
3. every pair of points from distinct groups occurs in exactly λ blocks of $\mathcal{A}$.

The *group-type* (or *type*) of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We often use an "exponential" notation to describe the type: a type $1^i 2^j \cdots$ denotes i occurrences of 1, j occurrences of 2, etc..

We need self-converse directed BIBDs with one hole. In what follows, the notation $IDB(k, \lambda; v, \omega)$ stands for a triple $(X, Y, \mathcal{A})$ where X is a v-set (of *points*) and Y is a ω-set, $Y \subseteq X$ and $\mathcal{A}$ is a collection of transitive ordered k-tuples (called *blocks*) of X such that every ordered pair of distinct points $(x, y) \in (X \times X) \backslash (Y \times Y)$ occurs in exactly λ blocks of $\mathcal{A}$ while any ordered pair of points in Y does not occur in any block, and hence Y is a *hole*. An $IDB(k, \lambda; v, \omega)$ $(X, Y, \mathcal{A})$ is called *self-converse*, denoted by $ISCDB(k, \lambda; v, \omega)$ (or $(X, Y, \mathcal{A}, f)$) if there exists an isomorphism f from $(X, Y, \mathcal{A})$ to its converse $(X, Y, \mathcal{A}^{-1})$ where $f(Y) = Y$.

Lemma 2.1 ([1]) *Suppose that t and u are positive integers. Then there exists a $(4, \lambda)$-GDD of type t^u if and only if the following conditions are all satisfied and $(t, u) \neq (2, 4), (6, 4)$.*

(1) $\lambda t(u-1) \equiv 0 \pmod 3$,

(2) $\lambda t^2 u(u-1) \equiv 0 \pmod{12}$,

(3) $u \geq 4$ *or* $u = 1$.

Lemma 2.2 ([1]) *A $(4, 1)$-GDD of type $2^u 5^1$ exists if and only if $u \equiv 0 \pmod 3$ and $u \geq 9$.*

Lemma 2.3 ([6]) *There exists a $(4, 1)$-GDD of type $6^5 9^1$.*

The following construction is a modification of Construction 2.3 in [9] and the proof can be found in [8].

Construction 2.4 ([8]) *Let V be a v-set of points and W a ω-set of points with $V \cap W = \emptyset$. Let π be an arbitrary permutation on W and f_j a permutation on G_j whose order $p(f_j) \leq 2$ for $1 \leq j \leq t$. Suppose that the following designs exist:*

1. *a (k, λ)-GDD $(V, \mathcal{G}, \mathcal{B})$ with $\mathcal{G} = \{G_j : j = 1, 2, \cdots, t\}$,*
2. *an $ISCDB(k, \lambda; |G_j| + \omega, \omega)$ $(G_j \cup W, W, \mathcal{B}_j, \pi \circ f_j)$ for $1 \leq j \leq t - 1$.*

Then there exists an $ISCDB(k, \lambda; v + \omega, |G_t| + \omega)$ with isomorphism $f = \pi \circ f_t \circ \cdots \circ f_2 \circ f_1$. Furthermore, if there is an $SCDB(k, \lambda; |G_t| + \omega)$ $(G_t \cup W, \mathcal{B}_t, \pi \circ f_t)$, then there exists an $SCDB(k, \lambda; v + \omega)$ with isomorphism f.

Let $(X, \mathcal{B}, f)$ be an $SCDB(k, \lambda; v)$. For any $x \in X$, if $f^t(x) = x$ but $f^s(x) \neq x$ when $s < t$, then we denote $p_f(x) = t$. When $p_f(x) = 1$, we call x a *fixed point*. The following Lemma is simple but very useful.

Lemma 2.5 *If the permutation f has a fixed point, then the existence of an $SCDB(k, \lambda; v)$ with the isomorphic mapping f is equivalent to the existence of an $ISCDB(k, \lambda; v, 1)$.*

Theorem 2.6 *There exists an $SCDB(4, \lambda; v)$ for $v \equiv 1 \pmod 3$, $v \neq 7$ and any λ.*

Proof In [8], there exists an $SCDB(4, 1; v)$ if and only if $v \equiv 1 \pmod 3$ and $v \neq 7$. Then an $SCDB(4, \lambda; v)$ can be obtained by repeating every block of the $SCDB(4, 1; v)$ λ times. □

By Theorem 2.6 and the necessary condition for the existence of a $DB(4, \lambda; v)$, we only need consider the existence of an $SCDB(4, \lambda; 7)$ for $\lambda \geq 2$ and an $SCDB(4, \lambda; v)$ for $\lambda \equiv 0 \pmod 3$ and $v \equiv 0, 2 \pmod 3$.

3 The Existence of $SCDB(4,3;v)$'s

In this section we will show that an $SCDB(4,3;v)$ exists for $v \equiv 0,2 \pmod 3$, and hence an $SCDB(4,\lambda;v)$ with $\lambda \equiv 0 \pmod 3$ exists. For convenience we define $f(\mathcal{B}^{-1}) = \{f(B^{-1}) : B \in \mathcal{B}\}$. First, we need the following results as auxiliary design for utilizing Construction 2.4.

Lemma 3.1 *For each pair* $(v,\omega) \in \{(11,2),(11,3),(14,2),(15,3),(35,11)\}$, *there exists an* $ISCDB$ $(4,3;v,\omega)$ *with isomorphism* f *whose order* $p(f) = 2$.

Proof Suppose that $X = Z_{v-\omega} \cup Y$ and $Y = \{\infty_1,\cdots,\infty_\omega\}$. The desired $ISCDB$ $(4,3;v,\omega)$ is $(X,Y,\mathcal{A}\cup f(\mathcal{A}^{-1}),f)$, where the block set $\mathcal{A}$ and the isomorphism f are listed below.

Case $(v,\omega) = (11,2)$

$\mathcal{A}$: $(\infty_1, 0, 1, 4)$ $(+1, \bmod 9)$,
$(0, 1, 3, 5)$ $(+1, \bmod 9)$,
$(\infty_2, 0, 1, 3)$ $(+1, \bmod 9)$.

$f = (0)(1)\cdots(8)(\infty_1\ \infty_2)$.

Case $(v,\omega) = (11,3)$

$\mathcal{A}$: $(0, 2, 4, 6)$, $(1, 3, 5, 7)$,
$(\infty_1, 0, 1, 3)$ $(+1, \bmod 8)$,
$(\infty_2, 0, 1, 5)$ $(+1, \bmod 8)$,
$(\infty_3, 0, 1, 3)$ $(+1, \bmod 8)$.

$f = (0)(1)\cdots(7)(\infty_1\ \infty_2)(\infty_3)$.

Case $(v,\omega) = (14,2)$

$\mathcal{A}$: $(0, 3, 6, 9)$, $(1, 4, 7, 10)$, $(2, 5, 8, 11)$,
$(0, 1, 7, 6)$, $(1, 2, 8, 7)$, $(2, 3, 9, 8)$,
$(3, 4, 10, 9)$, $(4, 5, 11, 10)$, $(5, 6, 0, 11)$,
$(\infty_1, 2, 4, 6)$ $(+1, \bmod 12)$,
$(0, 1, 3, 8)$ $(+1, \bmod 12)$,
$(\infty_2, 1, 2, 10)$ $(+1, \bmod 12)$.

$f = (0)(1)\cdots(11)(\infty_1\ \infty_2)$.

Case $(v, \omega) = (15, 3)$

$\mathcal{A}$: $(0, 3, 6, 9)$, $(1, 4, 7, 10)$, $(2, 5, 8, 11)$,
$(\infty_1, 0, 5, 7)$ $(+1, \bmod 12)$,
$(0, 1, 4, 9)$ $(+1, \bmod 12)$,
$(\infty_2, 0, 1, 2)$ $(+1, \bmod 12)$,
$(\infty_3, 0, 6, 8)$ $(+1, \bmod 12)$.

$f = (0)(1) \cdots (11)(\infty_1\ \infty_2)(\infty_3)$.

Case $(v, \omega) = (35, 11)$

$\mathcal{A}$: $(0, 6, 12, 18)$, $(1, 7, 13, 19)$,
$(2, 8, 14, 20)$, $(3, 9, 15, 21)$,
$(4, 10, 16, 22)$, $(5, 11, 17, 23)$,
$(0, 2, 5, \infty_1)$ $(+1, \bmod 24)$,
$(0, 1, 3, \infty_2)$ $(+1, \bmod 24)$,
$(0, 3, 8, \infty_3)$ $(+1, \bmod 24)$,
$(0, 1, 2, \infty_4)$ $(+1, \bmod 24)$,
$(0, 7, 14, \infty_5)$ $(+1, \bmod 24)$,
$(0, 6, 14, \infty_6)$ $(+1, \bmod 24)$,
$(0, 4, 13, \infty_7)$ $(+1, \bmod 24)$,
$(0, 5, 14, \infty_8)$ $(+1, \bmod 24)$,
$(0, 4, 12, \infty_9)$ $(+1, \bmod 24)$,
$(0, 4, 13, \infty_{10})$ $(+1, \bmod 24)$,
$(0, 6, 13, \infty_{11})$ $(+1, \bmod 24)$.

$f = (0)(1) \cdots (23)(\infty_1\ \infty_2) \cdots (\infty_9 \infty_{10})(\infty_{11})$. □

Theorem 3.2 *An $SCDB(4, 3; v)$ exists for $v \equiv 0$ or $1 \pmod 4$.*

Proof For $v \equiv 0$ or $1 \pmod 4$, there is a $B(4, 3; v)$ $(X, \mathcal{B})$ (see [3]). The desired $SCDB(4, 3; v)$ is obtained by writing each block of $\mathcal{B}$ twice — once in some order and the other in the reverse order, in which the isomorphism f is an identical permutation on X. □

What remains is to deal with the cases where $v \equiv 2, 3, 6, 11 \pmod{12}$. We will give some results with small order which play important roles on constructing new $SCDB(4, 3; v)$s.

Lemma 3.3 *There is an $SCDB(4, 3; 6)$ with isomorphism f whose order $p(f) = 2$.*

Proof Suppose $X = I_6$ and the isomorphism $f = (0\ 1)(2\ 3)(4\ 5)$, the block set $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup f(\mathcal{B}_1^{-1})$ where the blocks of $\mathcal{B}_0$ and $\mathcal{B}_1$ are listed below:

$$\begin{array}{llll} \mathcal{B}_0: & (0,2,3,1), & (3,4,5,2), & (5,1,0,4); \\ \mathcal{B}_1: & (0,2,4,1), & (0,2,3,4), & (1,2,0,5), \\ & (1,2,4,5), & (3,2,0,5), & (1,3,5,4). \end{array}$$

It is readily checked that $(X, \mathcal{B}, f)$ is the desired $SCDB(4,3;6)$. □

Lemma 3.4 *There is an $SCDB(4,3;v)$ with isomorphism f whose order $p(f) = 2$ for $v = 14, 18, 26$.*

Proof The desired $SCDB(4,3;v)$ $(X, \mathcal{B}, f)$ can be constructed by taking the point set $X = Z_v$, the isomorphism f: $x \to x + \frac{v}{2}$ for $x \in Z_v$, and the block set $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup f({\mathcal{B}_1}^{-1})$ where all blocks of $\mathcal{B}_0$ and $\mathcal{B}_1$ are listed below.

Case $v = 14$

$\mathcal{B}_0$: (0, 1, 8, 7) (+2, mod 14);
$\mathcal{B}_1$: (0, 1, 2, 3) (+2, mod 14), (0, 5, 12, 8) (+2, mod 14),
(0, 3, 4, 8) (+2, mod 14), (1, 6, 12, 4) (+2, mod 14),
(0, 2, 4, 7) (+2, mod 14), (0, 5, 10, 6) (+2, mod 14).

Case $v = 18$

$\mathcal{B}_0$: (0, 1, 10, 9) (+2, mod 18);
$\mathcal{B}_1$: (0, 2, 4, 7) (+2, mod 18), (0, 5, 6, 11) (+2, mod 18),
(0, 3, 4, 8) (+2, mod 18), (0, 6, 14, 12) (+2, mod 18),
(5, 0, 2, 9) (+2, mod 18), (0, 16, 1, 12) (+2, mod 18),
(1, 2, 3, 10) (+2, mod 18), (11, 1, 4, 14) (+2, mod 18).

Case $v = 26$

$\mathcal{B}_0$: (0, 1, 14, 13) (+2, mod 26);
$\mathcal{B}_1$: (0, 5, 3, 16) (+2, mod 26), (0, 6, 12, 19) (+2, mod 26),
(1, 3, 0, 10) (+2, mod 26), (0, 8, 17, 20) (+2, mod 26),
(3, 1, 23, 2) (+2, mod 26), (0, 8, 20, 18) (+2, mod 26),
(0, 18, 14, 10) (+2, mod 26), (0, 9, 11, 14) (+2, mod 26),
(0, 11, 20, 21) (+2, mod 26), (1, 5, 20, 10) (+2, mod 26),
(21, 6, 14, 17) (+2, mod 26), (16,18,15,11) (+2, mod 26).

□

Lemma 3.5 *There is an $SCDB(4,3;v)$ with isomorphism f whose order $p(f) = 2$ for $v = 11, 15, 23, 27$. Moreover, there exists an $SCDB(4,3;v,1)$.*

Proof Suppose that $X = Z_{v-1} \cup \{\infty\}$, the isomorphism f: $x \to x + \frac{v-1}{2}$ where $x \in Z_{v-1}$ and ∞ is a fixed point. The block set $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup f(\mathcal{B}_1^{-1})$, and all blocks of $\mathcal{B}_0$ and $\mathcal{B}_1$ are listed below, respectively. So $(X, \mathcal{B}, f)$ is the desired $SCDB(4,3;v)$ and hence an $ISCDB(4,3;v,1)$ exists by Lemma 2.5.

Case $v = 11$

$\mathcal{B}_0$: (0, 1, 6, 5) (+2, mod 10);

$\mathcal{B}_1$: (0, 1, 2, 3) (+2, mod 10), (0, 4, 8, ∞) (+2, mod 10), (0, 2, 4, 7) (+2, mod 10), (1, 5, ∞, 2) (+2, mod 10), (1, 3, 0, 6) (+2, mod 10).

Case $v = 15$

$\mathcal{B}_0$: (0, 1, 8, 7) (+2, mod 14);

$\mathcal{B}_1$: (0, 1, 2, 3) (+2, mod 14), (1, 3, 10, 6) (+2, mod 14), (0, 3, 4, 8) (+2, mod 14), (0, 9, ∞, 6) (+2, mod 14), (0, 2, 4, 7) (+2, mod 14), (0, 10, ∞, 6) (+2, mod 14), (0, 5, 8, 6) (+2, mod 14).

Case $v = 23$

$\mathcal{B}_0$: (0, 1, 12, 11) (+2, mod 22);

$\mathcal{B}_1$: (0, 3, 4, 8) (+2, mod 22), (0, 6, 12, 20) (+2, mod 22), (0, 2, 4, 5) (+2, mod 22), (0, 7, 10, 18) (+2, mod 22), (0, 5, 6, 11) (+2, mod 22), (0, 9, 13, 16) (+2, mod 22), (0, 7, 9, 12) (+2, mod 22), (0, 3, 15, ∞) (+2, mod 22), (9, 0, 14, 2) (+2, mod 22), (1, 14, 10, ∞) (+2, mod 22), (19, 3, 4, 5) (+2, mod 22).

Case $v = 27$

$\mathcal{B}_0$: (0, 1, 14, 13) (+2, mod 26);

$\mathcal{B}_1$: (0, 2, 7, 6) (+2, mod 26), (0, 8, 16, ∞) (+2, mod 26),
(0, 7, 6, 18) (+2, mod 26), (0, 20, 16, 10) (+2, mod 26),
(0, 9, 11, 14) (+2, mod 26), (13, ∞, 22, 6) (+2, mod 26),
(0, 9, 11, 14) (+2, mod 26), (24, 21, 22, 1) (+2, mod 26),
(0, 9, 12, 22) (+2, mod 26), (23, 5, 16, 12) (+2, mod 26),
(0, 6, 19, 11) (+2, mod 26), (16, 17, 15, 13) (+2,mod 26),
(11, 2, 20, 6) (+2, mod 26).

□

Lemma 3.6 *For $v = 35, 59, 83$, there exists an $SCDB(4,3;v)$ with isomorphism f whose order $p(f) = 2$.*

Proof By Lemma 2.1 there exist (4, 3)-GDDs of types 8^4, 8^7 and 8^{10}. We can apply Construction 2.4 with an $ISCDB(4,3;11,3)$ and an $SCDB(4,3;11)$ from Lemma 3.1 and Lemma 3.5 to obtain an $SCDB(4,3;35)$, an $SCDB(4,3;59)$ and an $SCDB(4,3;83)$. □

Lemma 3.7 *There exists an $SCDB(4,3;v)$ for $v = 38, 47$.*

Proof By Lemma 2.1 there exist (4, 3)-GDDs of types 9^4 and 9^5. Applying Construction 2.4 with an $ISCDB(4,3;11,2)$ and an $SCDB(4,3;11)$ from Lemmas 3.1 and 3.5 gives an $SCDB(4,3;38)$ and an $SCDB(4,3;47)$. □

Lemma 3.8 *There exists an $SCDB(4,3;39)$.*

Proof By Lemma 2.3 there exists a (4, 1)-GDD of type $6^5 9^1$ which produces a (4, 3)-GDD with the same type. Start with an $SCDB(4,3;6)$ and an $SCDB(4,3;9)$ from Lemma 3.3 and Theorem 3.2, and apply Construction 2.4 with $\omega = 0$ to give the desired result. □

Lemma 3.9 *There exists an* $SCDB(4,3;71)$.

Proof By Lemma 2.1 there is a $(4,3)$-GDD of type 10^7. We can apply Construction 2.4 with an $ISCDB(4,3;11,1)$ and an $SCDB(4,3;11)$ from Lemma 3.5 to obtain an $SCDB(4,3;71)$. □

Theorem 3.10 *For* $v \equiv 2 \pmod{12}$ *and* $v \geq 14$, *there exists an* $SCDB(4,3;v)$.

Proof By Lemma 2.1, Lemma 3.1 and Lemma 3.4, there exist a $(4,3)$-GDD of type 12^n with $n \geq 4$, and an $ISCDB(4,3;14,2)$ and an $SCDB(4,3;14)$ with isomorphism f whose order $p(f) = 2$. Applying Construction 2.4 with $\omega = 2$ gives an $SCDB(4,3;v)$ for $v \equiv 2 \pmod{12}$ and $v \geq 50$. For $v = 26, 38$, an $SCDB(4,3;v)$ exists by Lemma 3.4 and Lemma 3.7. The conclusion then follows. □

Theorem 3.11 *For* $v \equiv 3 \pmod{12}$ *and* $v \geq 15$, *there exists an* $SCDB(4,3;v)$.

Proof By Lemma 3.5 and Lemma 3.8, an $SCDB(4,3;v)$ exists for $v = 15, 27, 39$. Starting with a $(4,3)$-GDD of type 12^n for $n \geq 4$ from Lemma 2.1 and an $ISCDB(4,3;15,3)$ from Lemma 3.1 and an $SCDB(4,3;15)$ from Lemma 3.5, we can apply Construction 2.4 with $\omega = 3$ to obtain an $SCDB(4,3;v)$ for $v \equiv 3 \pmod{12}$ and $v \geq 51$. □

Theorem 3.12 *For* $v \equiv 6 \pmod{12}$, *an* $SCDB(4,3;v)$ *exists.*

Proof By Lemma 3.3 and Lemma 3.4, there exists an $SCDB(4,3;v)$ when $v = 6, 18$. For $v \equiv 6 \pmod{12}$ and $v \geq 30$, let $v = 6(2n+1)$ where $n \geq 2$. Then there exists a $(4,3)$-GDD of type 6^{2n+1} by Lemma 2.1. Applying Construction 2.4 with $\omega = 0$ gives an $SCDB(4,3;12n+6)$, i.e., an $SCDB(4,3;v)$. This completes the proof. □

Theorem 3.13 *For $v \equiv 11 \pmod{12}$, an $SCDB(4,3;v)$ exists.*

Proof For $v \leq 83$, there exists an $SCDB(4,3;v)$ by Lemmas 3.5, 3.6, 3.7 and 3.9. For $v \geq 95$, we divide two cases as follows:

(1) For $v \equiv 11 \pmod{24}$ and $v \geq 107$, there exists a $(4,3)$-GDD of type 24^k for $k \geq 4$ by Lemma 2.1. Starting with an $ISCDB(4,3;35,11)$ and an $SCDB(4,3;35)$ from Lemma 3.1 and Lemma 3.6, we can apply Construction 2.4 with $\omega = 11$ to obtain the desired $SCDB(4,3;v)$.

(2) For $v \equiv 23 \pmod{24}$ and $v \geq 95$, we have a $(4,3)$-GDD of type $2^{3k}5^1$ for $k \geq 3$ by Lemma 2.2. Give each point of the $(4,3)$-GDD a weight of 4 and apply Wilson's Fundamental Construction to obtain a $(4,3)$-GDD of type $8^{3k}20^1$. The required input design is a $(4,3)$-GDD of type 4^4. There exist an $ISCDB(4,3;11,3)$ and an $SCDB(4,3;23)$ by Lemma 3.1 and Lemma 3.5. We can apply Construction 2.4 with $\omega = 3$ to obtain an $SCDB(4,3;v)$. □

4 Existence of $SCDB(4,\lambda;7)$'s for $\lambda \geq 2$

Lemma 4.1 *For $\lambda \equiv 0 \pmod{2}$, there exists an $SCDB(4,\lambda;7)$.*

Proof Firstly, let us give the existence of an $SCDB(4,2;7)$. Let $X = I_7$, the isomorphism $f = (0\ 1)(2\ 3)(4\ 5)(6)$ and the block set $\mathcal{B} = \mathcal{B}_0 \cup f(\mathcal{B}_0^{-1})$ where all 7 blocks of $\mathcal{B}_0$ are listed as follows:

(0, 2, 1, 3), (0, 2, 4, 5), (1, 0, 4, 6), (2, 5, 4, 0),
(1, 5, 2, 6), (3, 0, 5, 6), (3, 4, 2, 6).

It is readily checked that $(X, \mathcal{B}, f)$ is an $SCDB(4,2;7)$. For $\lambda \equiv 0 \pmod{2}$, the desired $SCDB(4,\lambda;7)$ can be obtained by repeating each block of the $SCDB(4,2;7)$ $\frac{\lambda}{2}$ times. □

Lemma 4.2 *For $\lambda \equiv 0 \pmod{3}$, there exists an $SCDB(4,\lambda;7)$.*

Proof An $SCDB(4,3;7)$ can be constructed by taking the following 21 blocks based on $X = I_7$ and the isomorphism $f = (01)(23)(45)(6)$. The first three blocks are

$$(1,2,3,0), \quad (3,4,5,2), \quad (5,0,1,4).$$

The remaining 18 blocks are the blocks B and $f(B^{-1})$, where B consists of the following blocks:

$$\begin{array}{llll} (0,\ 3,\ 5,\ 6), & (1,\ 3,\ 4,\ 6), & (1,\ 0,\ 5,\ 6), & (1,\ 2,\ 5,\ 4), \\ (0,\ 2,\ 1,\ 3), & (3,\ 4,\ 2,\ 6), & (0,\ 2,\ 4,\ 5), & (0,\ 2,\ 4,\ 6), \\ (1,\ 2,\ 5,\ 6). & & & \end{array}$$

For $\lambda \equiv 0 \pmod 3$, the desired $SCDB(4,\lambda;7)$ can be obtained by repeating each block of the $SCDB(4,3;7)$ $\frac{\lambda}{3}$ times. □

Theorem 4.3 *There exists an* $SCDB(4,\lambda;7)$ *for* $\lambda \geq 2$.

Proof For $\lambda \geq 2$, λ can be written as $\lambda = \lambda_1 + \lambda_2$ where $\lambda_1 \equiv 0 \pmod 2$ and $\lambda_2 \equiv 0 \pmod 3$. By Lemma 4.1 and Lemma 4.2, there exist an $SCDB(4,\lambda_1;7)$ $(I_7, \mathcal{B}_1, f)$ and an $SCDB(4,\lambda_2;7)$ $(I_7, \mathcal{B}_2, f)$. So, it is easy to see that $(I_7, \mathcal{B}_1 \cup \mathcal{B}_2, f)$ is the desired $SCDB(4,\lambda;7)$. □

5 Concluding

Theorem 5.1 *There exists an* $SCDB(4,\lambda;v)$ *if and only if* $v \geq 4$ *when* $\lambda \equiv 0 \pmod 3$; $v \equiv 1 \pmod 3$ *when* $\lambda \equiv 1,2 \pmod 3$ *and* $(v,\lambda) \neq (7,1)$.

Proof The necessity follows from the necessary condition for the existence of a $DB(4,\lambda;v)$ and the non-existence of an $SCDB(4,1;7)$ in [8]. By Theorem 2.6, an $SCDB(4,\lambda;v)$ exists for $\lambda \equiv 1,2 \pmod 3$, $v \equiv 1 \pmod 3$ and $v \neq 7$. For $\lambda \equiv 0 \pmod 3$, there exists an $SCDB(4,\lambda;v)$ for $v \equiv 1 \pmod 3$ and $v \equiv 0,1 \pmod 4$ by

Theorems 2.6 and 3.2; an $SCDB(4,\lambda;v)$ exists for $v \equiv 2,3,6,11 \pmod{12}$ by Theorems 3.10-3.13. Furthermore, an $SCDB(4,\lambda;7)$ exists for any $\lambda > 1$ by Theorem 4.3. Therefore the conclusion holds. □

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