Two paths joining given vertices in $(2k+1)$-edge-connected graphs

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Abstract. Let $k \geq 3$ be odd and $G = (V(G), E(G))$ be a k-edge-connected graph. For $X \subseteq V(G)$, $e(X)$ denotes the number of edges between X and $V(G) - X$. We here prove that if $\{s_i, t_i\} \subseteq X_i \subseteq V(G)$ $(i = 1, 2)$, $X_1 \cap X_2 = \emptyset$, $e(X_1) \leq 2k-2$ and $e(X_2) \leq 2k-1$, then there exist paths P_1 and P_2 such that P_i joins s_i and t_i, $V(P_i) \subseteq X_i$ $(i = 1, 2)$ and $G - E(P_1 \cup P_2)$ is $(k-2)$-edge-connected, and in fact we give a generalization of this result and some other results about paths not containing given edges.

1 Introduction

We consider finite undirected multigraph without loops. Let G be a graph and let $V(G)$ and $E(G)$ be the set of vertices and edges of G, respectively. $\lambda(G)$ denotes the edge-connectivity of G. We allow repetition of vertices (but not edges) in a path or cycle. Let $\lambda(G) \geq k \geq 2$ and $T = \{s_1, t_1, s_2, t_2\} \subseteq V(G)$. When is the following true ?

(1.1) *There exist edge-disjoint paths P_1 and P_2 such that P_i joins s_i and t_i $(i = 1, 2)$ and $\lambda(G - E(P_1 \cup P_2)) \geq k - 2$.*

When k is even, if $|T| \leq 3$, then (1.1) is true ([4]). When k is odd, even if $|T| = 2$, (1.1) is not always true (Huck and Okamura [2]). In Theorem 1, it is given that if $k \geq 3$ is odd, $\lambda(G) \geq k$, for $i = 1, 2$, $\{s_i, t_i\} \subseteq X_i \subseteq V(G)$, $X_1 \cap X_2 = \emptyset$, and if the number of edges between X_i and $V(G) - X_i$ is n_i, $n_1 \leq 2k-2$ and $n_2 \leq 2k-1$, then there exist paths P_1 and P_2 such that P_i joins s_i and t_i, $V(P_i) \subseteq X_i$ $(i = 1, 2)$ and $\lambda(G - E(P_1 \cup P_2)) \geq k-2$. If $n_1 = n_2 = 2k-1$, then (1.1) is not always true. Figure 1 gives a counterexample, where $k = 5$ and $X_i = \{s_i, t_i, x_i\}$ $(i = 1, 2)$. A generalization of Theorem 1 is given in Theorem 2 and some other results about paths not containing given edges are given in Theorem 4. Related results for even k are given in [9].

A subgraph H in a k-edge-connected graph G such that $\lambda(G - E(H)) \geq k - 2$ is called 2-reducible. 2-reducible paths and cycles are investigated in [4-9].

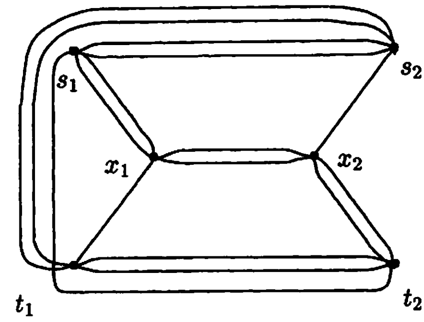

Figure 1

Notations and definitions

Let X, $Y \subseteq V(G)$. We set $\overline{X} := V(G) - X$. We denote by $\partial(X,Y;G)$ the set of edges with one end in X and the other in Y, and define $\partial(X;G) := \partial(X,\overline{X};G)$, $e(X,Y;G) := |\partial(X,Y;G)|$ and $e(X;G) := |\partial(X;G)|$. We set $N(X;G) := \{a \in \overline{X} \mid e(a,X) > 0\}$ and set $E(X;G) := \partial(X,X;G)$. When $X \cap Y = \emptyset$, $\lambda(X,Y;G)$ denotes the maximal number of edge-disjoint paths between X and Y. When $X = \{x\}$ and $Y = \{y\}$, we use $\lambda(x,y;G)$ for $\lambda(X,Y;G)$. We set $\lambda(X;G) := \min_{x \neq y \in X} \lambda(x,y;G)$ (note that $\lambda(G) = \lambda(V(G);G)$). In such expressions, we often omit G. If $|X| \geq 2$, $|\overline{X}| \geq 2$ and $e(X) = k$, we call X and $\partial(X)$ a k-set and a k-cut respectively. G/X denotes the graph obtained from G by identifying all the vertices in X and deleting any resulting loops. In G/X, X denotes the corresponding new vertex, each $x \in X$ denotes vertex X and for $Z \subseteq V(G)$ with $Z \cap X \neq \emptyset$, Z denotes $(Z - X) \cup \{X\}$. For $x, y \in V(G)$, we write $P = P[x,y]$ to denote that P is a path between x and y and we denote by $P(a,b)$ a subpath of P between a and b for $a, b \in V(P)$. We often denote a path by its edge set. For $x \in X$, $y \in \overline{X}$ and for paths $P_1[x,\overline{X}]$ in $G/\overline{X}$ with $\partial(X) \cap E(P_1) = \{f\}$ and $P_2[X,y]$ in G/X with $\partial(X) \cap E(P_2) = \{f\}$, we write $P := P_1 \cup P_2$ in G to denote that P is the path between x and y in G obtained from P_1 and P_2 by combining them at f.

For $K, L \subseteq E(G)$ and $W \subseteq V(G)$, we define

$$\mathcal{C}(G,K^+,L^-,W) := \left\{ C \;\middle|\; \begin{array}{l} C \text{ is a cycle in } G \text{ such that } K \subseteq E(C), \\ L \cap E(C) = \emptyset \text{ and } \lambda(W;G-E(C)) \geq k-2 \end{array} \right\},$$

$$\mathcal{P}(G,s,t,K^+,L^-,W) := \left\{ P \;\middle|\; \begin{array}{l} P = P[s,t] \text{ is a path in } G \text{ such that} \\ K \subseteq E(P),\ L \cap E(P) = \emptyset \text{ and} \\ \lambda(W;G-E(P)) \geq k-2 \end{array} \right\}.$$

In these notations, we omit K^+ or L^- if it is the emptyset.

Let $x \in X$. For $a, b \in N(x)$ with $a \neq b$ ($a = b$, respectively) and for $f \in \partial(x,a)$ and $g \in \partial(x,b) - f$, $G_x^{a,b}$ and $G^{f,g}$ denote the graph $(V(G), (E(G)+h) - \{f,g\})$, $((V(G), E(G) - \{f,g\})$, respectively), and is called a lifting of G at x, where h is a new edge between a and b. We call $G^{f,g}$ admissible if for each $y \neq z \in V(G) - x$, $\lambda(y,z;G^{f,g}) = \lambda(y,z;G)$.

Throughout this paper, we use $\alpha := \lfloor k/2 \rfloor$.

Our main results are the following.

THEOREM 1 *If $k \geq 3$ is odd, $\lambda(G) \geq k$, $\{s_i, t_i\} \subseteq X_i \subseteq V(G)$ $(i = 1,2)$, $X_1 \cap X_2 = \emptyset$, $e(X_1) \leq 2k-2$ and $e(X_2) \leq 2k-1$, then there exist paths $P_1[s_1,t_1]$ and $P_2[s_2,t_2]$ such that $V(P_i) \subseteq X_i$ $(i = 1,2)$ and $\lambda(G - E(P_1 \cup P_2)) \geq k-2$.*

THEOREM 2 *Suppose that:*

(i) *$k \geq 3$ is odd, $V(G) = W \cup S$, $W \cap S = \emptyset$, $\lambda(W) \geq k$, $\{s_1,t_1,s_2,t_2\} \subseteq W$, $L \subseteq E(G)$ and $e(b)$ is even for each $b \in S$,*

(ii) *$\{s_i,t_i\} \subseteq X_i \subseteq V(G)$ $(i = 1,2)$ and $X_1 \cap X_2 = \emptyset$.*

(1) *If $e(X_1) \leq 2k-2-2|L \cap E(X_1)|$ and $e(X_2) \leq 2k-1-2|L \cap E(X_2)|$, then there exist paths $P_1[s_1,t_1]$ and $P_2[s_2,t_2]$ such that $V(P_i) \subseteq X_i$, $L \cap E(P_i) = \emptyset$ (i = 1,2) and $\lambda(W; G - E(P_1 \cup P_2)) \geq k-2$.*

(2) *If $e(X_1) \leq 2k-1-2|L \cap E(X_1)|$ and $f \in \partial(s_2,t_2)$, then there exists a path $P[s_1,t_1]$ such that $V(P) \subseteq X_1$, $L \cap E(P) = \emptyset$ and $\lambda(W; G-f-E(P)) \geq k-2$.*

COROLLARY 3 *If $k \geq 3$ is odd, $\lambda(G) \geq k$, s_1,t_1,s_2,t_2,v are distinct virtices, $e(s_1) = e(t_1) = e(v) = k$, $e(s_1,t_1) = 0$, $f \in \partial(s_1,v)$, $g \in \partial(v,t_1)$ and $e(s_1,v) + e(v,t_1) \geq (k+3)/2$, then $G - \{f,g\}$ contains a path $P[s_2,t_2]$ such that $\lambda(G - \{f,g\} - E(P)) \geq k-2$.*

Theorem 1 is a corollary of Theorem 2 and to prove Theorem 2, we need the following.

THEOREM 4 *Suppose that $k \geq 3$ is odd, $V(G) = W \cup S, W \cap S = \emptyset$, $\lambda(W) \geq k$, $\{s,t,v\} \subseteq W$, $L \subseteq E(G)$ and $e(b)$ is even for each $b \in S$.*

(1) *If $|L| \leq (k-3)/2$ and $f \in \partial(s,v) - L$, then there is $P \in \mathcal{P}(G,s,t,f^+,L^-,W)$ such that $\lambda(s,t;G - E(P)) \geq k-1$.*

(2) *If $|L| \leq (k-3)/2$, $f \in \partial(v,s) - L$ and $g \in \partial(v,t) - L$, then $\mathcal{C}(G,\{f,g\}^+,L^-,W) \neq \emptyset$.*

(3) *If $|L| \leq (k-1)/2$, then there is $P \in \mathcal{P}(G,s,t,L^-,W)$ such that $\lambda(\{v,s,t\}; G - E(P)) \geq k-1$.*

(4) *If* $\{s,t\} \subseteq X \subseteq V(G) - v$, $e(X) \leq 2k - 2 - 2|L \cap E(X)|$ *and* $f \in \partial(v,s)$, *then there is* $P \in \mathcal{P}(G - f, s, t, L^-, W)$ *such that* $V(P) \subseteq X$ *and* $\lambda(v,t; G - f - E(P)) \geq k - 1$.

(5) *If* $\{s,t\} \subseteq X \subseteq V(G) - v$, $e(X) \leq 2k - 2 - 2|L \cap E(X)|$, $f \in \partial(v,s)$ *and* $g \in \partial(v,t)$, *then there is* $P \in \mathcal{P}(G - \{f,g\}, s, t, L^-, W)$ *such that* $V(P) \subseteq X$.

(6) *If* $\{s,t\} \subseteq X \subseteq V(G) - v$ *and* $e(X) \leq 2k - 1 - 2|L \cap E(X)|$, *then there is* $P \in \mathcal{P}(G, s, t, L^-, W)$ *such that* $V(P) \subseteq X$ *and* $\lambda(\{v,s,t\}; G - E(P)) \geq k - 1$.

Theorem 5 *If* $k \geq 3$ *is odd,* $\lambda(G) \geq k$, $V(G) = X \cup Y$, $X \cap Y = \emptyset$, $e(X) = k$, $s \in X$, $t \in V(G)$, $P_1 \in \mathcal{P}(G/Y, s, t, V(G))$, $L \subseteq E(Y)$ *and* $|L| \leq (k-3)/2$, *then there is* $P \in \mathcal{P}(G, s, t, L^-, V(G))$ *such that* $P/Y = P_1$.

2 Preliminaries

We prepare some lemmas. Lemma 1 follows by simple counting.

Lemma 1 *If* $X, Y \subseteq V(G)$, *then*

(2.1) $e(X - Y) + e(Y - X) = e(X) + e(Y) - 2e(X \cap Y, \overline{X} \cap \overline{Y})$,

(2.2) $e(X \cap Y) + e(\overline{X} \cap \overline{Y}) = e(X) + e(Y) - 2e(X - Y, Y - X)$.

Lemma 2 (Mader [3] and Frank [1]) *If* $x \in V(G)$, $3 \neq e(x) = k$ ($k = 2\alpha$ *or* $2\alpha + 1$) *and there is no cut-edge incident to* x, *then there are distinct edges* $\{f_1, \cdots, f_\alpha, g_1, \cdots, g_\alpha\} \subseteq \partial(x)$ *such that* G^{f_i, g_i} ($1 \leq i \leq \alpha$) *are admissible.*

Lemma 3 *If* $k \geq 3$ *is odd,* $V(G) = W \cup S$, $W \cap S = \emptyset$, $V(G) = X \cup Y$, $X \cap Y = \emptyset$, $e(X) = k + 1$, $W \cap X \neq \emptyset \neq W \cap Y$, $\lambda(W; G/X) \geq k$, $\lambda(W; G/Y) \geq k$, $x \in X \cap W$, $\lambda(x, Y) = k + 1$ *and* $e(b)$ *is even for each* $b \in S$, *then* $\lambda(W; G) \geq k$.

Proof. Let G be a minimum counterexample with respect to $|E(G)|$. To prove $S = \emptyset$, assume that there is $b \in S$. If $b \in X$ ($b \in Y$, respectively), there is an admissible lifting G^* of G/Y (G/X, respectively) at b by Lemma 2. Let G_b be a lifting of G such that $G_b/Y = G^*$ ($G_b/X = G^*$, respectively). $\lambda(W; G_b/X) \geq k$, $\lambda(W; G_b/Y) \geq k$, $\lambda(x, Y; G_b) = k + 1$ and $|E(G_b)| < |E(G)|$, and so $k \leq \lambda(W; G_b) \leq \lambda(W; G)$. Thus $S = \emptyset$. Since $\lambda(W; G) < k$, there is $Z \subseteq V(G)$ with $e(Z) \leq k - 1$. $X \cap Z$, $\overline{X} \cap \overline{Z}$, $X - Z$ and $Z - X$ are not emptyset, since $\lambda(G/X) \geq k$ and $\lambda(G/Y) \geq k$. By Lemma 1, $e(X - Z) + e(Z - X) \leq e(X) + e(Z) \leq 2k$ and $e(X \cap Z) + e(\overline{X} \cap \overline{Z}) \leq 2k$, and so $e(X - Z) = e(Z - X) = k$ and $e(X \cap Z) = e(\overline{X} \cap \overline{Z}) = k$. Then $\lambda(x, Y) = k$, a contradiction.

Lemma 4 *If $k \geq 3$ is odd, $\lambda(G) \geq k$, X and Y are k-sets, $X \cap Y \neq \emptyset$ and $X - Y \neq \emptyset$, then $Y \subseteq X$ or $\overline{Y} \subseteq X$.*

Proof. Otherwise $Y - X \neq \emptyset$ and $\overline{X} \cap \overline{Y} \neq \emptyset$. By Lemma 1, $e(X-Y) = e(Y-X) = k$ and $e(X \cap Y) = e(\overline{X} \cap \overline{Y}) = k$. Then $e(X) = e(X \cap Y) + e(X-Y) - 2e(X \cap Y, X-Y)$ is even, a contradiction.

Lemma 5 *If $k \geq 3$ is odd, $\lambda(G) \geq k$, X is a minimal k-set, $\{x, y\} \subseteq X$ and $h \in \partial(x, y)$, then h is contained in no k-cut.*

Proof. Otherwise there is a k-set Y with $|Y \cap \{x, y\}| = 1$, then $Y \subseteq X$ or $\overline{Y} \subseteq X$ by Lemma 4, contrary to the minimality of X.

Proof of Lemma 6 will be given in Section 3.

Lemma 6 *Suppose that $k \geq 3$ is odd, $V(G) = W \cup S$, $W \cap S = \emptyset$, $\lambda(W) \geq k$, $\{s, t, v\} \subseteq W$, $\{s, t\} \subseteq X \subseteq V(G) - v$, $L \subseteq E(G)$ and $e(b)$ is even for each $b \in S$.*

(7) *If $e(X) \leq 2k - 1 - 2|L \cap E(X)|$, $u \in W - X - v$ and $f \in \partial(v, u)$, then there is $P \in \mathcal{P}(G - f, s, t, L^-, W)$ such that $V(P) \subseteq X$.*

(8) *If $e(X) \leq 2k - 2 - 2|L \cap E(X)|$, $u \in X \cap W$, $e(u) = k$, $e(u, \overline{X}) \geq \alpha$ and $f \in \partial(v, u)$, then there is $P \in \mathcal{P}(G - f, s, t, L^-, W)$ such that $V(P) \subseteq X$.*

Note that $\alpha = (k-1)/2$ and $e(x, y) \leq \alpha$, if $k \geq 3$ is odd, $\lambda(G) \geq k$, $\{x, y\} \subseteq V(G)$ and $e(x) = e(y) = k$.

Lemma 7 *Suppose that $k \geq 3$ is odd, $\lambda(G) \geq k$, $\{s, t\} \subseteq X \subsetneq V(G)$, $e(x) = k$ for each $x \in X$, $L \subseteq E(X)$ and $e(X) \leq 2k - 2 - 2|L| + \varepsilon$ for integer ε. Then*

(1) $|L| \leq \alpha + (\varepsilon - 1)/2$.

(2) $|E(X)| \geq |L| + (|X| - 2)\alpha + (|X| - \varepsilon)/2$.

(3) *If $\varepsilon \leq 1$ and $x \in X$, then $\partial(x, X - x) - L \neq \emptyset$.*

(4) *If $\varepsilon \leq 1$ and $|X| = 2$, then $\partial(s, t) - L \neq \emptyset$.*

Proof. (1) By $k \leq e(X) \leq 2k - 2 - 2|L| + \varepsilon$, we have

$$|L| \leq (k-1)/2 + (\varepsilon - 1)/2 = \alpha + (\varepsilon - 1)/2.$$

(2) By $e(X) = k|X| - 2|E(X)| \leq 2k - 2 - 2|L| + \varepsilon$, we have

$$|E(X)| \geq |L| - k + 1 + ((2\alpha + 1)|X| - \varepsilon)/2 = |L| + (|X| - 2)\alpha + (|X| - \varepsilon)/2.$$

(3) Otherwise $e(x, X - x) \leq |L|$. By $e(x, \overline{X}) \geq k - |L|$, we have

$$e(X - x) = e(X) - e(x, \overline{X}) + e(x, X - x) \leq 2k - 1 - 2|L| - (k - |L|) + |L| = k - 1,$$

a contradiction.

(4) follows by (3).

LEMMA 8 *Suppose that $k \geq 3$ is odd, $\lambda(G) \geq k$, $X = \{s,t,x\} \subsetneq V(G)$, $e(s) = e(t) = e(x) = k$, $L \subseteq E(X)$ and $e(X) \leq 2k-2-2|L|+\varepsilon$ for integer ε.*

(1) If $\varepsilon \leq 2$ and $\partial(s,t) \subseteq L$, then $\partial(s,x) - L \neq \emptyset$ and $\partial(t,x) - L \neq \emptyset$.

(2) If $\varepsilon \leq 0$ and $e(x,\overline{X}) = \alpha$, then $\partial(s,t) - L \neq \emptyset$.

Proof. (1) If $\partial(s,x) \subseteq L$, then $e(s,\{t,x\}) \leq |L|$ and $|E(X)| = e(s,\{t,x\}) + e(t,x) \leq |L| + \alpha$, contrary to Lemma 7(2).

(2) Otherwise $|E(X)| = e(s,t) + e(x,\{s,t\}) \leq |L| + (\alpha+1)$, contrary to Lemma 7(2).

LEMMA 9 *Suppose that $k \geq 3$ is odd, $\lambda(G) \geq k$, $X = \{s,t,x_1,x_2\} \subsetneq V(G)$, $e(x) = k$ for each $x \in X$, $L \subseteq E(X)$, $e(X) \leq 2k-2-2|L|+\varepsilon$ for integer ε, and $\partial(s,t) \subseteq L$.*

(1) If $\varepsilon \leq 3$ and $e(x_1,x_2) = 0$, then for $i = 1$ or 2, $\partial(s,x_i) - L \neq \emptyset$ and $\partial(t,x_i) - L \neq \emptyset$.

(2) If $\varepsilon \leq 1$ and $e(x_1,\overline{X}) = \alpha$, then $\partial(s,x_2) - L \neq \emptyset$ and $\partial(t,x_2) - L \neq \emptyset$.

(3) If $\varepsilon \leq 1$, $\partial(s,x_2) \subseteq L$ and $\partial(t,x_1) \subseteq L$, then $\partial(s,x_1) - L \neq \emptyset$, $\partial(x_1,x_2) - L \neq \emptyset$ and $\partial(t,x_2) - L \neq \emptyset$.

Proof. (1) $|E(X)| = e(s,t) + e(s,x_1) + e(s,x_2) + e(t,x_1) + e(t,x_2)$. By Lemma 7(2), $|E(X)| \geq |L| + 2\alpha + 1$, and so at most one of $\partial(s,x_1)$, $\partial(s,x_2)$, $\partial(t,x_1)$ and $\partial(t,x_2)$ is contained in L.

(2) Let $X' := X - x_1$. Then $e(X') = e(X) + 1 \leq 2k - 2|L| \leq 2k - 2|L \cap E(X')|$. By Lemma 8(1), the result follows.

(3) By Lemma 7(3), $\partial(s,x_1) - L \neq \emptyset$ and $\partial(t,x_2) - L \neq \emptyset$. If $\partial(x_1,x_2) \subseteq L$, then $e(s,\{t,x_2\}) + e(t,x_1) + e(x_1,x_2) \leq |L|$ and we have $|E(X)| \leq |L| + e(s,x_1) + e(t,x_2) \leq |L| + 2\alpha$, contrary to Lemma 7(2).

LEMMA 10 *Suppose that $k \geq 3$ is odd, $\lambda(G) \geq k$, $X = \{s,t,x_1,x_2,x_3\} \subsetneq V(G)$, $e(x) = k$ for each $x \in X$, $L \subseteq E(X)$, $e(X) \leq 2k-2-2|L|+\varepsilon$ for integer ε, and $\partial(s,t) \subseteq L$.*

(1) If $\varepsilon \leq 4$ and $e(x_i,x_j) = 0$ $(1 \leq i \neq j \leq 3)$, then for some $1 \leq i \leq 3$, $\partial(s,x_i) - L \neq \emptyset$ and $\partial(t,x_i) - L \neq \emptyset$.

(2) If $\varepsilon \leq 2$, $e(x_1,\overline{X}) = \alpha$ and $e(x_2,x_3) = 0$, then for $i = 2$ or 3, $\partial(s,x_i) - L \neq \emptyset$ and $\partial(t,x_i) \neq \emptyset$.

(3) *If* $\varepsilon \le 0$ *and* $e(x_1, \overline{X}) = e(x_2, \overline{X}) = \alpha$, *then* $\partial(s, x_3) - L \ne \emptyset$ *and* $\partial(t, x_3) \ne \emptyset$.

Proof. (1) Otherwise by Lemma 7(3), we may let $\partial(x_i, s) - L \ne \emptyset$ $(i = 1, 2)$, $\partial(x_3, t) - L \ne \emptyset$ and $e(s, \{t, x_3\}) + e(t, \{x_1, x_2\}) \le |L|$. Then $|E(X)| \le |L| + e(s, \{x_1, x_2\}) + e(t, x_3) \le |L| + 3\alpha$, contrary to Lemma 7(2).

(2) Let $X' := X - x_1$. Then $e(X') = e(X) + 1 \le (2k - 2|L|) + 1 \le 2k - 2|L \cap E(X')| + 1$. By Lemma 9(1), the result follows.

(3) Let $X' := X - \{x_1, x_2\}$. Then $e(X') \le e(X) + 2 \le 2k - 2|L| \le 2k - 2|L \cap E(X')|$. By Lemma 8(1), the result follows.

Lemma 11 *Suppose that* $k \ge 3$ *is odd,* $\lambda(G) \ge k$, $\{s, t, v\} \subseteq V(G)$, $e(v) = k$, $u \in V(G) - v$, $f \in \partial(v, u)$, $L \subseteq E(G)$ *and each* $Y \subseteq V(G) - \{s, t, v\}$ *with* $u \in Y$ *is not a k-set.*

(1) *If* $|L| \le \alpha - 1$, *then there is* $P \in \mathcal{P}(G - f, s, t, L^-, V(G) - u)$ *such that* $\lambda(v, x; G - f - E(P)) \ge k - 1$ *for* $x = s$ *or* t *and* $e(u; G - f - E(P)) \ge k - 3$.

(2) *If* $\{s, t, u\} \subseteq X \subseteq V(G) - v$ *(possibly* $u = s$ *or* t*) and* $e(X) \le 2k - 2 - 2|L \cap E(X)|$, *then there is* $P \in \mathcal{P}(G - f, s, t, L^-, V(G) - u)$ *such that* $V(P) \subseteq X$, $\lambda(v, x; G - f - E(P)) \ge k - 1$ *for* $x = s$ *or* t *and* $e(u; G - f - E(P)) \ge k - 3$.

Proof. Since Theorem 4 will be proved without Lemma 11 in Section 3, we can use Theorem 4. In (1), let $X := V(G) - v$, then $e(X) = e(v) = k = 2k - 2 - 2(\alpha - 1) - 1 < 2k - 2 - 2|L \cap E(X)|$, and so it suffices to prove (2). If $u = s$ or t, then the result follows by Theorem 4(4). Thus let $u \ne s, t$ and let $T := \{s, t, v, u\}$. Let $G_1 = (G - f) + g$ and $G_2 = (G - f) + h$, where g is a new edge between v and s and h is a new edge between v and t. $\lambda(V(G_i) - u; G_i) \ge k$ $(i = 1$ or $2)$, say for $i = 1$, otherwise G has k-sets X_1 and X_2 with $X_1 \cap T = \{u, t\}$ and $X_2 \cap T = \{u, s\}$, contrary to Lemma 4. If $e(u) > k$, then $\lambda(G_1) \ge k$ and if $e(u) = k$, then $e(u; G_1)$ is even. Thus by Theorem 4(4), there is simple $P \in \mathcal{P}(G_1 - g, s, t, L^-, V(G_1) - u)$ such that $V(P) \subseteq X$, $\lambda(v, t; G_1 - g - E(P)) \ge k - 1$ and $e(u; G_1 - g - E(P)) \ge k - 3$. Then $e(u; G - f - E(P)) \ge k - 3$ and P is a required path.

Lemma 12 *Suppose that* $k \ge 3$ *is odd,* $V(G) = X \cup Y$, $X \cap Y = \emptyset$, $|X| \ge 2$, $e(X) = k$, $x \in X$, $\{v, y\} \subseteq Y$, $f \in \partial(x, y)$, $\lambda(G/X) \ge k - 2$ *and* $\lambda(v, X; G - f) = k - 1$.

(1) *If* $\lambda(G/Y - f) \ge k - 2$, *then* $\lambda(G) \ge k - 2$.

(2) *If* $e(x, Y) < \alpha$, $\lambda(V(G/Y) - x; G/Y - f) \ge k - 2$ *and* $e(x, G - f) \ge k - 3$, *then* $\lambda(G) \ge k - 2$.

Proof. In (2), we may let $e(x; G - f) = k - 3$, otherwise $\lambda(G/Y - f) \ge k - 2$ and the result follows by (1).

Case 1. $\lambda(G/X - f) \geq k - 2$.

By Lemma 3, in (1), $\lambda(G - f) \geq k - 2$, and so $\lambda(G) \geq k - 2$. In (2), since $e(x; G - f)$ is even, by Lemma 3, $\lambda(V(G) - x; G - f) \geq k - 2$. By $e(x; G) = k - 2$, we have $\lambda(G) \geq k - 2$.

Case 2. $\lambda(G/X - f) = k - 3$.

There is $Z \subseteq Y$ such that $e(Z) = k - 2$ and $y \in Z$. We choose maximal Z. Then $v \notin Z$. Since $e(Z; G - f)$ is even, by Lemma 3, $\lambda(V(G) - Z; G/Z - f) \geq k - 2$ in (1) and $\lambda(V(G) - Z - x; G/Z - f) \geq k - 2$ in (2). In (1), $\lambda(z, \overline{Z}; G) = k - 2$ for each $z \in Z$, and so $\lambda(G) \geq k - 2$. In (2), by $e(x, Z; G) \leq e(x, Y; G) < \alpha$, we have $e(Z + x) = e(Z) + e(x) - 2e(Z, x) \geq (k - 2) + (k - 2) - 2(\alpha - 1) = k - 1$. Thus $\lambda(z, \overline{Z + x}; G) = k - 2$ for each $z \in Z$, and so $\lambda(V(G) - x; G) = k - 2$. By $e(x; G) = k - 2$, we have $\lambda(G) = k - 2$.

Note that in what follows, k is odd and $\alpha = (k - 1)/2$.

3 Proof of Theorem 4 and Lemma 6

We denote (1),(2),(3),(4),(5) and (6) of Theorem 4 by (1),(2),(3),(4),(5) and (6) respectively and denote (7) and (8) of Lemma 6 by (7) and (8) respectively. In (8), if $u = s$ or t, then the result follows by (4), and thus we may let $u \neq s, t$. In (8), let $X' := X - u$, then $e(X') \leq e(X) + 1 \leq 2k - 1 - 2|L \cap E(X)| \leq 2k - 1 - 2|L \cap E(X')|$ and the result follows by (7). Let $\varepsilon = 0$ in (1),(2),(4) and (5), and let $\varepsilon = 1$ in (3),(6) and (7). We may assume

(3.1) *In* (1), (3), (4), (6) *and* (7), $e(x) = k$ *for each* $x \in W$. *In* (2) *and* (5), $e(x) = k$ *for each* $x \in W - v$ *and* $e(v) = k$ *or* $k + 1$.

Proof. Let $x \in W$ and $\partial(x) = \{g_1, \cdots, g_r\}$. In (1),(3),(4),(6) and (7) and if $x \neq v$ in (2) and (5) (respectively, $x = v$ in (2) and (5), $g_1 = f$ and $g_2 = g$) and $r \geq k + 1$ (respectively, $r \geq k + 2$), then we replace x and $\partial(x)$ by the graph in Figure 2 (respectively, Figure 3), in which each heavy edge represents α parallel edges, producing a new graph G'. If the result holds in G', then it also holds in G.

In (1), let $X := V(G) - s$, and in (2) and (3), let $X := V(G) - v$. Since

$$2k - 2 - 2|L \cap E(X)| + \varepsilon \geq 2k - 2 - 2|L| + \varepsilon \geq \begin{cases} k + 1 & \text{in (1) and (2)} \\ k & \text{in (3)} \end{cases},$$

$e(X) = k$ in (1) and (3) and $e(X) \leq k + 1$ in (2), we have $e(X) \leq 2k - 2-$

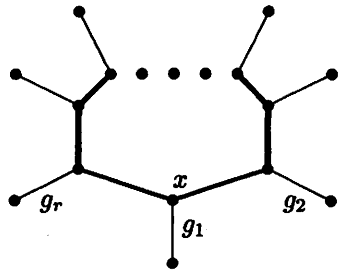

Figure 2

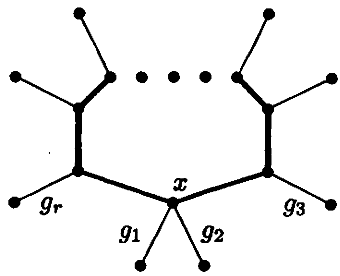

Figure 3

$2|L \cap E(X)| + \varepsilon$ and (1), (2) and (3) follow by (4), (5) and (6) respectively. In (5), if there is $P \in \mathcal{P}(G - f, s, t, L^-, W)$ such that $V(P) \subseteq X$ and $\lambda(v, t; G - f - E(P)) = k - 1$, then $P \in \mathcal{P}(G - \{f, g\}, s, t, L^-, W)$. Thus (5) follows by (4). We shall prove (4),(6) and (7) simultaneously. Let G be a counterexample which satisfies the additional condition (3.1) with $|E(G)|$ minimized. In this section, we let

in (4), $I(G) = I(G, X, L^-, W) := \{P \in \mathcal{P}(G - f, s, t, L^-, W) \mid V(P) \subseteq X$ and $\lambda(v, t; G - f - E(P)) = k - 1\}$,

in (6), $I(G) = I(G, X, L^-, W) := \{P \in \mathcal{P}(G, s, t, L^-, W) \mid V(P) \subseteq X$ and $\lambda(\{v, s, t\}; G - E(P)) = k - 1\}$,

in (7), $I(G) = I(G, X, L^-, W) := \{P \in \mathcal{P}(G - f, s, t, L^-, W) | V(P) \subseteq X\}$.

We let $T := \{s,t,v\}$ in (4) and (6), and let $T := \{s,t,v,u\}$ in (7). We have (3.2) by Lemma 7(1) and we may assume (3.3).

(3.2) *In* (4), $|L \cap E(X)| \leq \alpha - 1$. *In* (6) *and* (7), $|L \cap E(X)| \leq \alpha$.

(3.3) *If* $\{s,t\} \subseteq Y \subsetneq X$, *then* $e(Y) > e(X)$. *We may assume* $L \subseteq E(X)$.

Proof. Assume that $e(Y) \leq e(X)$. Then $e(Y) \leq 2k-2-2|L \cap E(X)|+\varepsilon \leq 2k-2-2|L \cap E(Y)|+\varepsilon$, since $|L \cap E(Y)| \leq |L \cap E(X)|$. If $I(G,Y,L^-,W) \neq \emptyset$, then $I(G) \supseteq I(G,Y,L^-,W) \neq \emptyset$.

(3.4) $S = \emptyset$.

Proof. Otherwise let $b \in S$. By Lemma 2, there is an admissible lifting G_b of G at b. For some $x_i \in N(b)$ and $h_i \in \partial(b,x_i)$ ($i = 1,2$), $G_b = G^{h_1,h_2}$. Let h be a new edge in G_b between x_1 and x_2 if $x_1 \neq x_2$. If $b \in X$ or $|X \cap \{x_1,x_2\}| \leq 1$, then by $e(X;G_b) \leq e(X;G)$, $\emptyset \neq I(G_b) \subseteq I(G)$. If $b \notin X$ and $\{x_1,x_2\} \subseteq X$, then let $L' := L + h$. $e(X;G_b) = e(X;G) - 2 \leq 2k-2-2|L|+\varepsilon-2 = 2k-2-2|L'|+\varepsilon$. Thus $\emptyset \neq I(G_b,X,L'^-,W) \subseteq I(G)$.

(3.5) *If* $Y \subseteq W - \{s,t\}$ *is a k-set, then* $Y \subseteq X$ *or* $Y \subseteq \overline{X}$.

Proof. Otherwise $Y \cap X \neq \emptyset \neq Y - X$ and we have $e(X-Y) \geq e(X)+1$ by (3.3). By (2.1), $e(Y-X) \leq e(Y)+e(X)-e(X-Y) \leq k-1$, a contradiction.

(3.6) $\partial(s,t) \subseteq L$.

Proof. If there is $h \in \partial(s,t) - L$, then $\{h\} \in I(G)$.

(3.7) *In* (6) *and* (7), *either* $\partial(s,x) \subseteq L$ *or* $\partial(t,x) \subseteq L$ *for each* $x \in X-T$.

Proof. Assume that there are $h_1 \in \partial(s,x) - L$ and $h_2 \in \partial(t,x) - L$ for some $x \in X-T$. By (3.5), $\lambda(\{v,s,t\};G-\{h_1,h_2\}) \geq k-1$ in (6) and $\lambda(G-\{f,h_1,h_2\}) \geq k-2$ in (7), and so we have $\{h_1,h_2\} \in I(G)$.

(3.8) *If* $\{x,y\} \subseteq W - T$, $h \in \partial(x,y)$ *and* h *is contained in no k-cut, then* $\{x,y\} \subseteq X$, $e(x,y) = \alpha$ *and* $\partial(x,y) = L$ *in* (6) *and* (7), *and we have a contradiction in* (4).

Proof. $\lambda(W-\{x,y\};G-h) \geq k$ and there is simple $P \in I(G-h,X,L^-,W-\{x,y\})$. Since $P \notin I(G)$, we have $e(x,y) = \alpha$ and $\lambda(\{x,y\};G-E(P)) = k-3$. If there is $h' \in \partial(x,y) - L$, then let s,x,y,t be in this order in P and let $P' := P(s,x) \cup \{h'\} \cup P(y,t)$ in G. Then $\lambda(\{x,y\};G-E(P')) \geq k-2$ and $P' \in I(G)$. Thus $\partial(x,y) = L$. We have a contradiction in (4).

(3.9) *If* Y *is a minimal k-set,* $\{x,y\} \subseteq Y - T$ *and* $h \in \partial(x,y)$, *then* $\{x,y\} \subseteq X$, $e(x,y) = \alpha$ *and* $\partial(x,y) = L$ *in* (6) *and* (7).

Proof. By Lemma 5, h is contained in no k-cut. The result follows by (3.8).

(3.10) *In* (4) *and* (6), *G has no k-set. In* (7), *if* Y *is a k-set and* $t \notin Y$, *then* $Y \subseteq \overline{X}$.

Proof. Assume that G has a k-set Y such that $|Y \cap T| \leq 1$ in (4) and (6) and $|Y \cap T| \leq 2$ in (7).

Case 1. $|Y \cap T| \leq 1$.

We choose minimal Y with this property. By (3.1) and (3.4), $|Y| \geq 3$, and so for some $x_1, x_2 \in Y - T$, $e(x_1, x_2) > 0$. By (3.9), in (6) and (7), $\{x_1, x_2\} \subseteq X$, $e(x_1, x_2) = \alpha$, $\partial(x_1, x_2) = L$ and $Y = \{x_1, x_2, y\}$ for some $y \in T$. By (3.5), $y = s$ or t. There is $P \in I(G/Y)$. Let z be the end vertex of P in Y. If $z = y$, then $P \in I(G)$. If $z = x_1$, then for $h' \in \partial(x_1, y)$, $P + h' \in I(G)$.

Case 2. $Y \cap T = \{s, v\}$ or $\{s, u\}$ in (7), say the former.

Let Y_1 and Y_2 be minimal k-sets with $Y_1 \subseteq Y$ and with $Y_2 \subseteq \overline{Y}$. Then $Y_1 \cap Y_2 = \emptyset$ and $Y_2 \cap T = \{t, u\}$.

(3.10.1) $|L| \leq \alpha - 1$.

Proof. Otherwise $e(X) \leq 2k - 1 - 2|L| \leq k$, contrary to Lemma 4.

(3.10.2) $|Y_1| = |Y_2| = 3$ *and* $V(G) = Y_1 \cup Y_2$.

Proof. If $|Y_1| \geq 4$, then $|Y_1 - T| \geq 3$, since $|Y_1|$ is odd. By $e(\{s, v\} \cup \overline{Y}_1) \leq 3k - 2$, we have for some $x_1, x_2 \in Y_1 - T$, $e(x_1, x_2) > 0$, contrary to (3.10.1) and (3.9). Thus $|Y_1| = |Y_2| = 3$. If $V(G) \neq Y_1 \cup Y_2$, then let Y_3 be a minimal k-set with $Y_1 \subsetneq Y_3 \subseteq \overline{Y_2}$ and let $z_1 \in Y_3 - Y_1$. Then for some $z_2 \in Y_3 - Y_1$, $e(z_1, z_2) > 0$. By (3.8), $e(z_1, z_2) = \alpha$ and $\partial(z_1, z_2) = L$, contrary to (3.10.1). Thus $V(G) = Y_1 \cup Y_2$.

Let $Y_i - T = \{x_i\}$ $(i = 1, 2)$. $|X| \geq 4$ by (3.6),(3.7) and by Lemmas 7(4) and 8(1), and so $\{x_1, x_2\} \subseteq X$. By (2.2), $e(X \cap Y_i) \leq e(X) + e(Y_i) - e(\overline{X} \cap \overline{Y}_i) \leq e(X)$ $(i = 1, 2)$, and so there are $g_1 \in \partial(s, x_1) - L$ and $g_2 \in \partial(t, x_2) - L$ by Lemma 7(4). By (3.7), $\partial(s, x_2) \subseteq L$ and $\partial(t, x_1) \subseteq L$. Then there is $h \in \partial(x_1, x_2) - L$ by Lemma 9(3). Let $P := \{g_1, h, g_2\}$. $\lambda(G/Y_i - f - E(P)) \geq k - 2$ $(i = 1, 2)$ by (3.5), and so $\lambda(G - f - E(P)) \geq k - 2$ and $P \in I(G)$.

Case 3. $Y \cap T = \{v, u\}$ in (7).

By (3.5), $Y \subseteq \overline{X}$.

(3.11) *In* (6) *and* (7), $I(G) \neq \emptyset$.

Proof. By (3.10), each edge in $E(X)$ is contained in no k-cut.

Case 1. $e(x_1, x_2) > 0$ for some $x_1, x_2 \in X - T$.

By (3.8), $e(x_1, x_2) = \alpha$ and $\partial(x_1, x_2) = L$. $e(X) = k$ by $e(X) \leq 2k - 1 - 2|L| = k$. Since $|X|$ is odd, there is $x_3 \in X - T - \{x_1, x_2\}$ and $N(x_3; G/\overline{X}) = \{s, t, \overline{X}\}$, contrary to (3.7).

Case 2. $N(x) \subseteq \{s, t\} \cup \overline{X}$ for each $x \in X - T$.

By $e(\{s, t\} \cup \overline{X}) \leq e(s) + e(t) + e(X) \leq 2k + (2k - 1 - 2|L|) \leq 4k - 1$, we have $|X - T| \leq 3$. By (3.6) and by Lemmas 7(4), 8(1), 9(1) and 10(1), for some $x \in X - T$, $\partial(s, x) - L \neq \emptyset$ and $\partial(x, t) - L \neq \emptyset$, contrary to (3.7).

We shall prove (4). $N(x) = T$ for each $x \in W - T$ by (3.8) and (3.10). Since $|W - T|$ is odd and $e(T) \leq 3k - 2$, we have $|W - T| = 1$, and so $|X| \leq 3$. By (3.6) and by Lemmas 7(4) and 8(1), $|X| = 3$ and for $x \in X - T$, there are $h_1 \in \partial(s, x) - L$ and $h_2 \in \partial(x, t) - L$. $e(v, x) = \alpha$ by $\{h_1, h_2\} \notin I(G)$, contrary to Lemma 8(2).

4 Proof of Theorem 5

If $t \in Y$, then let $g \in E(P_1) \cap \partial(X)$. By Theorem 4(1), there is $P_2 \in \mathcal{P}(G/X, X, t, g^+, L^-, W)$ such that $\lambda(X, t; G/X - E(P_2)) = k - 1$. Let $P := P_1 \cup P_2$ in G. By Lemma 3, $\lambda(G - E(P)) \geq k - 2$. If $t \in X$ and $E(P_1) \cap \partial(X) = \emptyset$, then let $P := P_1$. If $t \in X$ and $E(P_1) \cap \partial(X) = \{f, g\}$, then there is $C \in \mathcal{C}(G/X, \{f, g\}^+, L^-, W)$ by Theorem 4(2). Let $P := C \cup P_1$ in G. In each case, P is a required path.

5 Proof of Theorem 2

Theorem 2(2) follows by Lemma 6(7). We shall prove Theorem 2(1). In the same way as (3.1), we may assume

(5.1) $e(x) = k$ *for each* $x \in W$.

Let G be a counterexample to Theorem 2(1) which satisfies the additional condition (5.1) with $|E(G)|$ minimized. Let $\varepsilon_1 = 0$ and $\varepsilon_2 = 1$. Let $T := \{s_1, t_1, s_2, t_2\}$, $L_i := L \cap E(X_i)$, $G_i := G/\overline{X_i}$ $(i = 1, 2)$ and let

$J(G) = J(G, X_1, L_1^-, X_2, L_2^-, W) := \{(P_1, P_2) | P_i[s_i, t_i]$ is a simple path, $V(P_i) \subseteq X_i$, $L_i \cap E(P_i) = \emptyset$ $(i = 1, 2)$ and $\lambda(W; G - E(P_1 \cup P_2)) \geq k - 2\}$.

We may assume

(5.2) *If* $\{s_i, t_i\} \subseteq Y \subsetneqq X_i$, *then* $e(Y) > e(X_i)$ $(i = 1, 2)$.

Proof. Assume that $\{s_1, t_1\} \subseteq Y \subsetneqq X_1$ and $e(Y) \leq e(X_1)$. Then $e(Y) \leq$

$2k-2-2|L_1| \leq 2k-2-2|L \cap E(Y)|$. If $J(G, Y, (L \cap E(Y))^-, X_2, L_2^-, W) \neq \emptyset$, then $J(G) \supseteq J(G, Y, (L \cap E(Y))^-, X_2, L_2^-, W) \neq \emptyset$.

(5.3) $S = \emptyset$.

Proof. Otherwise let $b \in S$. By Lemma 2, for some $x_i \in N(b)$ and $h_i \in \partial(b, x_i)$ $(i = 1,2)$, there is an admissible lifting $G_b = G^{h_1,h_2}$. Let h be a new edge in G_b between x_1 and x_2 if $x_1 \neq x_2$.

Case 1. $\{x_1, x_2\} \subseteq X_i$ and $b \notin X_i$ $(i = 1 \text{ or } 2)$, say for i=1.

Let $L_1' := L_1 + h$. $e(X_1; G_b) = e(X_1; G) - 2 \leq 2k - 2 - 2|L_1'|$ and $e(X_2, G_b) \leq e(X_2, G)$. Thus $\emptyset \neq J(G_b, X_1, L_1'^-, X_2, L_2^-, W) \subseteq J(G)$.

Case 2. Either $|\{x_1, x_2\} \cap X_i| \leq 1$ $(i = 1,2)$ or $\{x_1, x_2, b\} \subseteq X_j$ $(j = 1$ or $2)$.

By $e(X_i, G_b) \leq e(X_i, G)$ $(i = 1,2)$, we have $\emptyset \neq J(G_b, X_1, L_1^-, X_2, L_2^-, W) \subseteq J(G)$.

(5.4) $e(x, \overline{X_i}) \leq \alpha$ *for each* $x \in X_i - T$ $(i = 1,2)$.

Proof. Otherwise $e(X_i - x) < e(X_i)$, contrary to (5.2).

(5.5) *If* $\{x, y\} \subseteq W - T$, $g \in \partial(x, y)$ *and* g *is contained in no* k*-cut, then* $e(x, y) = \alpha$ *and either* $\partial(x, y) = L_2$ *or* $|\{x, y\} \cap X_i| = 1$ $(i = 1,2)$.

Proof. There is $(P_1, P_2) \in J(G - g, X_1, L_1^-, X_2, L_2^-, W - \{x, y\})$. By $(P_1, P_2) \notin J(G)$, we have $\lambda(\{x, y\}; G - E(P_1 \cup P_2)) = k - 3$ and $e(x, y) = \alpha$ (note that P_1 and P_2 are simple). If for $i = 1$ or 2, $\{x, y\} \subseteq V(P_i)$ and there is $h \in \partial(x, y) - L_i$, say for i=1, then let s_1, x, y, t_1 be in this order in P_1 and let $P_3 := P_1(s_1, x) \cup \{h\} \cup P_1(y, t_1)$. Then $(P_3, P_2) \in J(G)$. Thus $\partial(x, y) = L_2$ or $|\{x, y\} \cap X_i| = 1$ $(i = 1,2)$.

(5.6) *If* $Y \subseteq W - T$, *then* Y *is not a* k*-set.*

Proof. Assume that $Y \subseteq W - T$ is a minimal k-set. For some $\{x_1, x_2, x_3\} \subseteq Y$, there are $g \in \partial(x_1, x_2)$ and $h \in \partial(x_2, x_3)$. By Lemma 5, g and h are contained in no k-cut. Thus by (5.5), $e(x_1, x_2) = e(x_2, x_3) = \alpha$. We may let $x_i \in X_i$ $(i = 1,2)$, otherwise $\partial(x_1, x_2) = L_2$ and $x_3 \in X_1$ by (5.5) and we can take $\{x_3, x_2\}$ instead of $\{x_1, x_2\}$. By $e(x_1, \overline{Y}) \leq \alpha$ and by (5.4), for some $x_4 \in X_1 \cap Y - x_1$, $e(x_1, x_4) > 0$, contrary to (5.5).

(5.7) $e(X_i) \geq k + 1$ $(i = 1,2)$.

Proof. Assume that $e(X_1) = k$ or $e(X_2) = k$.

Case 1. $e(X_2) = k$.

There is $P_2 \in \mathcal{P}(G_2, s_2, t_2, L_2^-, W)$ with $\lambda(s_2, \overline{X}_2; G_2 - E(P_2)) = k - 1$ by

Theorem 4(3). Then for some $g \in \partial(X_2)$, $\lambda(s_2, \overline{X}_2; G_2 - E(P_2) - g) = k-1$. Let $V(g) \cap \overline{X}_2 = \{u\}$. If $u \notin X_1$, then by Lemma 6(7), there is $P_1 \in \mathcal{P}(G/X_2 - g, s_1, t_1, L_1^-, W)$ with $V(P_1) \subseteq X_1$. $\lambda(G - E(P_1 \cup P_2)) \geq k-2$ by Lemma 12(1) and thus $(P_1, P_2) \in J(G)$. If $u \in X_1$ and $e(u, \overline{X}_1) \geq \alpha$, then by Lemma 6(8), there is $P_1 \in \mathcal{P}(G/X_2 - g, s_1, t_1, L_1^-, W)$ with $V(P_1) \subseteq X_1$, and thus $(P_1, P_2) \in J(G)$. If $u \in X_1$ and $e(u, \overline{X}_1) < \alpha$, then G/X_2 does not contain a k-set $Y \subseteq V(G/X_2) - \{s_1, t_1, X_2\}$ by (5.6), and so by Lemma 11(2), there is $P_1 \in \mathcal{P}(G/X_2 - g, s_1, t_1, L_1^-, W - u)$ such that $V(P_1) \subseteq X_1$ and $e(u; G/X_2 - g - E(P_1)) \geq k-3$. By Lemma 12(2), $\lambda(G - E(P_1 \cup P_2)) \geq k-2$, and thus $(P_1, P_2) \in J(G)$.

Case 2. $e(X_1) = k$ and $e(X_2) > k$.

By $V(G) \neq X_1 \cup X_2$, for some $u \in X_1$ and $v \in \overline{X}_1 \cap \overline{X}_2$, there is $g \in \partial(u, v)$. By Lemma 6(7), there is $P_2 \in \mathcal{P}(G/X_1 - g, s_2, t_2, L_2^-, W)$ with $V(P_2) \subseteq X_2$. By (5.6), G_1 contains no k-set $Y \subseteq V(G_1) - \{s_1, t_1, \overline{X}_1\}$. Thus by Lemma 11(1), there is $P_1 \in \mathcal{P}(G_1 - g, s_1, t_1, L_1^-, W - u)$ such that $\lambda(\overline{X}_1, x; G_1 - g - E(P_1)) \geq k-1$ for $x = s_1$ or t_1 and $e(u; G_1 - g - E(P_1)) \geq k-3$. If $e(u; G_1 - g - E(P_1)) \geq k-2$, then $\lambda(G_1 - g - E(P_1)) \geq k-2$, and so $\lambda(G - E(P_1 \cup P_2)) \geq k-2$ by Lemma 3. If $e(u; G_1 - g - E(P_1)) = k-3$, then by it is even and by Lemma 3, $\lambda(W - u; G - g - E(P_1 \cup P_2)) \geq k-2$, and so $\lambda(G - E(P_1 \cup P_2)) \geq k-2$. Thus $(P_1, P_2) \in J(G)$.

(5.8) $|L_i| \leq \alpha - 1$ $(i = 1, 2)$.

Proof. Otherwise $|L_2| = \alpha$ by Lemma 7(1). Then $e(X_2) \leq 2k - 1 - 2|L_2| = 2k - 1 - 2\alpha = k$, contrary to (5.7).

(5.9) *If Z is a k-set, then $|Z \cap \{s_1, t_1\}| = |Z \cap \{s_2, t_2\}| = 1$ or $X_1 \subset Z \subset \overline{X}_2$ or $X_2 \subset Z \subset \overline{X}_1$.*

Proof. Case 1. $|Z \cap T| \leq 1$.

We choose minimal Z with this property. For some $\{x, y\} \subseteq Z - T$, $e(x, y) > 0$. By Lemma 5, (5.5)and (5.8), $e(x, y) = \alpha$ and $|\{x, y\} \cap X_i| = 1$ $(i = 1, 2)$. $Z \cap T \cap X_i = \emptyset$ (i=1 or 2), say for $i = 2$. Then $e(X_2 - Z) > e(X_2)$ by (5.2), and so by (2.1), $e(Z - X_2) \leq e(Z) + e(X_2) - e(X_2 - Z) \leq k-1$, a contradiction.

Case 2. $|Z \cap T| = 2$.

If $Z \cap T = \{s_1, t_1\}$, then $Z - X_1 \neq \emptyset$ by (5.2) and (5.7). $X_1 \subseteq Z$, otherwise, $e(X_1 \cap Z) > e(X_1)$ by (5.2), and so by (2.2), $e(\overline{X}_1 \cap \overline{Z}) \leq e(X_1) + e(Z) - e(X_1 \cap Z) \leq k-1$, a contradiction. Similarly we have $X_2 \subseteq \overline{Z}$ and thus the result follows.

By (5.9), we have

(5.10) *If $x \in X_i - T$, $f \in \partial(s_i, x)$ and $g \in \partial(t_i, x)$, then there is no k-cut containing $\{f, g\}$ $(i = 1, 2)$.*

(5.11) *If $x_i \in X_i - T$, $\partial(s_i, x_i) - L_i \neq \emptyset$ and $\partial(t_i, x_i) - L_i \neq \emptyset$ $(i = 1, 2)$, then $e(x_1, x_2) = \alpha$.*

Proof. Let $f_i \in \partial(s_i, x_i) - L_i$ and $g_i \in \partial(t_i, x_i) - L_i$ $(i = 1, 2)$. By $J(G) = \emptyset$, we have $(\{f_1, g_1\}, \{f_2, g_2\}) \notin J(G)$, and so $\lambda(G - \{f_1, g_1, f_2, g_2\}) \leq k - 3$. By (5.10), there is a $(k+1)$-set Z with $\{f_1, g_1, f_2, g_2\} \subseteq \partial(Z)$. Let $x_1 \in Z$. By (5.2), $e(X_1 - Z) \geq e(X_1) + 1$, and so by (2.1), $e(Z - X_1) \leq e(X_1) + e(Z) - e(X_1 - Z) \leq k$. By (5.10), $Z - X_1$ is not a k-set, and so $|Z - X_1| = 1$ and $Z - X_1 = \{x_2\}$. Similarly we have $Z - X_2 = \{x_1\}$. Thus $Z = \{x_1, x_2\}$ and $e(x_1, x_2) = \alpha$.

(5.12) *If $e(x, y) = 0$ for each $\{x, y\} \subseteq X_i - T$ $(i = 1, 2)$, then $|X_1| \geq 6$ or $|X_2| \geq 6$.*

Proof. Assume that $|X_i| \leq 5$ $(i = 1, 2)$.
Case 1. $\partial(s_i, t_i) - L_i \neq \emptyset$ ($i = 1$ or 2), say for i=1.
Let $f \in \partial(s_1, t_1) - L_1$. If there is $g \in \partial(s_2, t_2) - L_2$, then $(f, g) \in J(G)$. Thus $\partial(s_2, t_2) \subseteq L_2$ and $|X_2| \geq 3$ by Lemma 7(4). There are $g \in \partial(s_2, x) - L_2$ and $h \in \partial(t_2, x) - L_2$ for some $x \in X_2 - T$ by Lemmas 8(1), 9(1) and 10(1). $\lambda(G - \{f, g, h\}) \geq k - 2$ by (5.10) and thus $(f, \{g, h\}) \in J(G)$.
Case 2. $\partial(s_i, t_i) \subseteq L_i$ $(i = 1, 2)$.
$|X_i| \geq 3$ $(i = 1, 2)$ by Lemma 7(4). $\partial(s_i, x_i) - L_i \neq \emptyset$ and $\partial(t_i, x_i) - L_i \neq \emptyset$ for some $x_i \in X_i - T$ $(i = 1, 2)$ by Lemmas 8(1), 9(1) and 10(1). Then $e(x_1, x_2) = \alpha$ by (5.11). By Lemma 8(2), $|X_1| \geq 4$. $\partial(s_1, x_3) - L_1 \neq \emptyset$ and $\partial(t_1, x_3) - L_1 \neq \emptyset$ for some $x_3 \in X_1 - T - x_1$ by Lemmas 9(2) and 10(2). Then $e(x_3, x_2) = \alpha$ by (5.11), contrary to (5.4).

(5.13) *If $X_1 \subseteq Y \subseteq \overline{X}_2$, then $e(Y) \geq k + 1$.*

Proof. Otherwise $e(Y) = k$. Let Y_1 and Y_2 be minimal k-sets with $X_1 \subseteq Y_1 \subseteq \overline{X}_2$ and $X_2 \subseteq Y_2 \subseteq \overline{X}_1$ (possibly $Y_1 = \overline{Y}_2$). Then $Y_1 \cap Y_2 = \emptyset$ by Lemma 4. By (5.9), if $Z \subsetneqq Y_i$, then Z is not a k-set, and so Y_i is a minimal k-set $(i = 1, 2)$. By Lemma 5, (5.5) and (5.8), $e(x, y) = 0$ for each $\{x, y\} \subseteq Y_i - T$ $(i = 1, 2)$. Thus $|Y_i| \leq 5$ $(i = 1, 2)$, contrary to (5.12).

(5.14) *G contains a k-set.*

Proof. Otherwise $e(x, y) = 0$ for each $\{x, y\} \subseteq X_i - T$ $(i = 1, 2)$ by (5.5) and (5.8). By (5.12), $|X_i| \geq 6$ ($i = 1$ or 2). Then $2k - 1 \geq e(X_i) = k|X_i| - 2|E(X_i)| \geq k|X_i| - 2(e(s_i) + e(t_i)) \geq 2k$, a contradiction.

By (5.14), let Z_1 be a minimal k-set. We may let $Z_1 \cap T = \{s_1, s_2\}$

by (5.9) and (5.13). Let Z_2 be a minimal k-set with $Z_2 \subsetneq \overline{Z}_1$. Then $Z_2 \cap T = \{t_1, t_2\}$. By Lemma 5, (5.5) and (5.8), we have

(5.15) *If* $\{x, y\} \subseteq X_i \cap Z_j - T$, *then* $e(x, y) = 0$ $(i, j = 1, 2)$.

(5.16) $e(X_i \cap Z_j) \leq 2k - 2 - 2|L \cap E(X_i \cap Z_j)| + \varepsilon_i$ *and* $|X_i \cap Z_j| \leq 3$ $(i, j = 1, 2)$.

Proof. By (2.2), $e(X_i \cap Z_j) \leq e(X_i) + e(Z_j) - e(\overline{X}_i \cap \overline{Z}_j) \leq e(X_i) \leq 2k - 2 - 2|L_i| + \varepsilon_i \leq 2k - 2 - 2|L \cap E(X_i \cap Z_j)| + \varepsilon_i$. If $|X_2 \cap Z_1| \geq 4$, then by (5.15), we have $e(X_2 \cap Z_1) \geq k|X_2 \cap Z_1| - 2e(s_2) \geq 2k$, a contradiction.

(5.17) *If* $Z_0 := \overline{Z}_1 \cap \overline{Z}_2 \neq \emptyset$, *then* $|Z_0| = 2$, $|Z_0 \cap X_i| = 1$ $(i = 1, 2)$, *and for* $Z_0 \cap X_i =: \{u_i\}$ $(i = 1, 2)$, $e(u_1, u_2) = \alpha$ *and* $e(u_i, X_i - u_i) = \alpha + 1$.

Proof. Let $Y_0 := Z_1$, Y_i be a minimal k-set with $Y_{i-1} \subsetneq Y_i \subseteq \overline{Z}_2$ $(1 \leq i \leq n)$ and let $Y_n = \overline{Z}_2$. For $1 \leq j \leq n$, let $x_j \in Y_j - Y_{j-1}$. By $e(x_j, Y_{j-1}) \leq \alpha$ and $e(x_j, \overline{Y}_j) \leq \alpha$, for some $y_j \in Y_j - Y_{j-1}$, $e(x_j, y_j) > 0$. Then $|X_i \cap \{x_j, y_j\}| = 1$ $(i = 1, 2)$ and $e(x_j, y_j) = \alpha$ by (5.5) and (5.8). Thus and by (5.4), $Z_0 \subseteq X_1 \cup X_2$, $|Z_0 \cap X_1| = |Z_0 \cap X_2|$ and $e(Z_0 \cap X_1, Z_0 \cap X_2) = \alpha |Z_0 \cap X_1|$. Since $e(Z_i) = e(Z_i \cap X_1) + e(Z_i - X_1) - 2e(X_1 \cap Z_i, Z_i - X_1)$, we have $2e(X_1 \cap Z_i, Z_i - X_1) \geq k$ and $e(X_1 \cap Z_i, Z_i - X_1) \geq \alpha + 1$ $(i = 1, 2)$. Then

$$2k - 2 \geq e(X_1) \geq e(X_1 \cap Z_1, Z_1 - X_1) + e(X_1 \cap Z_2, Z_2 - X_1) + e(X_1 \cap Z_0, X_2 \cap Z_0) \geq 2(\alpha + 1) + \alpha |Z_0 \cap X_1|,$$

and so $|Z_0 \cap X_1| = |Z_0 \cap X_2| = 1$. Let $Z_0 \cap X_i =: \{u_i\}$ $(i = 1, 2)$. By (5.4), $e(u_i, X_i - u_i) = \alpha + 1$ $(i = 1, 2)$.

(5.18) *If for simple paths* $P_1[s_1, t_1]$ *and* $P_2[s_2, t_2]$, $V(P_i) \subseteq X_i$, $L_i \cap E(P_i) = \emptyset$ *and* $|E(P_i) \cap \partial(Z_j)| = 1$ $(i, j = 1, 2)$, *then for some* $l = 0, 1$ *or* 2 *and some* $x_i \in V(P_i) \cap Z_l - T$ $(i = 1, 2)$, $e(x_1, x_2) = \alpha$.

Proof. $\lambda(G/\overline{Z}_1 - E(P_1 \cup P_2)) \leq k - 3$, $\lambda(G/\overline{Z}_2 - E(P_1 \cup P_2)) \leq k - 3$ or $\lambda(G/Z_1/Z_2 - E(P_1 \cup P_2)) \leq k - 3$, otherwise $\lambda(G - E(P_1 \cup P_2)) \geq k - 2$ and $(P_1, P_2) \in J(G)$. If $\lambda(G/\overline{Z}_1 - E(P_1 \cup P_2)) \leq k - 3$, then Z_1 contains $x_i \in Z_1 \cap X_i \cap V(P_i) - T$ $(i = 1, 2)$ and a $(k+1)$-set Y with $|\partial(Y) \cap E(P_1 \cup P_2)| = 4$ by (5.15). We may let $x_1 \in Y \subseteq Z_1 - s_1$. By (5.2), $e(X_1 - Y) > e(X_1)$. By (2.1), $e(Y - X_1) \leq e(X_1) + e(Y) - e(X_1 - Y) \leq k$, and so $e(Y - X_1) = \{x_2\}$. Similarly $e(Y - X_2) = \{x_1\}$ and we have $e(x_1, x_2) = \alpha$. For other cases, the proof is similar.

(5.19) $|E(X_i - Z_0)| \geq |L_i| + (|X_i - Z_0| - 2)\alpha + (|X_i - Z_0| - 1 - \varepsilon_i)/2$ $(i = 1, 2)$.

Proof. By Lemma 7(2), $|E(X_i)| \geq |L_i| + (|X_i| - 2)\alpha + (|X_i| - \varepsilon_i)/2$ $(i = 1, 2)$.

Thus we may let $Z_0 \neq \emptyset$. By (5.17), $|E(X_i)| = |E(X_i - Z_0)| + e(u_i, X_i - u_i) = |E(X_i - Z_0)| + \alpha + 1$ and $|X_i| = |X_i - Z_0| + 1$, and so we have

$$|E(X_i - Z_0)| + \alpha + 1 \geq |L_i| + (|X_i - Z_0| + 1 - 2)\alpha + (|X_i - Z_0| + 1 - \varepsilon_i)/2.$$

Thus (5.19) follows.

(5.20) *If $v \in X_i \cap Z_j \cap T$ and $x \in X_i \cap Z_j - T$, then $\partial(v, x) - L_i \neq \emptyset$ $(i, j = 1, 2)$.*

Proof. If $|X_i \cap Z_j| = 2$, then the result follows by (5.16) and Lemma 7(4). By (5.16), let $X_i \cap Z_j = \{v, x, y\}$. If $\partial(v, x) \subseteq L$, then $|E(X_i \cap Z_j)| \leq |L \cap E(X_i \cap Z_j)| + e(v, y) \leq |L \cap E(X_i \cap Z_j)| + \alpha$, contrary to Lemma 7(2).

(5.21) *If $|X_2 - Z_0| \geq 3$, then $G - L$ contains paths $P_1[s_1, t_1]$ and $P_2[s_2, t_2]$ such that for some $x_i \in X_i \cap Z_1 - T$ and $y_i \in X_i \cap Z_2 - T$, $V(P_i) \subseteq \{s_i, x_i, y_i, t_i\}$ (possibly $X_i \cap Z_j - T = \emptyset$) and in this order in P_i $(i = 1, 2)$.*

Proof. By (5.20), it suffices to prove that $\partial(X_i \cap Z_1, X_i \cap Z_2) - L_i \neq \emptyset$ $(i = 1, 2)$. If $\partial(X_i \cap Z_1, X_i \cap Z_2) \subseteq L_i$, then

$$\begin{aligned}|E(X_i - Z_0)| &\leq |L_i| + e(s_i, X_i \cap Z_1 - s_i) + e(t_i, X_i \cap Z_2 - t_i)\\ &\leq |L_i| + (|X_i \cap Z_1| - 1)\alpha + (|X_i \cap Z_2| - 1)\alpha = |L_i| + (|X_i - Z_0| - 2)\alpha.\end{aligned}$$

On the other hand, by (5.19), $|E(X_i - Z_0)| \geq |L_i| + (|X_i - Z_0| - 2)\alpha + 1/2$, a contradiction.

In what follows, let

$P_1^*[s_1, t_1]$ be a path given in (5.21) with its length minimized.

$P_2^*[s_2, t_2]$ be a path given in (5.21) with its length minimized if $|X_2 - Z_0| \geq 3$.

(5.22) $|X_2 - Z_0| \geq 3$.

Proof. Otherwise $X_2 - Z_0 = \{s_2, t_2\}$. By Lemmas 7(4) and 8(1), either there is $f \in \partial(s_2, t_2) - L_2$ or $X_2 \cap Z_0 = \{u_2\}$ and there are $g \in \partial(s_2, u_2) - L_2$ and $h \in \partial(t_2, u_2) - L_2$. Let $P_2 := \{f\}$ or $\{g, h\}$. For P_1^* and P_2, we have a contradiction by (5.18).

By (5.18),(5.21) and (5.22), we may let $x_i \in V(P_i^*) \cap Z_1 - T$ $(i = 1, 2)$ and $e(x_1, x_2) = \alpha$. Then $\partial(s_i, t_i) \subseteq L_i$ by the minimality of $|E(P_i^*)|$ $(i = 1, 2)$.

(5.23) $\partial(X_1 \cap Z_1 - x_1, X_1 \cap Z_2) - L_1 \neq \emptyset$.

Proof. Otherwise

$$|E(X_1 - Z_0)| \leq |L_1| + e(x_1, X_1 - x_1) + e(s_1, X_1 \cap Z_1 - \{s_1, x_1\}) + e(t_1,$$

$X_1 \cap Z_2 - t_1) \le |L_1| + (\alpha+1) + \alpha(|X_1 - Z_0| - 3) = |L_1| + (|X_1 - Z_0| - 2)\alpha + 1,$

By (5.19), $|X_1 - Z_0| = 3$. By Lemma 8(2), $X_1 \cap Z_0 = \{u_1\}$ and by Lemma 9(2), there are $f \in \partial(s_1, u_1) - L_1$ and $g \in \partial(t_1, u_1) - L_1$. For $\{f, g\}$ and P_2^*, we have a contradiction by (5.18).

By (5.20) and (5.23), $G - L$ contains a simple path $P_1[s_1, t_1]$ such that $V(P_1) \subseteq X_1 - Z_0 - x_1$. By (5.18) for P_1 and P_2^*, there are $z_1 \in V(P_1) \cap Z_2 - T$ and $y_2 \in V(P_2^*) \cap Z_2 - T$ and $e(z_1, y_2) = \alpha$.

(5.24) $\partial(X_2 \cap Z_1 - x_2, X_2 \cap Z_2 - y_2) - L_2 \neq \emptyset$.

Proof. Otherwise

$$|E(X_2 - Z_0)| \le |L_2| + e(x_2, y_2) + e(x_2, X_2 - Z_0 - \{x_2, y_2\}) + e(y_2, X_2 - Z_0 - \{x_2, y_2\}) + e(s_2, X_2 \cap Z_1 - \{s_2, x_2\}) + e(t_2, X_2 \cap Z_2 - \{t_2, y_2\}) \le |L_2| + e(x_2, y_2) + 2(\alpha + 1 - e(x_2, y_2)) + (|X_2 - Z_0| - 4)\alpha = |L_2| + 2\alpha + 2 - e(x_2, y_2) + (|X_2 - Z_0| - 4)\alpha \le |L_2| + (|X_2 - Z_0| - 2)\alpha + 1.$$

By (5.19), $|X_2 - Z_0| = 4$. If $Z_0 = \emptyset$, then by Lemma 9(2), $G - L$ contains a path $P_2[s_2, t_2]$ with $V(P_2) = \{s_2, y_2, t_2\}$, contrary to the minimality of $|E(P_2^*)|$. Thus $Z_0 \neq \emptyset$. If there are $f \in \partial(s_2, u_2) - L_2$ and $g \in \partial(t_2, u_2) - L_2$, then $(P_1^*, \{f, g\}) \in J(G)$. Thus let $\partial(s_2, u_2) \subseteq L_2$. Then

$$|E(X_2)| \le |L_2| + e(x_2, y_2) + e(x_2, X_2 - \{x_2, y_2\}) + e(y_2, X_2 - \{x_2, y_2\}) + e(u_2, t_2) \le |L_2| + e(x_2, y_2) + 2(\alpha + 1 - e(x_2, y_2)) + \alpha \le |L_2| + 3\alpha + 1,$$

contrary to Lemma 7(2).

By (5.24) and by the minimality of $|E(P_2^*)|$, $G - L$ contains a path $P_2[s_2, t_2]$ such that $V(P_2) = \{s_2, w_2, z_2, t_2\}$ for $w_2 \in X_2 \cap Z_1 - \{s_2, x_2\}$ and $z_2 \in X_2 \cap Z_2 - \{t_2, y_2\}$. By (5.18) for P_1^* and P_2, we have $y_1 \in V(P_1^*) \cap Z_2 - \{t_1, z_1\}$ and $e(y_1, z_2) = \alpha$. By (5.18) for P_1 and P_2, there is $w_1 \in V(P_1) \cap Z_1 - \{s_1, x_1\}$ and $e(w_1, w_2) = \alpha$. Then $e(X_1) \ge e(x_1, x_2) + e(z_1, y_2) + e(y_1, z_2) + e(w_1, w_2) \ge 4\alpha = 2k - 2$, and so $L_1 = \emptyset$, $Z_0 = \emptyset$ and $e(s_1, \overline{X}_1) = 0$. $e(s_1, X_1 \cap Z_2) = 0$ by the minimality of $|E(P_1^*)|$ and $e(s_1, X_1 \cap Z_1) \le 2\alpha$, then $e(s_1) \le k - 1$, a contradiction.

6 Proof of Corollary 3

Let $X := \{s_1, t_1, v\}$. Then $e(X) \le 3k - 2e(v, \{s_1, t_1\}) \le 3k - (k+3) \le 2k - 3$. The result follows by Theorem 1.

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