The Triangle Intersection Problem for Kite Systems

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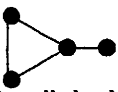
Abstract

The graph  is called a kite and the decomposition of K_n into kites is called a kite system. Such systems exists precisely when $n \equiv 0$ or 1 (mod 8). In 1975 C. C. Lindner and A. Rosa solved the intersection problem for Steiner triple systems. The object of this paper is to give a complete solution to the triangle intersection problem for kite systems(= how many triangles can two kite systems of order n have in common). We show that if $x \in \{0, 1, 2, ..., n(n-1)/8\}$, then there exists a pair of kite systems of order n having exactly x triangles in common.

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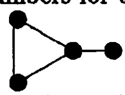
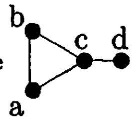
1 Introduction

A Steiner triple system (or simply triple system) of order n is a pair (X, T), where T is a collection of edge disjoint triangles (or triples) which partitions the edge set of K_n (= the complete graph on n vertices) with vertex set X. It is well-known [3] that the spectrum for triple systems (=the set of all n such that a triple system of order n exists) is precisely the set of all $n \equiv 1$ or 3(mod 6) and that if (X, T) is a triple system of order n, $|T| = n(n-1)/6$.

The Intersection problem for triple systems asks for which pairs (n, k) does there exists two triple systems of order n having exactly k triples in common. A complete solution to this problem was given in 1975 by C.C. Lindner and A.Rosa [5] who showed that, $J(3) = \{1\}$, $J(7) = \{0, 1, 3, 7\}$, $J(9) = \{0, 1, 2, 3, 4, 5, 6, 12\}$ and $J(n) = \{0, 1, 2, ..., x = n(n-1)/6\} \backslash \{x - 1, x - 2, x - 3, x - 5\}$ for all $n \geq 13$; where $J(n)$ denotes the set of all intersection numbers for triple systems of order n.

The graph  is called a kite and, not too surprisingly, a kite system of order n is a pair (X, K), where K is a collection of edge disjoint kites which partitions the edge set of K_n with vertex set X. In this case $|K| = n(n-1)/8$. The spectrum for kite system was determined in 1977 by J.C Bermont and J Schönheim [1]who showed that a kite system exists if and only if $n \equiv 0$ or 1(mod8).

In what follows we'll denote the kite 

by (a,b,c)-d or (b,a,c)-d.

Example 1.1 (*kite system of order 8*) $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $K = \{(1, 2, 4) - 5, (2, 3, 5) - 6, (3, 4, 6) - 7, (4, 5, 7) - 8, (8, 5, 6) - 1, (1, 6, 7) - 3, (8, 2, 7) - 3\}$.

Now given a kite system (X, K) of order n, if we delete the "tails" the result is a collection of edge disjoint triples $T(K)$, and of course $(X, T(K))$ is a partial triple system. For example if we delete the tails in Example1.1 the resulting collection of triples is $T(K) = \{(1, 2, 4), (2, 3, 5), (3, 4, 6), (4, 5, 7), (8, 5, 6), (1, 6, 7), (8, 2, 7)\}$. We remark that the partial triple system $(X, T(K))$ does not necessarily have order n. For example, if $K = \{(1, 2, 4)-8, (2, 3, 5)-8, (3, 4, 6)-8, (4, 5, 7)-8, (5, 6, 1)-8, (6, 7, 2)-8, (7, 1, 3)-8\}$, $(X \backslash \{8\}, T(K))$ is a partial triple system of order 7.

The triple intersection problem for kite systems asks for which pairs (n, k) does there exist a pair of kite systems (X, K_1) and (X, K_2) of order n such that $|T(K_1) \cap T(K_2)| = k$. The object of this paper is a complete solution of this problem. In particular we show that $J(n) =$

$\{0,1,2,...,n(n-1)/8\}$ for all $n \equiv 0$ or 1 (mod 8), without any exceptions, where $J(n)$ denotes the set of triple intersection numbers for kite systems of order n.

It is worth noting here that E. J. Billington and D. L. Kreher solved the intersection problem for kites [2]. What we are doing here is totally different.

We will break the construction into two sections: a section giving five examples followed by a section giving a recursive construction for all $n \equiv 0$ or 1 (mod 8) ≥ 24.

2 Examples

In this section we give solutions of the triple intersection problem for kite systems for orders $n = 8, 9, 16$ and 17, followed by an example necessary for the recursive construction in Section 3.

Example 2.1 (n=8) *Define 8 kite systems of order 8 as follows :*

$K_0 = \{(3,4,1)\text{–}5, (4,5,2)\text{–}6, (5,6,3)\text{–}7, (6,7,4)\text{–}8, (5,7,8)\text{–}2, (1,6,8)\text{–}3, (1,7,2)\text{–}3\}$, $K_1 = \{(2,4,1)\text{–}5, (5,6,3)\text{–}7, (6,7,4)\text{–}3, (6,8,1)\text{–}3, (2,5,7)\text{–}1, (4,5,8)\text{–}7, (3,8,2)\text{–}6\}$, $K_2 = \{(2,4,1)\text{–}5, (3,5,2)\text{–}6, (4,6,7)\text{–}2, (7,8,5)\text{–}4, (1,8,6)\text{–}5, (1,7,3)\text{–}6, (3,4,8)\text{–}2\}$, $K_3 = \{(2,4,1)\text{–}5, (3,5,2)\text{–}8, (3,6,4)\text{–}8, (7,8,5)\text{–}4, (1,6,8)\text{–}3, (1,3,7)\text{–}4, (2,7,6)\text{–}5\}$, $K_4 = \{(1,2,4)\text{–}8, (2,3,5)\text{–}8, (3,4,6)\text{–}8, (4,5,7)\text{–}1, (5,6,1)\text{–}8, (6,7,2)\text{–}8, (7,8,3)\text{–}1\}$ $K_5 = \{(2,4,1)\text{–}5, (2,7,3)-5, (7,4,5)-2, (5,6,8)-2, (1,7,6)-2, (1,3,8)-7, (3,6,4)-8\}$, $K_6 = \{(2,4,1)-5, (3,5,2)-6, (4,6,3)-7, (5,7,4)-8, (5,6,8)-7, (1,6,7)-2, (1,3,8)-2\}$, *and* $K_7 = \{(2,4,1)-5, (3,5,2)-6, (4,6,3)-7, (5,7,4)-8, (5,6,8)-1, (6,7,1)-3, (2,7,8)-3\}$.

Then $|T(K_7) \cap T(K_i)| = i$, *for all* $i = 0,1,2,3,4,5,6,7$, *and so* $J(8) = \{0,1,2,3,4,5,6,7\}$

Example 2.2 (n=9). *Define 10 kite systems of order 9 as follows:*

$K_0 = \{(1,3,4)\text{–}8, (2,4,5)\text{–}9, (3,5,6)\text{–}1, (4,6,7)\text{–}2, (5,7,8)\text{–}3, (6,8,9)\text{–}4, (7,9,1)\text{–}5, (1,8,2)\text{–}6, (2,9,3)\text{–}7\}$, $K_1 = \{(1,2,4)\text{–}8, (3,5,6)\text{–}1, (4,6,7)\text{–}2, (5,7,8)-2, (8,9,6)-2, (7,9,1)-5, (2,9,3)-7, (4,9,5)-2, (1,8,3)-4\}$, $K_2 = \{(1,2,4)-8, (2,3,5)-9, (4,7,6)-2, (5,8,7)-2, (6,9,8)-2, (1,7,9)-2, (4,9,3)\text{–}6, (1,8,3)\text{–}7, (1,6,5)\text{–}4\}$, $K_3 = \{(1,2,4)\text{–}8, (2,3,5)\text{–}6, (3,4,6)\text{–}1, 2\text{–}(5,7,8)\text{–}2, (6,8,9)\text{–}2, (7,9,1)\text{–}5, (5,9,4)\text{–}7, (2,6,7)\text{–}3, (1,8,3)\text{–}9\}$, $K_4 = \{(1,2,4) - 8, (2,3,5) - 9, (3,4,6) - 1, 9 - (5,7,4) - 9, (6,8,9) - 2, (7,9,1)\text{–}3, (1,5,8)\text{–}2, (7,8,3)\text{–}9, (2,7,6)\text{–}5\}$, $K_5 = \{(1,2,4)\text{–}8, (2,3,5)\text{–}9, (3,4,6) - 1, (4,5,7) - 3, (5,6,8) - 7, (7,9,1) - 5, (1,8,3) - 9, (2,8,9) - 4, (2,7,6)\text{–}9\}$, $K_6 = \{(1,2,4)\text{–}8, (3,5,2)\text{–}6, (3,4,6)\text{–}1, (5,7,4)\text{–}9, (5,6,8)\text{–}3, (6,7,9)\text{–}2, (1,7,3)\text{–}9, (5,9,1)\text{–}8, (2,7,8)\text{–}9\}$, $K_7 = \{(1,2,4)\text{–}8, (3,5,2)\text{–}6, (3,4,6) - 1, (4,5,7) - 2, (5,6,8) - 2, (6,7,9) - 4, (1,8,7) - 3, (3,8,9) -$

$2, (5,9,1)-3\}$, $K_8 = \{(1,2,4)-8, (2,5,3)-7, (3,4,6)-1, (4,5,7)-2, (5,6,8)-3, (6,7,9)-4, (7,8,1)-3, (8,9,2)-6, (1,5,9)-3\}$, *and* $K_9 = \{(1,2,4)-8, (2,3,5)-9, (3,4,6)-1, (4,5,7)-2, (5,6,8)-3, (6,7,9)-4, (7,8,1)-5, (8,9,2)-6, (1,9,3)-7\}$.

Then $|T(K_9) \cap T(K_i)| = i$, *for all* $i = 0,1,2,3,4,5,6,7,8,9$, *and so* $J(9) = \{0,1,2,3,4,5,6,7,8,9\}$

We will need the following definition before giving the next example. A trade of partial kite systems is a pair (B_1, B_2) where B_1 and B_2 are partial kite systems using exactly the same edges. Clearly if (X, K) is a kite system (or partial system) and (B_1, B_2) is a trade with $B_1 \subseteq K$, then $(X, (K \backslash B_1) \cup B_2)$ is a kite system (or a partial kite system).

Example 2.3 *(n=16) Define 6 non-intersecting trades* $C_i = (B_i, B_i^*)$, $1 \le i \le 6$, *as follows:*
$B_1 = \{(13,2,14)-16, (4,3,7)-16, (7,14,8)-16\}$ $B_1^* = \{(13,2,14)-8, (4,3,7)-8, (7,14,16)-8\}$ $B_2 = \{(3,6,5)-16, (3,13,15)-16, (5,15,11)-16\}$ $B_2^* = \{(3,6,5)-11, (3,13,15)-11, (5,15,16)-11\}$ $B_3 = \{(6,14,9)-16, (4,15,10)-16, (9,10,3)-16\}$ $B_3^* = \{(6,14,9)-3, (4,15,10)-3, (9,10,16)-3\}$ $B_4 = \{(8,12,4)-16, (1,7,6)-16, (4,6,2)-16\}$ $B_4^* = \{(8,12,4)-2, (1,7,6)-2, (4,6,16)-2\}$ $B_5 = \{(2,15,12)-16, (6,11,13)-16, (12,13,1)-16\}$ $B_5^* = \{(2,15,12)-1, (6,11,13)-1, (12,13,16)-1\}$ $B_6 = \{(2,3,1)-14, (5,10,14)-15, (9,7,15)-1, (1,4,5)-7, (8,10,2)-5, (11,12,7)-2, (2,9,11)-3, (12,14,3)-8, (5,13,8)-11, (1,8,9)-13, (11,14,4)-9, (7,10,13)-4, (1,11,10)-12, (5,9,12)-6, (8,15,6)-10\}$ $B_6^* = \{(14,15,1)-2, (14,10,5)-9, (9,15,7)-11, (1,4,5)-13, (2,10,8)-13, (2,5,7)-12, (8,11,3)-1, (9,11,2)-3, (3,14,12)-11, (4,13,9)-12, (8,9,1)-11, (4,14,11)-1, (7,13,10)-11, (6,15,8)-5, (6,10,12)-5\}$.

Observe that $|T(B_6) \cap T(B_6^*)| = 10$ *and* $|T(B_i) \cap T(B_i^*)| = 2$ *for* $i = 1,2,3,4,5$. *Now define* $K_{30} = \cup_{i=1}^{6} B_i$, $K_{29} = B_1^* \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$, $K_{28} = B_1^* \cup B_2^* \cup B_3 \cup B_4 \cup B_5 \cup B_6$, $K_{27} = B_1^* \cup B_2^* \cup B_3^* \cup B_4 \cup B_5 \cup B_6$, $K_{26} = B_1^* \cup B_2^* \cup B_3^* \cup B_4^* \cup B_5 \cup B_6$, $K_{25} = B_1^* \cup B_2^* \cup B_3^* \cup B_4^* \cup B_5^* \cup B_6$, $K_{24} = B_1^* \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6^*$, $K_{23} = B_1^* \cup B_2^* \cup B_3 \cup B_4 \cup B_5 \cup B_6^*$, $K_{22} = B_1^* \cup B_2^* \cup B_3^* \cup B_4 \cup B_5 \cup B_6^*$, $K_{21} = B_1^* \cup B_2^* \cup B_3^* \cup B_4^* \cup B_5 \cup B_6^*$, $K_{20} = \cup_{i=1}^{6} B_i^*$

Let $(X, K)\alpha_i$ *be the kite system obtained from* (X, K) *by applying the permutation* α_i *to each block of* K *where:*
$\alpha_1 = (1\ 12)$, $\alpha_2 = (8\ 9\ 10)$, $\alpha_3 = (1\ 9\ 10)$, $\alpha_4 = (6\ 7)(8\ 9)(10\ 11)(12\ 13)$, $\alpha_5 = (1\ 9\ 10\ 12)$, $\alpha_6 = (1\ 5\ 9\ 10\ 12)$, $\alpha_7 = (1\ 5\ 6\ 9\ 10\ 11)$, $\alpha_8 = (1\ 2\ 3)(4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15)$, $\alpha_9 = (1\ 5\ 6\ 9\ 10\ 11\ 12)$, $\alpha_{10} = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15)$, $\alpha_{11} = (2\ 16)\alpha_{10}$, $\alpha_{12} = (14\ 16)\alpha_{10}$

Define $K_{19} = (K_{29})\alpha_1$, $K_{18} = (K_{28})\alpha_1$, $K_{17} = (K_{27})\alpha_1$, $K_{16} = (K_{27})\alpha_2$, $K_{15} = (K_{30})\alpha_3$, $K_{14} = (K_{29})\alpha_3$, $K_{13} = (K_{28})\alpha_3$, $K_{12} = (K_{30})\alpha_4$, $K_{11} = (K_{30})\alpha_5$, $K_{10} = (K_{29})\alpha_5$, $K_9 = (K_{28})\alpha_5$, $K_8 = (K_{30})\alpha_6$, $K_7 =$

$(K_{30})\alpha_7$, $K_6 = (K_{29})\alpha_7$, $K_5 = (K_{30})\alpha_8$, $K_4 = (K_{30})\alpha_9$, $K_3 = (K_{29})\alpha_9$ $K_2 = (K_{30})\alpha_{10}$, $K_1 = (K_{30})\alpha_{11}$ $K_0 = (K_{30})\alpha_{12}$.

Then $|T(K_{30}) \cap T(K_i)| = i$, *for all* $0 \leq i \leq 30$, *and so* $J(16) = \{0, 1, 2, ..., 30\}$.

Example 2.4 (n=*17*)

A kite system of order 17 *has* 34 *kites. Let* $C = \{(5+i, 11+i, 1+i) - (6+i) | 1 \leq i \leq 17\}$, $C^* = \{(5+i, 11+i, 1+i) - (6+i) | 1 \leq i \leq 17\}$ *and* $E = \{(2+i, 4+i, 1+i) - (9+i) | 1 \leq i \leq 17\}$ (*sums are computed modulo* 17 *and* 17 = 0). *Then* (C, C^*) *is a trade and* $|T(C) \cap T(C^*)| = 0$. *Moreover each of the following* E_j *forms a trade with* E *such that* $|T(E) \cap T(E_j)| = j$.

$E_0 = \{(3+i, 4+i, 1+i) - (9+i), | 1 \leq i \leq 17\}$, $E_1 = \{(2,4,1) - 16, (5,6,3) - 11, (4,6,7) - 16, (5,7,8) - 16, (8,9,6) - 14, (7,10,9) - 1, (8,11,10) - 2, (11,12,9) - 17, (12,13,10) - 1, (13,14,11) - 2, (12,14,15) - 7, (13,15,16) - 2, (16,17,14) - 5, (17,1,15) - 6, (2,3,17) - 8, (4,13,5) - 2, (4,12,3) - 1\}$, $E_2 = \{(2,4,1) - 9, 10 - (3,11,2) - 10, (2,16,17) - 3, (5,7,8) - 17, (8,9,6) - 3, (9,10,7) - 4, (10,11,8) - 16, (11,12,9) - 17, (12,13,10) - 1, (11,13,14) - 16, (12,15,14) - 17, (13,15,16) - 7, (15,17,1) - 16, (4,12,3) - 1, (4,13,5) - 2, (6,14,5) - 3, (7,15,6) - 4\}$, $E_3 = \{(2,4,1) - 9, (3,5,2) - 16, (4,6,3) - 1, (5,8,7) - 16, (6,8,9) - 17, (7,9,10) - 1, (8,10,11) - 2, (9,11,12) - 3, (12,13,10) - 2, (13,14,11) - 3, (12,15,14) - 17, (13,15,16) - 1, (1,15,17) - 2, (5,13,4) - 12, (5,6,14) - 16, (6,15,7) - 4, (8,16,17) - 3\}$, $E_4 = \{(2,4,1) - 10, (2,5,3) - 17, (4,6,3) - 1, (4,7,5) - 8, (7,9,10) - 2, (8,10,11) - 3, (9,11,12) - 4, (10,12,13) - 5, (13,14,11) - 2, (14,15,12) - 3, (15,16,13) - 4, (14,16,17) - 2, (15,17,1) - 16, (5,14,6) - 9, (7,15,6) - 8, (7,8,16) - 2, (8,17,9) - 1\}$, $E_5 = \{(2,4,1) - 9, (3,5,2) - 10, (4,6,3) - 11, (5,7,4) - 12, (6,8,5) - 13, (7,9,10) - 1, (8,10,11) - 2, (9,11,12) - 3, (10,12,13) - 4, (11,13,14) - 5, (12,15,14) - 6, (13,15,16) - 2, (14,16,17) - 2, (15,17,1) - 3, (7,15,6) - 9, (7,8,16) - 1, (8,9,17) - 3\}$, $E_6 = \{(2,4,1) - 16, (3,5,2) - 10, (4,6,3) - 11, (5,7,4) - 12, (6,8,5) - 13, (7,9,6) - 14, (8,11,10) - 7, (9,12,11) - 2, (10,13,12) - 3, (11,14,13) - 4, (12,15,14) - 5, (13,16,15) - 6, (14,17,16) - 2, (1,15,17) - 2, (8,16,7) - 15, (9,8,17) - 3, (9,10,1) - 3\}$, $E_7 = \{(2,4,1) - 16, (3,5,2) - 16, (4,6,3) - 11, (5,7,4) - 12, (6,8,5) - 13, (7,9,6) - 14, (8,10,7) - 15, (9,11,12) - 3, (10,12,13) - 4, (11,13,14) - 5, (12,14,15) - 6, (13,15,16) - 7, (14,16,17) - 2, (1,15,17) - 3, (9,17,8) - 16, (9,10,1) - 3, (2,10,11) - 8\}$, $E_8 = \{(2,4,1) - 16, (3,5,2) - 16, (3,6,4) - 12, (4,7,5) - 13, (5,8,6) - 14, (6,9,7) - 15, (7,10,8) - 16, (8,11,9) - 17, (10,12,13) - 4, (11,13,14) - 5, (12,14,15) - 6, (13,15,16) - 7, (14,16,17) - 8, (1,15,17) - 3, (9,10,1) - 3, (10,11,2) - 17, (3,11,12) - 9\}$.

Define $K_{34} = E \cup C$, $K_i = E_i \cup C^*$ *and* $K_{17+i} = E_i \cup C$ *for all* $0 \leq i \leq 8$.

Now consider the following 8 *mutually disjoint trade pairs* (A_i, A_i^*) *and* 8 *mutually disjoint trade pairs* (B_i, B_i^*) *for* $0 \leq i \leq 7$. *First of all* $A_i \subseteq K_{34}$ *and* $|T(A_i^*) \cap T(K_{34})| = 2$. *Also* $B_i \subseteq K_{17}$, $|T(B_i) \cap T(K_{34})| = 2$ *and* $|T(B_i^*) \cap T(K_{34})| = 1$.

$A_i = \{(2+i, 4+i, 1+i) - (9+i), (17+i, 6+i, 13+i) - (1+i), (13+i, 2+i, 9+i) - (14+i)\}$, $A_i^* = \{(1+i, 4+i, 2+i) - (9+i), (17+i, 6+i, 13+i) - (2+i), (1+i, 13+i, 9+i) - (14+i)\}$, $A_{i+4} = \{(11+i, 13+i, 10+i) - (1+i), (9+i, 15+i, 5+i) - (10+i), (5+i, 11+i, 1+i) - (6+i)\}$ *and* $A_{i+4}^* = \{(10+i, 13+i, 11+i) - (1+i), (9+i, 15+i, 5+i) - (11+i), (5+i, 10+i, 1+i) - (6+i)\}$, *for* $0 \leq i \leq 3$.

$B_i = \{(16+i, 2+i, 1+i) - (9+i), (17+i, 6+i, 13+i) - (1+i), (13+i, 2+i, 9+i) - (14+i)\}$, $B_i^* = \{(1+i, 16+i, 2+i) - (9+i), (17+i, 6+i, 13+i) - (2+i), (1+i, 13+i, 9+i) - (14+i)\}$, $B_{i+4} = \{(8+i, 11+i, 10+i) - (1+i), (9+i, 15+i, 5+i) - (10+i), (5+i, 11+i, 1+i) - (6+i)\}$ *and* $B_{i+4}^* = \{(10+i, 8+i, 11+i) - (1+i), (9+i, 15+i, 5+i) - (11+i), (5+i, 10+i, 1+i) - (6+i)\}$ *for* $0 \leq i \leq 3$.

We can define the remaining K_j*'s by interchanging the blocks of the trades as follows:* $K_{34-j} = K_{34} \cup \{A_0^*, ..., A_{j-1}^*\} \backslash \{A_0, ..., A_{j-1}\}$ *and* $K_{17-j} = K_{17} \cup \{B_0^*, ..., B_{j-1}^*\} \backslash \{B_0, ..., B_{j-1}\}$ *for each* $1 \leq j \leq 8$

Then $|T(K_{34}) \cap T(K_i)| = i$, *for all* $0 \leq i \leq 34$, *and so* $J(17) = \{0, 1, ..., 34\}$

Denote by $X(1,2,3)$ the set $\{X \times \{1\}, X \times \{2\}, X \times \{3\}\}$. A tripartite kite system of order n is a pair $(X(1,2,3), K)$, where K is an edge disjoint collection of kites which partition the edge set of $K_{n,n,n}$ with vertex set $X(1,2,3)$.

Example 2.5 (*tripartite kite systems of order* 4 *intersecting in* 0 *and* 12 *triples*).

Let $X = \{1, 2, 3, 4\}$ *and define* K_1 *and* K_2 *as follows:*
$K_1 = \{((2,2), (4,3), (1,1)) - (1,2), ((1,2), (3,3), (2,1)) - (2,2), ((4,2), (2,3), (3,1)) - (3,2), ((3,2), (1,3), (4,1)) - (4,2), ((1,1), (2,3), (3,2)) - (3,3), ((2,1), (1,3), (4,2)) - (4,3), ((3,1), (4,3), (1,2)) - (1,3), ((4,1), (3,3), (2,2)) - (2,3), ((1,1), (4,2), (3,3)) - (3,1), ((2,1), (3,2), (4,3)) - (4,1), ((3,1), (2,2), (1,3)) - (1,1), ((4,1), (1,2), (2,3)) - (2,1)\}$
$K_2 = \{((2,2), (3,3), (1,1)) - (1,2), ((1,2), (4,3), (2,1)) - (2,2), ((4,2), (1,3), (3,1)) - (3,2), ((3,2), (2,3), (4,1)) - (4,2), ((1,1), (4,3), (3,2)) - (3,3), ((2,1), (3,3), (4,2)) - (4,3), ((3,1), (2,3), (1,2)) - (1,3), ((4,1), (1,3), (2,2)) - (2,3), ((4,1), (1,2), (3,3)) - (3,1), ((3,1), (2,2), (4,3)) - (4,1), ((2,1), (3,2), (1,3)) - (1,1), ((1,1), (4,2), (2,3)) - (2,1)\}$

Then $|T(K_1) \cap T(K_1)| = 12$, *and* $|T(K_1) \cap T(K_2)| = 0$

3 The $8n(8n+1)$-Construction

Let $2n \geq 6$, $X = \{1, 2, 3, ..., 2n\}$, and H a partition of X in sets of size 2 if $2n \equiv 0$ or 2 (mod 6) and of size 2 and 4 with one set of size 4 if $2n \equiv 4$

(mod 6). The sets in H are called holes. Let (X, H, T) be a group divisible design (GDD) with groups H and blocks T of size 3 (see[4]).

Let $S = X \times \{1,2,3,4\}$ or $\{\infty\} \cup (X \cup \{1,2,3,4\})$ and define a collection of kites K on S follows:

(1) If H contains a hole h of size 4 define a copy of Example 2.3 or 2.4 on $h \times \{1,2,3,4\}$ or $\{\infty\} \cup (h \times \{1,2,3,4\})$ and put these kites in K.

(2) For each hole h of size 2 define a copy of Example 2.1 or 2.2 on $h \times \{1,2,3,4\}$ or $\{\infty\} \cup (h \times \{1,2,3,4\})$ and put these kites in K.

(3) For each triple $\{a,b,c\} \in T$ define a copy of Example 2.5 on $K_{4,4,4}$ with parts $X \times \{a\}$, $X \times \{b\}$, and $X \times \{c\}$ and put these kites in K.

Then (S, K) is a kite system of order $8n$ or $8n+1$, as the case maybe.

Now let (S, K_1) and (S, K_2) be a pair of kite systems of order $N = 8n$ or $8n+1$ constructed using the $8n$ or $8n+1$ Construction. Since we can use any kite system in (1) and (2) and any tripartite kite system in (3) we can write any $N \in J(n)$ as $x + (n-1)J(8 \text{ or } 9) + 4\binom{n}{2}\{0, 12\}$, where $x \in J(8), J(9), J(16)$, or $J(17)$ as the case may be. We have the following theorem.

Theorem 3.1 *$J(n) = \{0, 1, 2, ..., n(n-1)/8\}$ for all $n \equiv 0$ or $1 (mod\ 8) \geq 8$.*

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