# On Potentially $K_k$-graphic Sequences *

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**Abstract**

In this paper, we characterize the potentially $K_4$-graphic sequences. This characterization implies the value $\sigma(K_4, n)$, which was conjectured by P. Erdös, M. S. Jacobson and J. Lehel [1] and was confirmed by R. J. Gould, M. S. Jacobson and J. Lehel [2] and Jiong-Sheng Li and Zixia Song [5], independently.

## 1 Introduction

An $n$-term nonincreasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be *graphic* if it is the degree sequence of a simple graph $G$ of order $n$ and such a graph $G$ is referred to as a *realization* of $\pi$. Denote by $\sigma(\pi)$ the sum of all the terms of $\pi$. Let $H$ be a simple graph. A graphic sequence $\pi$ is said to be *potentially $H$-graphic* if it has a realization $G$ containing $H$ as a subgraph.

In [1], Erdös, Jacobson and Lehel considered the following problem about potentially $K_k$-graphic sequences: determine the smallest positive even integer $\sigma(K_k, n)$ such that every $n$-term graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ without zero terms and with degree sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$ at least $\sigma(K_k, n)$ is potentially $K_k$-graphic. The graph $G(n, k)$ on $n$ vertices

*This paper is a result of an undergraduate research project by a math major student Mr. Morgan Warner under the supervision of Dr. Rong Luo at WVU. This undergraduate research project was administrated by Dr. John Goldwasser and Dr. Cun-Quan Zhang and was partially supported by the undergraduate Academic Enrichment Program of the Eberly College of Arts and Sciences, West Virginia University.

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with $k-2$ vertices with degree $n-1$ and every edge incident to one of these $k-2$ vertices has degree sequence $\pi_0 = ((n-1)^{k-2}, (k-2)^{n-k+2})$ (where $(n-1)^{k-2}$ means $n-1$ repeats $k-2$ times), but does not contain a $K_k$. Notice that $\pi_0$ has unique realization $G(n,k)$. So, $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$. They further conjectured that the lower bound is the exact value of $\sigma(K_k, n)$ for $n$ sufficiently large. They also proved the conjecture for $k=3$ and $n \geq 6$, because it is false for $n=4$ and 5. Gould et al. [2] and Li and Song [5] independently confirmed this conjecture for $k=4$ and $n \geq 8$. The conjecture is confirmed in [6] and [7] for any $k \geq 5$ and for $n$ sufficiently large. Li et al. [7] and Mubayi [10] also independently determined the values $\sigma(K_k, 2k)$ for any $k \geq 3$. Recently, Li and Yin [9] determined the values $\sigma(K_k, n)$ for any $k \geq 3$ and $n \geq k$. In [2], Gould, Jacobson and Lehel generalized the above problem: given simple graph $H$, determine the smallest positive even integer $\sigma(H,n)$ such that every $n$-term graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ without zero terms and with degree sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$ at least $\sigma(H,n)$ is potentially $H$-graphic. They determined the values $\sigma(pK_2, n)$ and $\sigma(C_4, n)$ where $pK_2$ is a matching of $p$ edges and $C_4$ is a cycle of length 4.

Motivated by the above problems, we consider the following problem: characterize the potentially $H$-graphic sequences without zero terms. In [8], R. Luo characterized the potentially $C_k$-graphic sequences for each $k = 3, 4, 5$. In [11], Niu characterized the potentially $(K_4 - e)$-graphic sequences. In this paper, we characterize the potential $K_4$-graphic sequences. From this characterization, it is straightforward to find the values of $\sigma(K_4, n)$.

## 2 Lemmas

In order to prove our main theorem, we need the following results.

**Lemma 2.1** *(D. J. Kleitman and D. L. Wang [4] and Hakimi [3]) $\pi$ is graphic if and only if $\pi'$ is graphic.*

The following corollary is obvious.

**Corollary 2.2** *Let $H$ be a simple graph. If $\pi'$ is potentially $H$-graphic, then $\pi$ is also potentially $H$-graphic.*

**Theorem 2.3** *(Gould et al. [2]) Let $\pi = (d_1, \cdots, d_n)$ be a graphic sequence and $G$ be a graph with the vertex set $V(G) = \{v_1, \cdots, v_k\}$. If $H$ is a realization of $\pi$ with $G \subseteq H$, then there is a realization $H'$ of $\pi$ with $G \subseteq H'$ so that the two multisets $\{d_H(v_1), \cdots, d_H(v_k)\}$ and $\{d_1, \cdots, d_k\}$ are the same.*

Lemma 2.1, Corollary 2.2 and Theorem 2.3 will be applied repeatedly and implicitly in the proof of our main theorem.

**Lemma 2.4** *Let* $\pi = (d_1, d_2, \cdots, d_n)$ *be a graphic sequence with* $n \geq 4$ *and* $d_1 = d_2 = d_3 = d_4 = 3$. *Then* $\pi$ *is potentially* $K_4$*-graphic if and only if* $(d_5, \cdots, d_n)$ *is graphic.*

**Proof.** This follows immediately from Theorem 2.3. ■

Let $\pi = (d_1, d_2, \cdots, d_n)$ be a nonincreasing positive integer sequence. The *residual sequence*, denoted by $\pi'_k$, obtained from $\pi$ by laying off $d_k$ is defined as follows:

$$\pi'_k = \begin{cases} (d_1 - 1, \cdots, d_{k-1} - 1, d_{k+1} - 1, \cdots, d_{d_k+1} - 1, d_{d_k+2}, \cdots, d_n), if\ d_k \geq k, \\ (d_1 - 1, \cdots, d_{d_k} - 1, d_{d_k+1}, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n), if\ d_k \leq k - 1. \end{cases}$$

**Theorem 2.5** *(Rong Luo [8]) Let* $\pi = (d_1, d_2, \cdots, d_n)$ *be a graphic sequence with* $n \geq 3$. *Then* $\pi$ *is potentially* $C_3$*-graphic if and only if* $d_3 \geq 2$ *except for 2 cases:* $\pi = (2^4)$ *and* $\pi = (2^5)$.

**Lemma 2.6** *Let* $\pi = (d_1, d_2, \cdots, d_n)$ *be a graphic sequence with* $d_4 \geq 3$ *and* $n \geq 4$. *If* $\pi$ *is not* $K_4$*-graphic and* $\pi'_1 \neq (2^4), (2^5)$, *then* $n - 2 \geq d_1 \geq d_2 \geq d_3 \geq d_4 = \cdots = d_{d_1+2} \geq \cdots \geq d_n$.

**Proof.** By way of contradiction, we assume that there exists an integer $4 \leq t \leq d_1 + 1$ so that $d_t > d_{t+1}$. Since $d_4 \geq 3$ and $\pi \neq (2^4), (2^5)$, the residual sequence $\pi'_1 = (d'_1, d'_2, \cdots, d'_{n-1})$ satisfies the conditions in Theorem 2.5. Notice that $d'_i = d_{i+1} - 1$ for each $i = 1, 2, \cdots, t$. Therefore, $\pi'_1$ has a realization $G$ containing a $K_3$ so that the degrees of vertices of $K_3$ in $G$ are $d'_1, d'_2, d'_3$. Thus we can obtain a realization $H$ of $\pi$ from $G$ by adding a vertex to $G$ which is adjacent to each vertex whose degree is decreased by one in $G$. Then, $H$ contains $K_4$ as a subgraph. ■

**Lemma 2.7** *Let* $\pi = (d_1, d_2, \cdots, d_n)$ *be a graphic sequence with* $d_1 = n - 1$ *and* $n \geq 4$. *Then* $\pi$ *is potentially* $K_4$*-graphic if and only if* $d_4 \geq 3$ *and* $\pi \neq (n - 1, 3^s, 1^{n-s-1})$ *for each* $s = 4, 5$.

**Proof.** This follows from Lemma 2.6. ■

# 3 Main Theorem

**Theorem 3.1** *Let* $\pi = (d_1, d_2, \cdots, d_n)$ *be a graphic sequence without zero terms and with* $d_4 \geq 3$ *and* $n \geq 4$. *Then* $\pi$ *is* $K_4$*-graphic if and only if*

*$d_4 \geq 3$ and $\pi \neq (n-1, 3^s, 1^{n-s-1})$ for each $s = 4, 5$ except the following sequences:*

$n = 5 : (4, 3^4),\ (3^4, 2)$;
$n = 6 : (4^6),\ (4^2, 3^4),\ (4, 3^4, 2),\ (3^6),\ (3^5, 1),\ (3^4, 2^2)$;
$n = 7 : (4^7),\ (4^3, 3^4),\ (4, 3^6),\ (4, 3^5, 1),\ (3^6, 2),\ (3^5, 2, 1)$;
$n = 8 : (3^7, 1),\ (3^6, 1^2)$.

Before we present the proof of Theorem 3.1, we give an application of the theorem.

**Theorem 3.2** *(Gould, Jacobson and Lehel[2] and Li and Song[5])*
*$\sigma(K_4, n) = 4n - 4$ for $n \geq 8$.*

**Proof.** In [1], by taking the extremal example $\pi_0 = ((n-1)^{k-2}, (k-2)^{n-k+2})$, Erdös et al. gave a lower bound for $\sigma(K_k, n)$, i.e., $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$. In particular, $\sigma(K_4, n) \geq 4n-4$. It is sufficient to show that $\sigma(K_4, n) \leq 4n-4$ for $n \geq 8$. Let $\pi$ be a graphic sequence with $\sigma(\pi) \geq 4n-4$ and without zero terms. We only need to prove that $\pi$ is potentially $K_4$-graphic.

(I) $\pi \neq (n-1, 3^s, 1^{n-1-s})$ for each $s = 4,\ 5$ since $\sigma(\pi) \geq 4n-4$ and $n \geq 8$.

(II) We claim that $d_4 \geq 3$.

By way of contradiction, we assume that $d_4 \leq 2$. Let $G$ be a realization of $\pi$ with the vertex set $V = \{v_1, \cdots, v_n\}$ so that $d(v_i) = d_i$ for each $i = 1, \cdots, n$. Denote $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = V \setminus V_1$. Denote by $[V_1, V_2]$ the set of edges with one endvertex in $V_1$ and the other endvertex in $V_2$. Then, $|[V_1, V_2]| \leq d_4 + \cdots d_n$. On the other hand, it is easy to see that $d_1 - 2 + d_2 - 2 + d_3 - 2 \leq |[V_1, V_2]|$. Therefore, $\sigma(\pi) = d_1 + d_2 + d_3 + d_4 + \cdots + d_n \leq 2(d_4 + \cdots + d_n) + 6 \leq 4(n-3) + 6 = 4n - 6 < 4n - 4 \leq \sigma(\pi)$ since $d_4 \leq 2$. This contradiction implies that $d_4 \geq 3$.

(III) For $n = 8$, we have that $\sigma(\pi) \geq 4 \times 8 - 4 = 28$. Therefore, $\pi \neq (3^7, 1),\ (3^6, 1^2)$. By Theorem 3.1, $\pi$ is potentially $K_4$-graphic.
■

**Proof of Theorem 3.1.** The necessary condition is obvious. Therefore we only need to prove the sufficient condition. Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence satisfying the conditions of the theorem. It suffices to show that $\pi$ has a realization containing $K_4$ as a subgraph. If $d_1 = n-1$, then, by Lemma 2.7, $\pi$ is potentially $K_4$-graphic. Therefore, we assume that $d_1 \leq n-2$. We consider the following cases:

Case 1: $n = 4$. This case is obvious.

Case 2: $n = 5$.

Then $d_1 = 3$. Therefore, $d_1 = d_2 = d_3 = d_4 = 3$. Since $\sigma(\pi)$ is even, we have that $d_5 = 2$. Therefore, $\pi = (3^4, 2)$, which is an exception.

Case 3. $n = 6$.

Then $3 \leq d_1 \leq 4$. If $d_1 = 3$, then $d_1 = d_2 = d_3 = d_4 = 3$. Since $\pi \neq (3^6)$, $(3^5, 1)$, $(3^4, 2^2)$, $\pi$ must be $(3^4, 1^2)$, which is potentially $K_4$-graphic. Now assume that $d_1 = 4$. Then $\pi_1' \neq (2^5)$ otherwise $\pi = (4, 3^4, 2)$, which is an exception. It is easy to see that $\pi_1' \neq (2^4)$. By Lemma 2.6, we may assume that $d_4 = d_5 = d_6$. Therefore, $\pi = (4^6), (4^2, 3^4)$, which are exceptions.

Case 4: $n = 7$.

Then $3 \leq d_1 \leq 5$. If $\pi_1'$ has at most five positive terms, then $d_6 = d_7 = 1$ since $d_1 \leq 5$, and therefore, $\pi_1'$ contains 1 as a term. Thus $\pi_1' \neq (2^4), (2^5)$. By Lemma 2.6, we may assume that $d_4 = d_5 = \cdots = d_{d_1+2}$. Since $d_1 \geq 3$, we have that $d_4 = d_5 \geq 3$. We consider the following three subcases.

Subcase 4.1. $d_1 = 3$. Then $d_1 = d_2 = d_3 = d_4 = d_5 = 3$. Notice that $d_5 + d_6 + d_7$ is even. Since $3 = d_5 \geq d_6 \geq d_7 \geq 1$, we have that $(d_5, d_6, d_7)$ is one the following sequences: $(3^2, 2), (3, 2, 1)$. Therefore, $\pi = (3^6, 2)$ *or* $(3^5, 2, 1)$, both of which are exceptions.

Subcase 4.2. $d_1 = 4$. Then $d_4 = d_5 = d_6$. If $d_4 = 4$, then $\pi = (4^6, 2)$ since $\pi \neq (4^7)$. It is easy to see that $(4^6, 2)$ is potentially $K_4$-graphic. If $d_4 = 3$, then $\pi$ must be either $(4^2, 3^4, 2)$ or $(4^3, 3^3, 1)$ since $\pi \neq (4^3, 3^4), (4, 3^6), (4, 3^5, 1)$. It is easy to see that both $(4^2, 3^4, 2)$ and $(4^3, 3^3, 1)$ are potentially $K_4$-graphic.

Subcase 4.3. $d_1 = 5$. Then $d_4 = d_5 = d_6 = d_7 \geq 3$. Notice that $d_2 + d_3$ is odd. Since $\sigma(\pi)$ is even, we have that $d_4 \leq 4$. If $d_4 = 4$, then $d_2 = 5$ and $d_3 = 4$. Therefore $\pi = (5^2, 4^5)$, which is potentially $K_4$-graphic. If $d_4 = 3$, then either $d_2 = 4, d_3 = 3$ or $d_2 = 5, d_3 = 4$. That is, either $\pi = (5, 4, 3^5)$ or $\pi = (5^2, 4, 3^4)$. It is easy to see that both of them are potentially $K_4$-graphic.

Case 5: $n = 8$.

If $\pi_1'$ has at most six positive terms, then it must contain 1 as a term since $d_1 \leq n-2$. Therefore, $\pi_1' \neq (2^4), (2^5)$. By Lemma 2.6, we may assume that $d_4 = d_5 = \cdots = d_{d_1+2}$. Since $d_1 \geq 3$, we have that $d_4 = d_5 \geq 3$. We may further assume that $\pi_8'$ is not $K_4$-graphic. If in the sequence $\pi_8'$, $d_4' = 2$, then $d_8 \leq 2$. Since $d_2 \geq d_3 \geq d_4 = d_5 \geq 3$, we have that $d_8 = 2$. Then $d_2 = 3$ and $d_6 = d_7 = d_8 = 2$. Therefore, $d_1 + 2 \leq 5$. Thus $d_1 = 3$ and $\pi = (3^4, 2^4)$, which is potentially $K_4$-graphic. Therefore, in the sequence $\pi_8'$, $d_4' \geq 3$ and $\pi_8'$ has seven positive terms.

Then, by Case 4, $\pi_8'$ must be one of the following sequences:

$$(4^7),\ (4^3, 3^4),\ (4, 3^6),\ (4, 3^5, 1),\ (3^6, 2),\ (3^5, 2, 1);$$

$\pi'_8 = (4^7)$ implies that $\pi$ is one of the sequences $(5^4, 4^4), (5^3, 4^4, 3), (5^2, 4^5, 2), (5, 4^6, 1)$. It is easy to see that these are all $K_4$-graphic.

Notice that $\pi \neq (3^7, 1)$, $(3^6, 1^2)$. $\pi'_8 = (4, 3^5, 1)$ *or* $(3^6, 2)$*or* $(3^5, 2, 1)$ implies that $\pi$ is one of the following sequences:

$$(5, 3^5, 1^2),\ (4^2, 3^4, 1^2),\ (4, 3^4, 2, 1^2),\ (4, 3^5, 2, 1),\ (4, 3^6, 2), (4^2, 3^4, 2^2).$$

$\pi'_8 = (4^3, 3^4)$ *or* $(4, 3^6)$ implies that $\pi$ is one of the following sequences:

$$(5, 4^2, 3^4, 1),\ (5, 3^6, 1),\ (4^4, 3^3, 1),\ (4^2, 3^5, 1),\ (5^2, 4, 3^4, 2), (5, 4^3, 3^3, 2), (4^5, 3^2, 2),$$
$$(4^3, 3^4, 2), (5^3, 3^5), (5^2, 4^2, 3^4), (5, 4^4, 3^3), (4^6, 3^2), (5, 4^2, 3^5), (4^4, 3^4).$$

It is easy to check that all the above 20 sequences are potentially $K_4$-graphic.

Case 6: $n \geq 9$.

I. We claim that $\pi'_1 \neq (2^4), (2^5)$.

Otherwise, $d_n = 1$ and $\pi'_1$ contains at least one term with the value 1 since $d_1 \leq n-2$, a contradiction.

II. We may assume that $d_4 = d_5 = \cdots = d_{d_1+2}$. Since $d_1 \geq 3$, we have that $d_4 = d_5 \geq 3$.

III. In $\pi'_n$, we claim that $d'_4 \geq 3$.

By way of contradiction, we assume that $d'_4 \leq 2$. Then $d'_4 = 2$. Therefore $d_n \leq 2$. Then $d_n = 2$ since $d_2 \geq d_3 \geq d_4 = d_5 \geq 3$. Thus, $\pi'_n = (d_1 - 1, d_2 - 1, d_3, d_4, \cdots, d_{n-1})$. Since $d_3 \geq d_4 = d_5 \geq 3$ and $d'_4 = 2$, we have that $d_1 = d_2 = d_3 = d_4 = d_5 = 3$ and $2 \geq d_6 \geq \cdots \geq d_n = 2$. Therefore, $\pi = (3^5, 2^{n-5})$, contradicting the fact that $\sigma(\pi)$ is even.

We use induction on $n$ to prove this case. We first prove the case $n = 9$. Let $\pi$ be a graphic sequence satisfying $d_4 \geq 3$ and $\pi \neq (8, 3^4, 1^4)$ *or* $(8, 3^5, 1^3)$. We will show that $\pi$ is potentially $K_4$-graphic.

IV. By III and Case 5, we may assume that $\pi'_9$ is one of the following four sequences: $(3^7, 1)$, $(3^6, 1^2)$, $(7, 3^4, 1^3)$, $(7, 3^5, 1^2)$ since otherwise $\pi'_9$ is potentially $K_4$-graphic by Case 5 and therefore, so is $\pi$.

Since each of the four sequences contains 1 as one of its terms, we have that $d_9 = 1$. Therefore, $\pi$ must be one of the following:

$$(4, 3^6, 1^2),\ (4, 3^5, 1^3), (8, 3^4, 1^4), (8, 3^5, 1^3).$$

Since $\pi$ is not $(8, 3^4, 1^4)$ or $(8, 3^5, 1^3)$, $\pi$ is either $(4, 3^6, 1^2)$ or $(4, 3^5, 1^3)$, and it is easy to see that both of them are potentially $K_4$-graphic.

IV. Now we assume that $n \geq 10$. Since $\pi \neq (n-1, 3^s, 1^{n-s-1})$ for each $s = 4, 5$, $\pi'_n \neq (n-2, 3^s, 1^{n-s-2})$ for each $s = 4, 5$ either. By III and induction hypothesis, $\pi'_n$ is potentially $K_4$-graphic and therefore, so is $\pi$.

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