On Harmonious Labelings of the Balanced Quadruple Shells *

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Abstract

A multiple shell MS $\{ n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r} \}$ is a graph formed by t_i shells of widths n_i, $1 \leq i \leq r$, which have a common apex. This graph has $\sum_{i=1}^{r} t_i(n_i - 1) + 1$ vertices. A multiple shell is said to be balanced with width w if it is of the form MS$\{w^t\}$ or MS$\{w^t, (w+1)^s\}$. Deb and Limaye have conjectured that all multiple shells are harmonious, and shown that the conjecture is true for the balanced double shells and balanced triple shells. In this paper, the conjecture is proved to be true for the balanced quadruple shells.

Keywords: *harmonious graph, multiple shell, vertex labeling, edge labeling*

1 Introduction

In 1980 Graham and Sloane [1] gave a variation on graceful labeling of graphs. A simple , finite graph G with n vertices and $q(\geq n)$ edges is said to be harmonious if there is an injection $f : V(G) \rightarrow Z_q$, Z_q is the integer group modulo q , such that the induced function $g : E(G) \rightarrow Z_q$ defined by $g(xy)=[f(x) + f(y)]$ mod q , $xy \in E(G)$ is a bijection. Such a labeling of the vertices and edges is called a harmonious labeling of graph. In a harmonious labeling the vertex labels are distinct and the induced edge

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labels are $0, 1, 2, \ldots, q-1$.

Graham and Sloane proved that odd cycles C_{4m+1}, C_{4m+3}, wheels W_n, $n \geq 3$ and Petersen graph are harmonious, most graphs including even cycles are not harmonious. L.Bolian and Z.Xiankun [4]proved that the graph C_n' obtained by joining a path to a vertex of C_n is harmonious if and only if it has even number of edges, the helm H_n is harmonious when n is odd. S.C.Shee [5] gave harmonious labeling of graph obtained by identifying center of the star S_m with a vertex of an odd cycle C_n. The first author [7]of this paper has proved that the disjoint union $C_{2k} \cup C_{2j+1}$ of cycles C_{2k} and C_{2j+1} $(k \geq 2, j \geq 1, (k,j) \neq (2,1))$ is harmonious. For the literature on harmonious graphs we refer to [3] and the relevant reference given in them.

A shell $S_{n,n-3}$ of width n is a graph obtained by taking $n-3$ concurrent chords in a cycle C_n on n vertices. The vertex at which all the chords are concurrent is called apex. The two vertices adjacent to the apex have degree 2, apex has degree $n-1$ and all the other vertices have degree 3.

A multiple shell MS$\{n_1^{t_1}, n_2^{t_2}, \ldots, n_r^{t_r}\}$ is a graph formed by t_i shells of widths n_i, $1 \leq i \leq r$, which have a common apex. This graph has $\sum_{i=1}^{r} t_i(n_i - 1) + 1$ vertices. A multiple shell is said to be balanced with width w if it is of the form MS $\{w^t\}$ or MS$\{w^t, (w+1)^s\}$. If a multiple shell has in all k shells having a common apex, then it is called k-tuple shell, i.e. double shell if $k = 2$, triple shell if $k = 3$ etc.

Suppose S is a balanced multiple shell on n vertices with k shells having a common apex. If $n = kt$ then S=MS$\{t, (t+1)^{k-1}\}$. On the other hand if $n = kt + r$, $r \neq 0$, then S=MS$\{(t+1)^{k-r+1}, (t+2)^{r-1}\}$.

Deb and Limaye [2] gave harmonious labeling of many families of cycle related graphs, such as shell graphs, cycles with maximum possible number of concurrent alternate chords, some families of multiple shells. They have conjectured that all multiple shells are harmonious, and shown that the conjecture is true for the balanced double shells and balanced triple shells. In this paper, the conjecture is proved to be true for the balanced quadruple shells.

2 Balanced Quadruple Shells

Now, we consider the case of balanced quadruple shells. We have

Theorem 2.1. All the balanced quadruple shells are harmonious.

Proof Let G be a balanced quadruple shell on n vertices and m edges .

Case 1: $n \equiv 1 \ mod\ 4$. Let $n = 4p + 1$, $p \geq 3$, then $m = 8p - 4$.
In this case we must have four shells of size $p + 1$.

Let

$$V(G) = \{v_1, v_2, \ldots, v_{4p+1}\}.$$

The four cycles are

$$\begin{aligned} C_1 &= \{v_1, v_2, \ldots, v_p, v_{2p+1}\}, \\ C_2 &= \{v_{p+1}, v_{p+2}, \ldots, v_{2p}, v_{2p+1}\}, \\ C_3 &= \{v_{2p+2}, v_{2p+3}, \ldots, v_{3p+1}, v_{2p+1}\}, \\ C_4 &= \{v_{3p+2}, v_{3p+3}, \ldots, v_{4p+1}, v_{2p+1}\}. \end{aligned}$$

This means that v_{2p+1} is the common apex and the $4p - 8$ chords are

$$\{ v_{2p+1}v_i \mid 2 \leq i \leq p-1 \vee p+2 \leq i \leq 2p-1 \vee 2p+3 \leq i \leq 3p \vee 3p+3 \leq i \leq 4p \}.$$

We label the vertices as follows:

$$f(v_i) = \begin{cases} 8p - (i+9)/2, & 1 \leq i \leq 2p-1 \text{ and } i \bmod 2 = 1, \\ 6p - 4 - i/2, & 2 \leq i \leq 2p-2 \text{ and } i \bmod 2 = 0, \\ 5p - 5 + i/2, & 2p+2 \leq i \leq 4p-2 \text{ and } i \bmod 2 = 0, \\ 3p + (i-7)/2, & 2p+3 \leq i \leq 4p-1 \text{ and } i \bmod 2 = 1, \\ -3 + i, & 2p \leq i \leq 2p+1, \\ -3p - 1 + i, & i = 4p, \\ -p - 4 + i, & i = 4p+1. \end{cases}$$

It is easy to verify that f is an injective function from the vertex set $V(G)$ of G to the set $\{0, 1, \ldots m - 1\}$.

Denote by

$$g(v_iv_j) = [f(v_i) + f(v_j)] \ mod\ m.$$

Now,we show that g is an one-to-one function from the edge set $E(G)$ of G to the set $\{0, 1, \ldots m - 1\}$.

Let

$$D_1 = \{g(v_i, v_{i+1}) | 1 \leq i \leq 4p, i \neq p, 2p, 2p+1, 3p+1\} = D_{11} \cup D_{12} \cup D_{13} \cup D_{14},$$

where

$$\begin{aligned} D_{11} &= \{g(v_i, v_{i+1}) | 1 \leq i \leq p-1\} &&= \{6p-6, 6p-7, \ldots, 5p-4\}, \\ D_{12} &= \{g(v_i, v_{i+1}) | p+1 \leq i \leq 2p-1\} &&= D_{121} \cup D_{122}, \\ D_{121} &= \{g(v_i, v_{i+1}) | p+1 \leq i \leq 2p-2\} &&= \{5p-6, 5p-7, \ldots, 4p-3\}, \\ D_{122} &= \{g(v_i, v_{i+1}) | i = 2p-1\} &&= \{p-3\}, \\ D_{13} &= \{g(v_i, v_{i+1}) | 2p+2 \leq i \leq 3p\} &&= \{2p-2, 2p-1, \ldots, 3p-4\}, \\ D_{14} &= \{g(v_i, v_{i+1}) | 3p+2 \leq i \leq 4p\} &&= D_{141} \cup D_{142} \cup D_{143}, \\ D_{141} &= \{g(v_i, v_{i+1}) | 3p+2 \leq i \leq 4p-2\} &&= \{3p-2, 3p-1, \ldots, 4p-6\}, \\ D_{142} &= \{g(v_i, v_{i+1}) | i = 4p-1\} &&= \{6p-5\}, \\ D_{143} &= \{g(v_i, v_{i+1}) | i = 4p\} &&= \{4p-4\}. \end{aligned}$$

Let

$$D_2 = \{g(v_{2p+1}, v_i) | 1 \leq i \leq 4p+1, i \neq 2p+1\}$$
$$= D_{21} \cup D_{22} \cup D_{23} \cup D_{24} \cup D_{25} \cup D_{26} \cup D_{27},$$

where

$$\begin{aligned}
D_{21} &= \{g(v_{2p+1}, v_i) | 1 \leq i \leq 2p-1 \text{ and } i \bmod 2 = 1\} \\
&= \{2p-3, 2p-4, \ldots, p-2\}, \\
D_{22} &= \{g(v_{2p+1}, v_i) | 2 \leq i \leq 2p-2 \text{ and } i \bmod 2 = 0\} \\
&= \{8p-7, 8p-8, \ldots, 7p-5\}, \\
D_{23} &= \{g(v_{2p+1}, v_i) | i = 2p\} = \{4p-5\}, \\
D_{24} &= \{g(v_{2p+1}, v_i) | 2p+2 \leq i \leq 4p-2 \text{ and } i \bmod 2 = 0\} \\
&= D_{241} \cup D_{242}, \\
D_{241} &= \{g(v_{2p+1}, v_i) | i = 2p+2, 2p+4\} = \{8p-6, 8p-5\}, \\
D_{242} &= \{g(v_{2p+1}, v_i) | 2p+6 \leq i \leq 4p-2 \text{ and } i \bmod 2 = 0\} \\
&= \{0, 1, \ldots, p-4\}, \\
D_{25} &= \{g(v_{2p+1}, v_i) | 2p+3 \leq i \leq 4p-1 \text{ and } i \bmod 2 = 1\} \\
&= \{6p-4, 6p-3, \ldots, 7p-6\}, \\
D_{26} &= \{g(v_{2p+1}, v_i) | i = 4p\} = \{3p-3\}, \\
D_{27} &= \{g(v_{2p+1}, v_i) | i = 4p+1\} = \{5p-5\}.
\end{aligned}$$

Let D be the labels set of all edges, then we have

$$\begin{aligned}
D &= D_1 \cup D_2 \\
&= D_{242} \cup D_{122} \cup D_{21} \cup D_{13} \cup D_{26} \cup D_{141} \cup D_{23} \cup D_{143} \cup D_{121} \\
&\quad \cup D_{27} \cup D_{11} \cup D_{142} \cup D_{25} \cup D_{22} \cup D_{241} \\
&= \{0, 1, \ldots, p-4\} \cup \{p-3\} \cup \{2p-3, 2p-4, \ldots, p-2\} \\
&\quad \cup \{2p-2, 2p-1, \ldots, 3p-4\} \cup \{3p-3\} \\
&\quad \cup \{3p-2, 3p-1, \ldots, 4p-6\} \cup \{4p-5\} \\
&\quad \cup \{4p-4\} \cup \{5p-6, 5p-7, \ldots, 4p-3\} \\
&\quad \cup \{5p-5\} \cup \{6p-6, 6p-7, \ldots, 5p-4\} \\
&\quad \cup \{6p-5\} \cup \{6p-4, 6p-3, \ldots, 7p-6\} \\
&\quad \cup \{8p-7, 8p-8, \ldots, 7p-5\} \cup \{8p-6, 8p-5\} \\
&= \{0, 1, 2, \ldots, 8p-6, 8p-5\}.
\end{aligned}$$

It is obvious that the labels of each edge are different. So, g maps E onto $\{0,1,\ldots,m-1\}$. According to the definition of harmonious graph, we can conclude that the balanced quadruple shells are harmonious for $n = 4p+1$.

Case 2: $n \equiv 2 \bmod 4$. Let $n = 4p+2$, $p \geq 3$, then $m = 8p-2$.
In this case we must have three shells of size $p+1$ and one shell of size $p+2$.

Let

$$V(G) = \{v_1, v_2, \ldots, v_{4p+2}\}.$$

The four cycles are

$$\begin{array}{rcl} C_1 & = & \{v_1, v_2, \ldots, v_p, v_{2p+1}\}, \\ C_2 & = & \{v_{p+1}, v_{p+2}, \ldots, v_{2p}, v_{2p+1}\}, \\ C_3 & = & \{v_{2p+2}, v_{2p+3}, \ldots, v_{3p+1}, v_{2p+1}\}, \\ C_4 & = & \{v_{3p+2}, v_{3p+3}, \ldots, v_{4p+2}, v_{2p+1}\}. \end{array}$$

This means that v_{2p+1} is the common apex and the $4p-7$ chords are

$\{ v_{2p+1}v_i \mid 2 \le i \le p-1 \vee p+2 \le i \le 2p-1 \vee 2p+3 \le i \le 3p \vee 3p+3 \le i \le 4p+1 \}$.

We label the vertices as follows:

$$f(v_i) = \begin{cases} 8p - (i+5)/2, & 1 \le i \le 2p-1 \text{ and } i \bmod 2 = 1, \\ 6p - 1 - i/2, & 2 \le i \le 2p \text{ and } i \bmod 2 = 0, \\ 3p - 1 + i/2, & 2p+2 \le i \le 4p-2 \text{ and } i \bmod 2 = 0, \\ 5p + (i-5)/2, & 2p+3 \le i \le 4p-1 \text{ and } i \bmod 2 = 1, \\ -3 + i, & i = 2p+1, \\ -p + i, & i = 4p, \\ 0, & i = 4p+1, \\ -2(p+1) + i, & i = 4p+2. \end{cases}$$

By a proof similar to the one in Case 1, we have that this assignment provides a harmonious labeling for $n = 4p+2$.

Case 3: $n \equiv 3 \ mod\ 4$. Let $n = 4p+3$, $p \ge 2$, then $m = 8p$.
In this case we must have two shells of size $p+1$ and two shells of size $p+2$. Let

$$V(G) = \{v_1, v_2, \ldots, v_{4p+3}\}.$$

The four cycles are

$$\begin{array}{rcl} C_1 & = & \{v_1, v_2, \ldots, v_p, v_{2p+1}\}, \\ C_2 & = & \{v_{p+1}, v_{p+2}, \ldots, v_{2p}, v_{2p+1}\}, \\ C_3 & = & \{v_{2p+2}, v_{2p+3}, \ldots, v_{3p+1}, v_{2p+1}\}, \\ C_4 & = & \{v_{3p+2}, v_{3p+3}, \ldots, v_{4p+3}, v_{2p+1}\}. \end{array}$$

This means that v_{2p+1} is the common apex and the $4p-6$ chords are

$\{ v_{2p+1}v_i \mid 2 \le i \le p-1 \vee p+2 \le i \le 2p-1 \vee 2p+3 \le i \le 3p \vee 3p+3 \le i \le 4p+2 \}$.

We label the vertices as follows:

$$f(v_i) = \begin{cases} 8p - (i+1)/2, & 1 \le i \le 2p-1 \text{ and } i \bmod 2 = 1, \\ 6p - i/2, & 2 \le i \le 2p \text{ and } i \bmod 2 = 0, \\ 5p - 1 + i/2, & 2p+2 \le i \le 4p \text{ and } i \bmod 2 = 0, \\ 3p + (i-3)/2, & 2p+3 \le i \le 4p+1 \text{ and } i \bmod 2 = 1, \\ -1 + i, & i = 2p+1, \\ -3p - 2 + i, & i = 4p+2, \\ -p - 4 + i, & i = 4p+3. \end{cases}$$

By a proof similar to the one in Case 1, we have that this assignment provides a harmonious labeling for $n = 4p + 3$.

Case 4: $n \equiv 0 \ mod \ 4$. Let $n = 4p$, $p \geq 3$, then $m = 8p - 6$.
In this case we must have one shells of size p and three shells of size $p + 1$. Let

$$V(G) = \{v_1, v_2, \ldots, v_{4p}\}.$$

The four cycles are

$$\begin{array}{rcl} C_1 & = & \{v_1, v_2, \ldots, v_p, v_{2p+1}\}, \\ C_2 & = & \{v_{p+1}, v_{p+2}, \ldots, v_{2p}, v_{2p+1}\}, \\ C_3 & = & \{v_{2p+2}, v_{2p+3}, \ldots, v_{3p+1}, v_{2p+1}\}, \\ C_4 & = & \{v_{3p+2}, v_{3p+3}, \ldots, v_{4p}, v_{2p+1}\}. \end{array}$$

This means that v_{2p+1} is the common apex and the $4p - 9$ chords are

$$\{\, v_{2p+1}v_i \mid 2 \leq i \leq p-1 \vee p+2 \leq i \leq 2p-1 \vee 2p+3 \leq i \leq 3p \vee 3p+3 \leq i \leq 4p-1 \,\}.$$

We label the vertices as follows:

$$f(v_i) = \begin{cases} 8p - (i + 13)/2, & 1 \leq i \leq 2p - 1 \text{ and } i \bmod 2 = 1, \\ 2p - 2 - i/2, & 2 \leq i \leq 2p - 2 \text{ and } i \bmod 2 = 0, \\ -p - 1 + i/2, & 2p + 2 \leq i \leq 4p - 2 \text{ and } i \bmod 2 = 0, \\ p + (i - 7)/2, & 2p + 3 \leq i \leq 4p - 1 \text{ and } i \bmod 2 = 1, \\ 3p - 5 + i, & 2p \leq i \leq 2p + 1, \\ 2p - 5 + i, & i = 4p. \end{cases}$$

By a proof similar to the one in Case 1, we have that this assignment provides a harmonious labeling for $n = 4p$.

According to the Cases 1 – 4, we can say that the balanced quadruple shells are harmonious. □

In Figure 1, we illustrate our harmonious labelings for MS$\{10^4\}$, MS$\{9^3, 10\}$, MS$\{9^2, 10^2\}$ and MS$\{9, 10^3\}$.

Conclusion

Deb and Limaye [2] have conjectured that all multiple shells are harmonious , and shown that the conjecture is true for the balanced double shells and balanced triple shells . In this paper, the conjecture is proved to be true for the balanced quadruple shells . The conjecture still remains open for $k \geq 5$.

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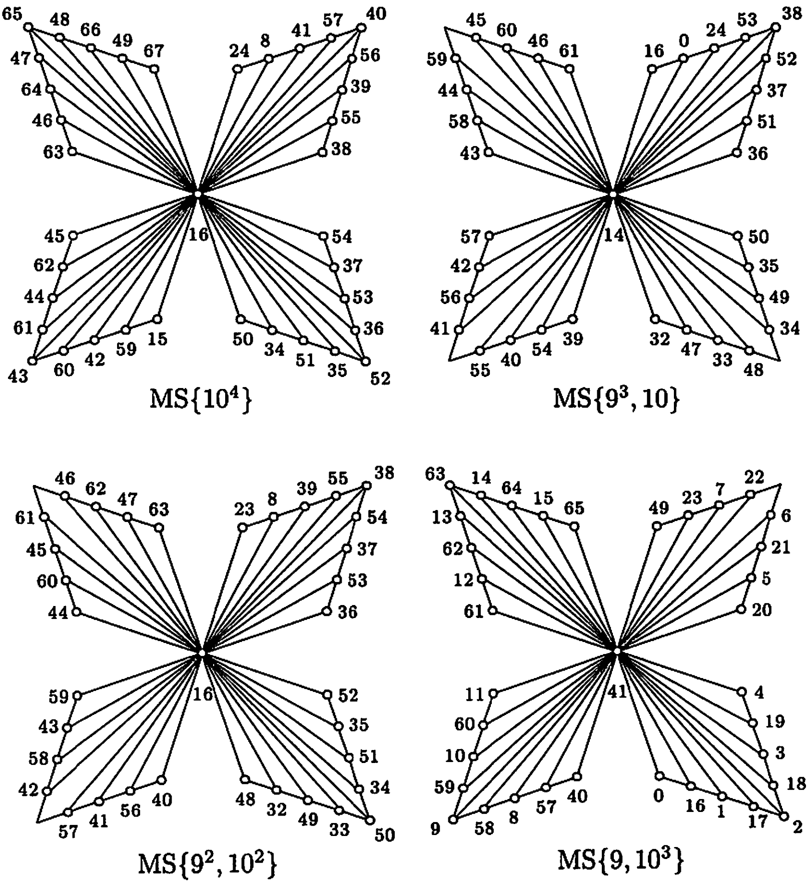

Figure 1 : Harmonious labelings of MS$\{10^4\}$, MS$\{9^3,10\}$, MS $\{9^2,10^2\}$ and MS $\{9,10^3\}$