Supereulerian planar graphs

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Abstract

We investigate the supereulerian graph problems within planar graphs, and we prove that if a 2-edge-connected planar graph G is at most three edges short of having two edge-disjoint spanning trees, then G is supereulerian except a few classes of graphs. This is applied to show the existence of spanning Eulerian subgraphs in planar graphs with small edge cut conditions. We also determine several extremal bounds for planar graphs to be supereulerian.

1 Introduction

Graphs in this note are finite and loopless. Undefined terms and notations are from [1]. As in [1], the edge-connectivity of a graph G is denoted by $\kappa'(G)$ and $d_G(v)$ denotes the degree of a vertex v in G. A graph G is **essentially** k**-edge-connected** if $|E(G)| \geq k+1$ and if for every $E_0 \subseteq E(G)$ with $|E_0| < k$, $G - E_0$ has exactly one component H with $E(H) \neq \emptyset$. The greatest integer k such that G is essentially k-edge-connected is the **essential edge-connectivity** $\kappa_e(G)$ of G. For each $i = 0, 1, 2, \cdots$, denote $D_i(G) = \{v \in V(G) | d_G(v) = i\}$, and for any integer $t \geq 1$, denote $D_t^*(G) =$

$\bigcup_{i \geq t} D_i(G)$. The **edge arboricity** $a_1(G)$ of G is the minimum number of edge-disjoint forests whose union equals G. The **girth** of G, denoted by $g(G)$, is the length of a shortest cycle of G, or ∞ if G is acyclic. We use $H \subseteq G$ ($H \subset G$) to denote the fact that H is a subgraph of G (proper subgraph of G). Let V, W be disjoint subsets of $V(G)$. Then $[V, W]_G$ denotes the set of edges in G that has one end in V and the other end in W. Let $X \subseteq E(G)$. The **contraction** G/X is obtained from G by contracting each edge of X and deleting the resulting loops. If $H \subseteq G$, we write G/H for $G/E(H)$. Note that even if G is a simple graph, contracting some edges of G may result in a graph with multiple edges. A graph with at least two vertices is called a **nontrivial** graph.

A subgraph H of a graph G is **dominating** if $G - V(H)$ is edgeless. Let $O(G)$ denote the set of odd degree vertices of G. A graph G is **Eulerian** if $O(G) = \emptyset$ and G is connected. A graph G is **supereulerian** if G has a spanning Eulerian subgraph.

In [4] Catlin defined collapsible graphs. Given a subset R of $V(G)$, a subgraph Γ of G is called an R**-subgraph** if both $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph G is **collapsible** if for any even subset R of $V(G)$, G has an R-subgraph. Catlin showed in [4] that every vertex of G lies in a unique maximal collapsible subgraph of G. The **reduction** of G, denoted by G', is obtained from G by contracting all maximal collapsible subgraphs of G. A graph G is **reduced** if G has no nontrivial collapsible subgraphs, or equivalently, if $G = G'$, the reduction of G. A **nontrivial vertex** in G' is a vertex that is the contraction image of a nontrivial connected subgraph of G. Note that if G has an $O(G)$-subgraph Γ, then $G - E(\Gamma)$ is a spanning Eulerian subgraph of G. Therefore, every collapsible graph is supereulerian.

Jaeger in [12] showed that if G has two edge-disjoint spanning trees, then G is supereulerian. Defining $F(G)$ to be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees, Catlin [4] and Catlin *et al.* [8] improved Jaeger's result. We put these former results in the following theorem.

Theorem 1.1 *Let G be a graph. Each of the following holds.*

(i) (Jaeger, [12]) If $F(G) = 0$, then G is supereulerian.

(ii) (Catlin, [4]) If $F(G) \leq 1$ and if G is connected, then G is collapsible if and only if the reduction of G is not a K_2.

(iii) (Catlin, Han and Lai, [8]) If $F(G) \leq 2$ and if G is connected, then either G is collapsible, or the reduction of G is a K_2 or a $K_{2,t}$ for some integer $t \geq 1$.

Theorem 1.1(iii) was conjectured in [4] and [5]. In [5], Catlin also conjectured that if $F(G) \leq 3$ and if G is 3-edge-connected, then G is collapsible if and only if the reduction of G is not the Petersen graph. Noting that planar graphs cannot be contracted to the Petersen graph, we in this note prove the following.

Theorem 1.2 *Let G be a 3-edge-connected planar graph. If $F(G) \leq 3$, then G is collapsible.*

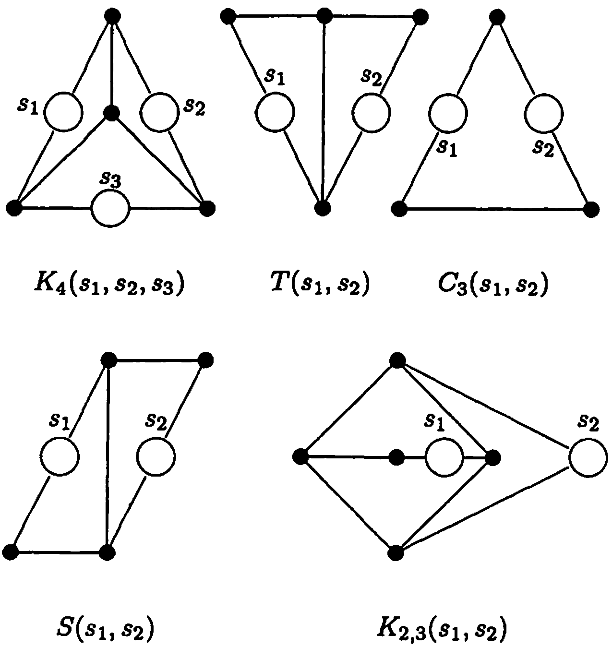


Figure 1

Let $s_i \geq 1 (i = 1, 2, 3)$ be integers. Denote $K_4(s_1, s_2, s_3)$, $T(s_1, s_2)$, $C_3(s_1, s_2)$, $S(s_1, s_2)$ and $K_{2,3}(s_1, s_2)$ to be the graphs depicted in Figure 1, where the s_i $(i = 1, 2, 3)$ vertices and the two vertices connected by the two lines shown in each of the graphs forms a K_{2,s_i} graph. Denote

$$\mathcal{F}_1 = \left\{ K_4(s_1, s_2, s_3), T(s_1, s_2), C_3(s_1, s_2), S(s_1, s_2), \right.$$

$$\left. K_{2,3}(s_1, s_2) \;\middle|\; s_i \geq 1 \, (i = 1, 2, 3) \text{ is an integer} \right\}$$

and $\mathcal{F} = \mathcal{F}_1 \cup \{K_{2,t} \mid t \geq 2\}$. Clearly, each graph G in $\mathcal{F}$ is reduced and $F(G) \leq 3$. We prove the following results.

Theorem 1.3 *Let G be a 2-edge-connected planar graph. If $F(G) \leq 3$, then either G is collapsible or the reduction of G is a graph in $\mathcal{F}$.*

Theorem 1.4 *Let G be an essentially 3-edge-connected planar reduced graph with $\kappa'(G) \geq 2$. If $F(G) \leq 5$, then G has a dominating Eulerian subgraph H such that $D_3^*(G) \subseteq H$.*

Theorem 1.5 *Let G be a simple graph of order n with $\kappa'(G) \geq 2$.*

(i) (Catlin and Li, [9]) If for every edge cut $S \subseteq E(G)$ with $|S| \leq 3$ we have that every component of $G - S$ has order at least $\frac{n}{5}$, then G is supereulerian if and only if G cannot be contracted to $K_{2,3}$.

(ii) (Broersma and Xiong, [2]) If $n \geq 13$ and if for every edge cut $S \subseteq E(G)$ with $|S| \leq 3$ we have that every component of $G - S$ has order at least $\frac{n-2}{5}$, then G is supereulerian if and only if G cannot be contracted to $K_{2,3}$ or $K_{2,5}$.

Applying Theorem 1.4, we can improve Theorem 1.5 within planar graphs.

Theorem 1.6 *Let G be a simple planar graph of order n with $\kappa'(G) \geq 3$. If for every edge cut $S \subseteq E(G)$ with $|S| = 3$ we have that every component of $G - S$ has order at least $\frac{n}{16}$, then G is supereulerian.*

In [3], Cai considered this problem: Find the best possible bound $f(n)$ for a simple graph G with n vertices such that if $|E(G)| \geq f(n)$, then G is supereulerian. Let Q_3 denote the cube ($K_2 \times C_4$) and $Q_3 - v$ denote the cube minus a vertex. Cai proved the following result.

Theorem 1.7 *(Cai, [3]) Let G be a simple graph of order $n \geq 5$ and $\kappa'(G) \geq 2$. If*

$$|E(G)| \geq \binom{n-4}{2} + 6,$$

then exactly one of the following holds:

(i) G is supereulerian.

(ii) G can be contracted to a $K_{2,3}$.

(iii) G is the graph $K_{2,5}$ or $Q_3 - v$.

In [3], Cai conjectured that when restricted to the 3-edge-connected simple graphs, the lower bound can be improved. In [7] and [10], Catlin and Chen settled this conjecture.

Theorem 1.8 *(Catlin and Chen, [7], [10]) Let G be a simple graph with $n \geq 11$ vertices and with $\kappa'(G) \geq 3$. If*

$$|E(G)| \geq \binom{n-9}{2} + 16,$$

then G is collapsible.

Graphs contractible to the Petersen graph indicate the sharpness of this result. In this note, we prove the following related results among planar graphs.

Theorem 1.9 *Let G be a planar graph with n vertices, with $\kappa'(G) \geq 3$ and $g(G) \geq 4$. If $|E(G)| \geq 2n - 5$, then G is collapsible.*

Theorem 1.9 cannot be relaxed to 2-edge-connected planar graphs since $K_{2,t}$ is not collapsible.

Theorem 1.10 *Let G be a planar graph with $\kappa'(G) \geq 3$ such that every edge of G is in a face of degree at most 6. If either G has at most two faces of degree 5 and no faces of degree bigger than 5, or G has exactly one face of degree 6 and no other faces of degree bigger than 4, then G is collapsible.*

Theorem 1.10 is related to a former conjecture of Paulraja ([15], [16]): If G is a 2-connected graph with $\delta(G) \geq 3$ such that every edge of G lies in a cycle of length at most 4, then G is supereulerian. This conjecture was proved in [13].

Theorem 1.11 *If G is a 2-edge-connected simple planar graph with order $n \geq 6$ and $|E(G)| \geq 3n - 8$, then $F(G) = 0$.*

Theorem 1.12 *If G is a 2-edge-connected simple planar graph with $n \geq 9$ vertices and $|E(G)| \geq 3n - 12$ edges, then exactly one of the following holds:*

(i) G is supereulerian.

(ii) G has a maximal collapsible subgraph H with order $n - 4$ such that G/H is a $K_{2,3}$.

In this paper, we present the proofs of Theorem 1.2 and Theorem 1.3 in Section 2. Theorems 1.3 and 1.2 will be applied to prove Theorems 1.4 and 1.6 in Section 3, to prove Theorems 1.9 and 1.10 in Section 4, respectively. The proofs for Theorems 1.11 and 1.12 are in Section 5.

2 Proofs of Theorem 1.2 and Theorem 1.3

The proofs need the help of a reduction technique of the 4-cycle, first introduced by Catlin in [6]. Let G be a graph and let $C = v_1v_2v_3v_4v_1$ be a 4-cycle of G. Let G_π denote the graph obtained from $G - E(C)$ by identifying v_1 and v_3 to form a single vertex w_1, and by identifying v_2 and v_4 to form a single vertex w_2, and by joining w_1 and w_2 with a new edge e_π.

Theorem 2.1 *Each of the following holds:*

(i) (Catlin, [6]) If G_π is collapsible, then G is collapsible; if G_π is

supereulerian, then G is supereulerian.

(ii) (Catlin, Han and Lai, [8]) If G is reduced, then $F(G_\pi) = F(G)-1$.

(iii) (Catlin, [6]) If G is $K_{3,3}$ minus an edge, then G is collapsible.

Applying Theorem 2.1(i), we have the following lemma.

Lemma 2.1 *Let L_1, L_2, L_3 be the graphs given in Figure 2. Then L_1, L_2 and L_3 are collapsible.*

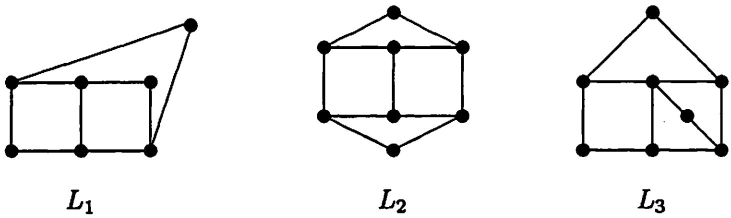


Figure 2

Theorem 2.2 *Let G be a 2-edge-connected graph with $|E(G)| > 0$.*

(i) (Catlin, [4]) If G is the reduction of a graph and $H \subseteq G$, then both G and H are reduced.

(ii) (Catlin, [4]) If G is reduced, then G cannot have any cycles of length less than 4.

(iii) (Catlin, [6]) If G is reduced, then $a_1(G) \leq 2$; if $a_1(G) \leq 2$, then $F(G) = 2|V(G)| - |E(G)| - 2$.

(iv) (Catlin, [4]) Let G′ denote the reduction of a graph G. Then G is supereulerian if and only if G′ is supereulerian; and G is collapsible if and only if $G' = K_1$.

Proof of Theorem 1.2 Let G be a 3-edge-connected planar graph with $F(G) \leq 3$ and let G' denote its reduction. By Theorem 2.2(iv), we only need to show that $G' = K_1$.

By contradiction. Suppose that G' is nontrivial. Since G is 3-edge-connected and planar, G' is also 3-edge-connected and planar. Since every spanning tree of G will become a connected spanning subgraph in any contraction of G, $F(G') \leq F(G) \leq 3$. We may assume that G' is embedded on the plane.

By Theorem 2.2(i) and (ii), G' is reduced and cannot have any cycles of length 2 or 3, and so G' is a simple plane graph each of whose face has degree at least 4. Let f denote the number of faces of G' and let f_i denote the number of faces of G' having degree i, where $i \geq 1$ is an integer. Note that $f_1 = f_2 = f_3 = 0$, and so we have

$$4f + \sum_{i=5}^{\infty}(i-4)f_i = \sum_{i=4}^{\infty} if_i = 2|E(G')|. \tag{1}$$

By (1) and by Euler's formula,

$$2|E(G')| = 2|V(G')| + 2f - 4 = 2|V(G')| + |E(G')| - 4 - \frac{1}{2}\sum_{i=5}^{\infty}(i-4)f_i. \tag{2}$$

Thus $|E(G')| = 2|V(G')| - 4 - \frac{1}{2}\sum_{i=5}^{\infty}(i-4)f_i$. On the other hand, since $F(G') \le 3$ and by Theorem 2.2(iii), $|E(G')| \ge 2|V(G')| - 5$. It follows that

$$2|V(G')| - 5 \le |E(G')| = 2|V(G')| - 4 - \frac{1}{2}\sum_{i=5}^{\infty}(i-4)f_i. \tag{3}$$

If $|E(G')| = 2|V(G')| - 4$, then by Theorem 2.2(iii), $F(G') = 2$, and so by Theorem 1.1(iii) and by the fact that G' is 3-edge-connected, G' must be a collapsible graph. Thus $G' = K_1$ by Theorem 2.2(iv). This contradicts the assumption that G' is nontrivial. Therefore, to obtain our final contradiction, we only need to show that $|E(G')| = 2|V(G')| - 5$ is impossible.

If $|E(G')| = 2|V(G')| - 5$, then by (3), we have

$$\text{either } f_4 = f - 2 \text{ and } f_5 = 2, \text{ or } f_4 = f - 1 \text{ and } f_6 = 1. \tag{4}$$

Since $\kappa'(G') \ge 3$, we must have $f_4 \ge 1$. Let $C = v_1v_2v_3v_4v_1$ denote a 4-cycle of G' and consider G'_π. Since $\kappa'(G') \ge 3$, G'_π is connected. Moreover,

$$\text{if } \kappa'(G'_\pi) \le 2, \text{ then } e_\pi = w_1w_2 \text{ is in an edge cut of size at most 2 in } G'_\pi. \tag{5}$$

Suppose first that $\kappa'(G'_\pi) \ge 2$. Then by Theorem 2.1(ii), $F(G'_\pi) \le 2$. It follows by Theorem 1.1(iii) that either G'_π is collapsible, whence G' is collapsible by Theorem 2.1(i), contrary to the assumption that G' is reduced; or the reduction of G'_π is a $K_{2,t}$ for some integer $t \ge 2$, whence G' has an edge cut of size 2, contrary to the fact that $\kappa'(G') \ge 3$.

Therefore by (5), e_π must be the only cut edge of G'_π. Let G'_1 and G'_2 be the two components of $G'_\pi - e_\pi$ with $w_1 \in V(G'_1)$ and $w_2 \in V(G'_2)$. Then $G' - E(C)$ has two components G_1 and G_2 with $v_1, v_3 \in V(G_1)$ and $v_2, v_4 \in V(G_2)$ such that G'_1 can be obtained from G_1 by identifying v_1 and v_3, and G'_2 can be obtained from G_2 by identifying v_2 and v_4. Since G' is reduced, both G_1 and G_2 are reduced.

If $F(G_1) \leq 2$, then by Theorem 1.1(iii) and by the fact that G_1 is reduced, G_1 is either a K_2 or a $K_{2,t}$, for some integer $t \geq 1$. If $G_1 = K_2$, then G' has a 3-cycle, contrary to Theorem 2.2(ii); if $G_1 = K_{2,t}$, then G' has an edge cut of two edges, contrary to the assumption that $\kappa'(G') \geq 3$. Therefore $F(G_1) \geq 3$. Similarly, $F(G_2) \geq 3$.

By Theorem 2.2(iii), $|E(G_i)| \leq 2|V(G_i)| - 5$ for both $i = 1$ and $i = 2$. It follows by $F(G') \leq 3$ and by Theorem 2.2(iii) that

$$\begin{aligned} 2|V(G')| - 9 &= |E(G')| - 4 = \sum_{i=1}^{2} |E(G_i)| \\ &\leq 2(\sum_{i=1}^{2} |V(G_i)| - 5) = 2|V(G')| - 10, \end{aligned}$$

a contradiction. Thus G' must be a K_1 and so G is collapsible. □

Proof of Theorem 1.3 Suppose that G is not collapsible, and G' is the reduction of G. Then $G' \neq K_1$ and $F(G') \leq 3$. By Theorem 1.2, $\kappa'(G') = 2$. We apply induction on $n = |V(G')|$ to prove $G' \in \mathcal{F}$.

Clearly, $n \geq 4$. If $n = 4$, then $G = K_{2,2}$ and the result holds. We suppose that the result holds for fewer vertices.

Note that $\kappa'(G') = 2$. Let $X \subseteq E(G')$ be an edge cut of G' with $|X| = 2$. Pick an $e \in X$, and denote $[e] = \{e' \in E(G') | \{e, e'\}$ is an edge cut of $G'\}$ $\cup \{e\}$. Then for any $\{e_1, e_2\} \subseteq [e]$, $\{e_1, e_2\}$ is also an edge cut of G. Let $|[e]| = k \geq 2$. Then there are k connected subgraphs $H_1, H_2, \cdots, H_k$ such that H_i, H_{i+1} $(i = 1, 2, \cdots, k-1)$ and H_1, H_k are joined by one edge in $[e]$(see Figure 3), and each H_i $(i = 1, 2, \cdots, k)$ is either a K_1 or 2-edge-connected.

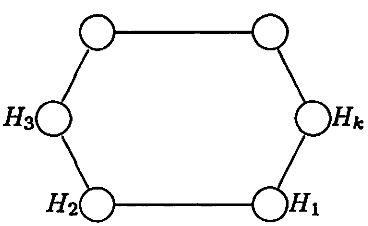


Figure 3

Thus, we have

$$\sum_{i=1}^{k} F(H_i) = \sum_{i=1}^{k} (2|V(H_i)| - |E(H_i)| - 2)$$

$$= 2\sum_{i=1}^{k}|V(H_i)| - \sum_{i=1}^{k}|E(H_i)| - 2k$$
$$= 2|V(G')| - (|E(G')| - k) - 2k = 2|V(G')| - |E(G')| - k$$
$$= F(G') - k + 2.$$

We break it into four cases.

Case 1 $k \geq 5$.

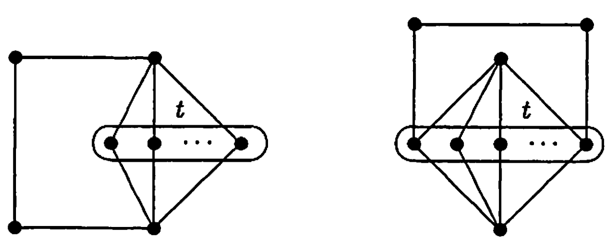
Note that $F(G') \leq 3$, we have $F(H_i) = 0$ $(i = 1, 2, \cdots, k)$ and $k = 5$. Thus $H_i = K_1$ $(i = 1, 2, \cdots, k)$ and $G' = C_5 = C_3(1,1)$.

Case 2 $k = 4$.

Then $\sum_{i=1}^{4} F(H_i) = F(G') - 2$. If $F(G') \leq 2$, then $F(H_i) = 0$ $(i = 1, 2, 3, 4)$ and $G' = C_4 = K_{2,2}$. If $F(G') = 3$, then $\sum_{i=1}^{4} F(H_i) = 1$. Without loss of generality, we assume that $F(H_1) = 1$. By Theorem 1.1(ii), either H_1 is nontrivial and collapsible, contrary to the fact that G' is reduced, or $H_1 = K_2$, contrary to the assumption that $\kappa'(G') \geq 2$ or $k = 4$.

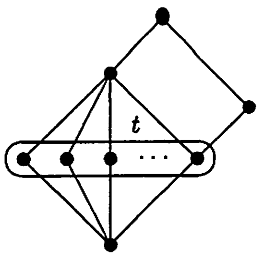
Case 3 $k = 3$.

Then $\sum_{i=1}^{3} F(H_i) = F(G') - 1$. Note that a triangle is collapsible, we have $F(G') = 3$, and there doesn't exist some H_i such that $F(H_i) = 1$. Without loss of generality, we let $F(H_1) = 2$ and $F(H_2) = F(H_3) = 0$. Then $H_2 = H_3 = K_1$. Note that H_1 is 2-edge-connected, we have $H_1 = K_{2,t}$ $(t \geq 2)$ by Theorem 1.1(iii). Thus G' must be one of the following graphs shown in Figure 4.



$G' = C_3(t, 1)$

$G' = C_3(2, 1)$(when $t = 2$)
$G' = K_{2,3}(1, t-2)$(when $t \geq 3$)



$G' = S(t-1, 1)$

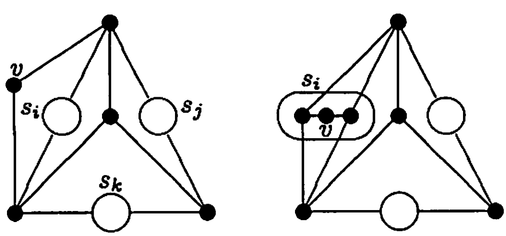
Figure 4

Case 4 $k = 2$.

Let H_1, H_2 be the two components of $G' - [e]$ and we assume that $F(H_1) \le F(H_2)$. Since $\sum_{i=1}^{2} F(H_i) = F(G') \le 3$, $F(H_1) \le 1$. Note that for $i = 1, 2$, either $H_i = K_1$ or H_i is 2-edge-connected, we have $H_1 = K_1$ by Theorem 1.1(ii), and $H_2 \ne K_1$ since C_2 is collapsible. Therefore $F(H_2) \le 3$ and $\kappa'(H_2) \ge 2$.

Note that H_2 is reduced, we have $H_2 \in \mathcal{F}$ by induction. Thus there exists a vertex $v \in V(G')$ such that $d_{G'}(v) = 2$ and v is one vertex of a 4-cycle of G'. Let $G_1 = G' - v$. Then $G_1 \in \mathcal{F}$ by induction.

When $G_1 = K_4(s_1, s_2, s_3)$, there are 4 possible way for v to join G_1(see Figure 5). Let $\{s_i, s_j, s_k\} = \{s_1, s_2, s_3\}$. Then for the first graph in Figure 5, in which case $G' = K_4(s_i + 1, s_j, s_k)$. By Lemma 2.1 and Theorem 2.1(iii), each of the other graphs in Figure 5 contains a collapsible graph, and so G' could not be these three graphs.



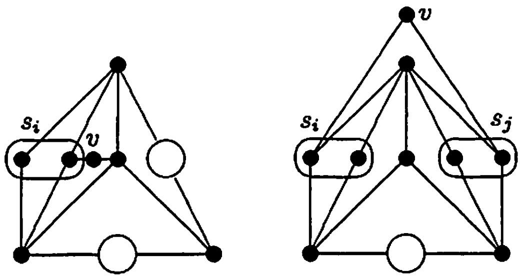


Figure 5

Similarly, we can check other 5 cases. If $G_1 = K_{2,t}$ $(t \geq 2)$, then $G' = K_{2,t+1}$. If $G_1 = T(s_1, s_2)$, then $G' \in \{T(s_1+1, s_2), T(s_1, s_2+1), K_4(s_1, s_2, 1), S(2, s_1), S(2, s_2)\}$. If $G_1 = C_3(s_1, s_2)$, then $G' \in \{C_3(s_1+1, s_2), C_3(s_1, s_2+1), K_{2,3}(s_1, 1), K_{2,3}(s_2, 1)\}$. If $G_1 = S(s_1, s_2)$, then $G' \in \{S(s_1+1, s_2), S(s_1, s_2+1), T(2, s_1), T(2, s_2), K_4(1, 1, s_1), K_4(1, 1, s_2)\}$. If $G_1 = K_{2,3}(s_1, s_2)$, then $G' \in \{K_{2,3}(s_1+1, s_2), K_{2,3}(s_1, s_2+1)\}$. □

3 Proofs of Theorems 1.4 and 1.6

The following theorem and lemma are needed in the proof of Theorem 1.4.

Theorem 3.1 *(Chen et al., [11]) If G is a 3-edge-connected planar graph with $|V(G)| \leq 23$, then G is supereulerian.*

Lemma 3.1 *Let $C = v_1v_2v_3v_4v_1$ be a cycle of a graph G with $N(v_i) - V(C) = \{u_i\}$ $(i = 1, 2, 3, 4)$, and with either $u_1 \neq u_2$ or $u_1 \neq u_3$. If $a_1(G) \leq 2$, then $a_1(G_\pi) \leq 2$.*

Proof Let (E_1, E_2) be a partition of $E(G)$ such that each $G[E_i]$ $(i = 1, 2)$ is acyclic, and let

$$E = E(C) \cup \{u_1v_1, u_2v_2, u_3v_3, u_4v_4\}, \; E_1' = E_1 - E, \; E_2' = E_2 - E$$

$$E_{11} = E_1' \cup \{u_1v_1, u_4v_4, v_1v_2, v_2v_3\}, \quad E_{12} = E_2' \cup \{u_2v_2, u_3v_3, v_3v_4, v_4v_1\}$$

$$E_{21} = E_1' \cup \{u_1v_1, u_2v_2, v_2v_3, v_3v_4\}, \quad E_{22} = E_2' \cup \{u_3v_3, u_4v_4, v_1v_2, v_4v_1\}$$

$$E_{31} = E_1' \cup \{u_2v_2, u_3v_3, v_3v_4, v_4v_1\}, \quad E_{32} = E_2' \cup \{u_1v_1, u_4v_4, v_1v_2, v_2v_3\}$$

$$E_{41} = E_1' \cup \{u_3v_3, u_4v_4, v_1v_2, v_4v_1\}, \quad E_{42} = E_2' \cup \{u_1v_1, u_2v_2, v_2v_3, v_3v_4\}$$

Then for each $i = 1, 2, 3, 4$, (E_{i1}, E_{i2}) is also a partition of $E(G)$ such that each $G[E_{ij}]$ $(j = 1, 2)$ is acyclic. Let $e_\pi = w_1 w_2$ be the new edge in G_π, and for $i = 1, 2, 3, 4$, let

$$E'_{i1} = E_{i1} - E(C), \quad E'_{i2} = (E_{i2} - E(C)) \cup \{e_\pi\}$$

Then (E'_{i1}, E'_{i2}) is a partition of $E(G_\pi)$. Suppose that $a_1(G_\pi) \geq 3$. Note that $G_\pi[E'_{i1}]$ is acyclic by the construction of (E_{i1}, E_{i2}), $G_\pi[E'_{12}]$, $G_\pi[E'_{22}]$, $G_\pi[E'_{32}]$, $G_\pi[E'_{42}]$ contain cycles $u_2 P_1 u_3 w_1 w_2 u_2$, $u_3 P_2 u_4 w_2 w_1 u_3$, $u_4 P_3 u_1 w_1 w_2 u_4$, $u_1 P_4 u_2 w_2 w_1 u_1$, respectively, where P_1, P_2, P_3, P_4 respectively are (u_2, u_3)-path, (u_3, u_4)-path,(u_4, u_1)-path and (u_1, u_2)-path in G. As either $u_1 \neq u_2$ or $u_1 \neq u_3$, $G[E_2]$ contains a cycle $C \subseteq P_1 \cup P_2 \cup P_3 \cup P_4$, a contradiction. □

Proof of Theorem 1.4 By contradiction, suppose that G is a smallest counterexample. Then G is reduced.

Claim 1 $\kappa'(G) \geq 3$.

Since $\kappa_e(G) \geq 3$, that is, G is essentially 3-edge-connected, we only need to prove that $d_G(v) \geq 3$ for any $v \in V(G)$. Suppose that there exists $v \in V(G)$ such that $d_G(v) = 2$. Let $e_1 = vu_1$, $e_2 = vu_2$. Note that G is reduced, G doesn't contain triangle. Thus $u_1 u_2 \notin E(G)$. Let $G_1 = G/e_1$. Then $\kappa'(G_1) \geq 2$, $\kappa_e(G_1) \geq 3$ and $F(G_1) \leq 5$. Note that G is smallest, there exists a dominating Eulerian subgraph H' in G_1 such that $D_3^*(G_1) \subseteq V(H')$. By $u_1 u_2 \notin E(G)$ and $\kappa_e(G) \geq 3$, we have $d_G(u_1) \geq 3$ and $d_G(u_2) \geq 3$. Thus $d_{G_1}(u_i) \geq 3$ $(i = 1, 2)$ and $u_1, u_2 \in H$. Let $H = \begin{cases} H', & \text{if } u_1 u_1 \notin E(H') \\ G[(E(H') - \{u_1 u_2\}) \cup \{vu_1, vu_2\}], & \text{if } u_1 u_2 \in E(H') \end{cases}$. Then H is a dominating Eulerian subgraph of G such that $D_3^*(G) \subseteq H$, a contradiction.

Claim 2 $\kappa_e(G) \geq 4$.

Suppose that S is a 3-edge cut and G_1, G_2 are two components of $G - S$ with $F(G_1) \leq F(G_2)$ and $E(G_1) \neq \emptyset, E(G_2) \neq \emptyset$. Then $F(G_1) + F(G_2) = F(G) + 1 \leq 6$ by Theorem 2.2(iii). Thus we have $F(G_1) \leq 3$.

If G_1 has an cut edge e, let H_1 and H_2 be two components of $G_1 - e$ and H_2 be the component adjacent to at least two edges of S. Then either $[V(H_1), V(G) - V(H_1)]_G = e$ or $[V(H_1), V(G) - V(H_1)]_G$ is a 2-edge cut in G, contrary to Claim 1. So we have $\kappa'(G_1) \geq 2$. Note that G_1 is reduced and $|V(G_1)| \geq 2$, by Theorem 1.3, $G_1 \in \mathcal{F}$. Since $|D_2(G_1)| \leq 3$ and by planarness of G, $G_1 = K_4(1, 1, 1)$. Similarly, $G_2 = K_4(1, 1, 1)$. So G must be the graph shown in Figure 6. Clearly, G is supereulerian, a contradiction.

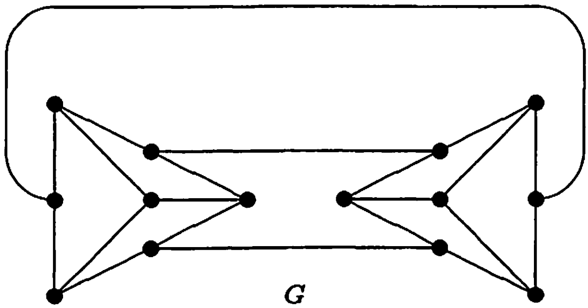


Figure 6

Claim 3 G has at least 13 4-faces.

By Claim 1 and Theorem 3.1, we have $n = |V(G)| \geq 24$. Let x be the number of 4-faces, m the number of the edges, f the number of faces. Then $2m \geq 4x + 5(f - x)$. Thus $2m \geq 5f - x$. Note that $n - m + f = 2$, we have $5n - 3m \geq 10 - x$. Since $F(G) = 2n - m - 2 \leq 5$, we have $x \geq n - 11 \geq 13$. Thus G has at least 13 4-faces.

Claim 4 No two 4-faces $C_1 = v_1v_2v_3v_4v_1$ and $C_2 = v_1v_2v_5v_6v_1$ in G satisfy $d_G(v_i) = 3(i = 1, 2\cdots, 6)$.

By contradiction. Suppose that there exist two 4-faces $C_1 = v_1v_2v_3v_4v_1$ and $C_2 = v_1v_2v_5v_6v_1$ in G such that $d_G(v_i) = 3$ $(i = 1, 2\cdots, 6)$. Let $H = G - \{v_1, v_2, \cdots, v_6\}$. Then we can get the new graph $(G_\pi)_\pi$ by using $\pi-$collapsible 2 times (see the graphs in Figure 7).

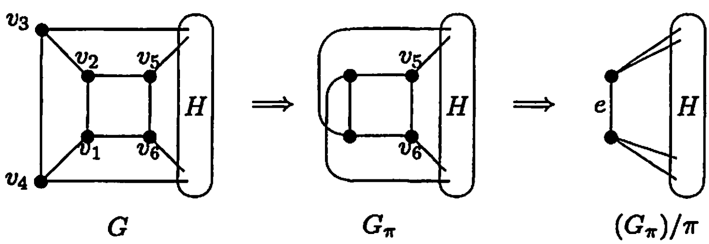


Figure 7

Let $e = w_1w_2$ denote the new edge in $(G_\pi)_\pi$. Clearly, $(G_\pi)_\pi$ is 2-edge-connected, otherwise, e is a cut edge of $(G_\pi)_\pi$ and $X = \{v_1v_2, v_3v_4, v_5v_6\}$ is a 3-edge cut in G in which both sides of $G-X$ have edges, contrary to Claim 2. Next we want to prove that G is 3-edge-connected. Suppose that G is not 3-edge-connected. Then $\{e, e_1\}$ is a 2-edge cut, where $e_1 = w_3w_4 \in E(H)$.

Let H_1, H_2 be two components of $(G_\pi)_\pi - \{e, e_1\}$, and $w_1, w_3 \in V(H_1)$, $w_2, w_4 \in V(H_2)$. Then $E(H_1 - \{w_1\}) = \emptyset$ and $E(H_2 - \{w_2\}) = \emptyset$ by Claim 2. Thus $N_{(G_\pi)_\pi}(w_1) = w_3$ and $N_{(G_\pi)_\pi}(w_2) = w_4$. Hence $V(H) = \{w_3, w_4\}$, $N_G(v_3) \cap H = N_G(v_5) \cap H = \{w_3\}$ and $N_G(v_6) \cap H = N_G(v_4) \cap H = \{w_4\}$. So G is supereulerian, a contradiction. Thus G is 3-edge-connected. Clearly, $(G_\pi)_\pi$ is planar with $F((G_\pi)_\pi) = F(G) - 2 \le 3$ by Lemma 3.1 and Theorem 2.2(iii). By Theorem 1.3, $(G_\pi)_\pi$ is supereulerian. Thus G is supereulerian by Theorem 2.1(i), a contradiction. So Claim 4 holds.

Claim 5 Suppose that $C = v_1v_2v_3v_4v_1$ is a 4-face of G. Then for $i = 1, 2, 3, 4$, $d_G(v_i) = 3$.

Let $G_1 = (G - \{v_1v_2, v_3v_4\})/\{v_1v_4, v_2v_3\}$ and $G_2 = (G - \{v_1v_4, v_2v_3\})/\{v_1v_2, v_3v_4\}$. First we prove that either $\kappa_e(G_1) \ge 3$ or $\kappa_e(G_2) \ge 3$. Suppose that $\kappa_e(G_i) = 2$ for $i = 1, 2$. Then G must have the structure in Figure 8.

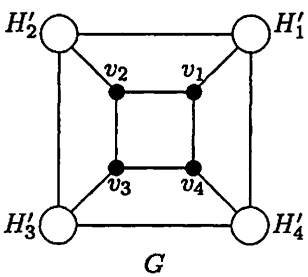


Figure 8

For $i = 1, 2, 3$, $|[V(H_i'), V(H_{i+1}')]_G| = 1$ and $|[V(H_4'), V(H_1')]_G| = 1$. Let $H_i = H_1' \cup \{v_i\}$ $(i = 1, 2, 3, 4)$. Then

$$\begin{aligned}\sum_{i=1}^{4} F(H_i) &= \sum_{i=1}^{4}(2|V(H_i)| - |E(H_i)| - 2) \\ &= 2n - (m - 8) - 8 = 2n - m = F(G) + 2 \le 7.\end{aligned}$$

Note that $|V(H_i)| \ge 2$, there are at least three of these H_i's, say H_1, H_2, H_3, such that $F(H_i) \le 2$ $(i = 1, 2, 3)$. Since $\kappa'(G) \ge 3$, we have $H_i \ne K_{2,t}$ $(i = 1, 2, 3)$. Thus $H_1 = H_2 = H_3 = K_2$ by Theorem 1.1(iii). This contradicts Claim 4. So without loss generality, we assume that $\kappa_e(G_1) \ge 3$ and w_1, w_2 are two new vertices. By the assumption of G, G_1 has a dominating Eulerian subgraph H' such that $D_3^*(G_1) \subseteq H'$. If either $w_1 \in H'$ or $w_2 \in H'$, then we can always get a dominating Eulerian subgraph H of G such that $D_3^*(G) \subseteq H$, it is impossible. Thus $w_1, w_2 \notin H'$. Therefore Claim 5 holds.

By Claim 3, let C_1, C_2, C_3, C_4, C_5 be five 4-faces of G. By Claims 4,5, no two of these 4-faces have common vertices or edges. Applying π−collapsible to each of these five 4-faces, we get the graph G_3 with $F(G_3) = 0$ by Lemma 3.1 and Theorem 2.2(iii). Note that G_3 is connected, G_3 is supereulerian by Theorem 1.1(i). Thus G is also supereulerian, a contradiction. □

Proof of Theorem 1.6 Let G' be the reduction of G. If $G' = K_1$, then G is supereulerian. Next we suppose that $G' \neq K_1$. Then G' is 3-edge-connected and nontrivial. Denote $d_i = |D_i(G')|$ $(i \geq 3)$.

If $d_3 \geq 17$, then we assume that $v_1, v_2, \cdots, v_{17}$ are the vertices of $V(G')$ in $D_3(G')$, i.e. $d_{G'}(v_i) = 3$ for each i, and the corresponding pre-images are H_1, H_2, $\cdots$, H_{17}. Each H_i is joined to the rest of G by an edge cut consisting of $d_{G'}(v_i) = 3$ edges. By the hypothesis of Theorem 1.6, $|V(H_i)| \geq \frac{n}{16}$, and

$$n = |V(G)| \geq \sum_{i=1}^{17} |V(H_i)| \geq \frac{17n}{16},$$

a contradiction. So $d_3 \leq 16$.

By Theorem 1.4, we only need to consider $F(G') \geq 6$. Note that $|V(G')| = \sum_{i\geq 3} d_i$, $2|E(G')| = \sum_{v\in V(G)} d_G(v) = \sum_{i\geq 3} id_i$, and $F(G') = 2|V(G')|$ $-|E(G')| - 2$, we have the following

$$d_3 \geq 16 + \sum_{i\geq 5}(i-4)d_i.$$

Thus $n = d_3 = 16$. By Theorem 3.1, G is supereulerian. □

4 Proofs of Theorems 1.9 and 1.10

We shall apply Theorem 1.2 to prove both Theorems 1.9 and 1.10. First, we need one more lemma in this section.

Lemma 4.1 *Let G be a planar graph such that every face of G has degree at least 4. Then $|E(G)| \leq 2|V(G)| - 4$.*

Proof Let f denote the number of faces of G. Since every face of G has degree at least 4, $4f \leq 2|E(G)|$ and so the lemma follows from Euler's formula. □

Proof of Theorem 1.9 Let G be a planar graph with $\kappa'(G) \geq 3$ and with n vertices. Assume that $|E(G)| \geq 2n - 5$. Let G' denote the reduction of G. By Theorem 2.2(iv), it suffices to show $G' = K_1$.

By contradiction, assume that G' is nontrivial. Then by Theorem 2.2(ii), G' also has girth at least 4. Note that G' is planar and $\kappa'(G') \geq 3$, let $H_1, \cdots H_l$ denote the nontrivial maximal collapsible subgraphs of G and let p denote the number of vertices of G'. Then by Lemma 4.1, each $|E(H_i)| \leq 2|V(H_i)| - 4$,

$$2n - 5 \leq |E(G)| = \sum_{i=1}^{l} |E(H_i)| + |E(G')| \leq 2\sum_{i=1}^{l} |V(H_i)| - 4l + |E(G')|,$$

and so $|E(G')| \geq 2p - 5 + 2l$. By Theorem 1.1(iii) and $\kappa'(G') \geq 3$, we have $F(G') \geq 3$. Thus, by Theorem 2.2(iii), $2p - 5 \geq |E(G')| \geq 2p - 5 + 2l$, and so $l = 0$ and $F(G') = 3$. Therefore by Theorem 1.2, G' must be collapsible, contrary to the fact that G' does not have nontrivial collapsible subgraphs. This proves Theorem 1.9. □

Proof of Theorem 1.10 Let G' be the reduction of G. Again we argue by contradiction and assume that $G' \not\cong K_1$. Note that $\kappa'(G') \geq 3$, by Euler's formula, by Theorem 2.2(iii) and by (1) in the process of the proof of Theorem 1.2, one concludes that $F(G') = 3$ and so Theorem 1.10 follows from Theorem 1.2. □

5 Proofs of Theorems 1.11 and 1.12

A few more lemmas and a former theorem of Nash-Williams and Tutte are needed in the proofs in this section.

Theorem 5.1 *(Nash-Williams [14] and Tutte [17]) A graph $G = (V, E)$ contains l edge-disjoint spanning trees if and only if for each partition $(V_1, V_2, \cdots, V_k)$ of V, the number of edges which have end in different parts of the partition is at least $l(k-1)$.*

Proof of Theorem 1.11 Suppose $V = (V_1, V_2, \cdots, V_k)$ is any partition of V. Without loss of generality, let $|V_1| = |V_2| = \cdots = |V_l| = 1$, $|V_{l+1}| = \cdots = |V_{l+m}| = 2$, and for $l + m + 1 \leq j \leq k$, $|V_j| \geq 3$. Since G is a 2-connected simple planar graph, $|E(G[V_j])| \leq 3|V_j| - 6$ for $l+m+1 \leq j \leq k$; $|E(G[V_i])| \leq 1$ for $l+1 \leq i \leq l+m$; and $E(G[V_j]) = \emptyset$ for $1 \leq j \leq l$. Then, we have

$$\begin{aligned}\sum_{1\leq i<j\leq k} |[V_i, V_j]_G| &= |E(G)| - \sum_{j=1}^{k} |E(G[V_j])| \\ &\geq 3n - 8 - m - [3(n - 2m - l) - 6(k - l - m)] \\ &= 6k - 3l - m - 8 \\ &= 2(k-1) + 4k - 3l - m - 6.\end{aligned}$$

We consider the following cases:

Case 1 $l = 0$.
By $n \geq 6$ and $k \geq 2$, $4k - 3l - m - 6 = 3k + (k - m) - 6 \geq 0$.

Case 2 $m = 0$.
If $l = k$, then $4k - 3l - m - 6 = k - 6 \geq 0$ since $k = n \geq 6$. If $l < k$, then $4k - 3l - m - 6 = 3(k - l) + k - 6 \geq 0$ except for $l = 1, k = 2$. But when $l = 1, k = 2$, since G is a 2-edge-connected simple graph, $|[V_1, V_2]_G| \geq 2 = 2(k - 1)$.

Case 3 $l > 0$ and $m > 0$.
If $k = l+m$, then $4k-3l-m-6 = 3m+l-6 = (2m+l)+m-6 = n+m-6 \geq 0$ since $n = 2m + l \geq 6$. If $k > l + m$, then $k - l \geq m + 1 \geq 2$ and $k - m \geq 0$. It follows that $4k - 3l - m - 6 = 3(k - l) + (k - m) - 6 \geq 0$.

Therefore, in any case, $\sum_{1 \leq i < j \leq k} |[V_i, V_j]_G| \geq 2(k - 1)$, and so G must have two edge-disjoint spanning trees by Theorem 5.1. □

We shall prove a stronger result than Theorem 1.12, as stated below.

Theorem 5.2 *If G is a 2-edge-connected simple planar graph with $n \geq 9$ vertices and with $|E(G)| \geq 3n - 12$ edges, then exactly one of the following holds:*

(i) G is collapsible.

(ii) The reduction of G is a 4-cycle.

(iii) The reduction of G is isomorphic to $K_{2,3}$ with exactly one nontrivial vertex whose pre-image is a maximal planar graph of $n - 4$ vertices.

We need two more lemmas.

Lemma 5.1 *If G is a simple planar graph with $n \geq 9$ vertices and with $|E(G)| \geq 3n - 12$, then G is not reduced.*

Proof If G is reduced, then by Theorem 2.2(ii) and Lemma 4.1, $2n - 4 \geq |E(G)| \geq 3n - 12$, whence $n \leq 8$, contrary to the assumption that $n \geq 9$. Therefore, G is not reduced. □

Lemma 5.2 *Let $C > 0$ be a constant, and let G be a simple planar graph with n vertices and with l nontrivial maximal collapsible subgraphs. Let G' denote the reduction of G. If $|E(G)| \geq 3n - C$, then*

$$|E(G')| \geq 3|V(G')| - C + 3l.$$

Proof Let $H_1, \cdots H_l$ be nontrivial maximal collapsible subgraphs of G. Let $G' = G/(\bigcup_{i=1}^{l} E(H_i))$ be the reduction of G with n' vertices. Since each H_i is a nontrivial planar graph for each i with $1 \leq i \leq l$,

$$|E(H_i)| \leq 3|V(H_i)| - 6. \tag{6}$$

Note that $|V(G')| = n - \sum_{i=1}^{l} |V(H_i)| + l$. It follows from (6) that

$$\begin{aligned} |E(G')| &= |E(G)| - \sum_{i=1}^{l} |E(H_i)| \geq |E(G)| - \sum_{i=1}^{l} (3|V(H_i)| - 6) \\ &\geq 3n - C - 3(n - |V(G')| + l) + 6l = 3|V(G')| - C + 3l. \end{aligned}$$

This proves the lemma. □

Proof of Theorem 5.2 Since the 4-cycle and $K_{2,3}$ are not collapsible, (i),(ii) and (iii) are mutually exclusive. We assume that both Theorem 5.2(i) and Theorem 5.2(ii) are false, and want to prove that Theorem 5.2(iii) must hold.

By Lemma 5.1, G is not reduced. Let $H_1, \cdots H_l$ be the nontrivial maximal collapsible subgraphs of G. Let $G' = G/(\bigcup_{i=1}^{l} E(H_i))$ be the reduction of G with n' vertices. By Theorem 2.2(i), G' is reduced and so by Lemma 5.2 with $C = 12$ and $l \geq 1$, we have

$$|E(G')| \geq 3|V(G')| - 12 + 3l. \tag{7}$$

By Theorem 2.2(ii) and Lemma 4.1,

$$2|V(G')| - 4 \geq |E(G')|. \tag{8}$$

It follows from (7) and (8) that

$$2|V(G')| - 4 \geq |E(G')| \geq 3|V(G')| - 12 + 3l. \tag{9}$$

By (9), $|V(G')| \leq 8 - 3l$. Since any 2-edge-connected graph with 3 vertices is collapsible, and since any simple graph with 4 vertices is either collapsible or is isomorphic to the 4-cycle, we must have $l = 1$ and $|V(G')| = 5$. Therefore, equalities must hold everywhere in (9), and so by Theorem 1.1(iii) and by $|V(G')| = 5$, $G' \cong K_{2,3}$ with exactly one nontrivial vertex, named by H.

Note that $|E(H)| \geq |E(G)| - 6 \geq 3n - 18$ and that $|V(H)| = n - 4$. It follows that $|E(H)| \geq 3(n-4) - 6 = 3|V(H)| - 6$, and so H must be a maximal planar graph with $n - 4$ vertices. □

Proof of Theorem 1.12 Note that 4-cycles are supereulerian, and so Theorem 1.11 follows from Theorem 2.2(iv) and Theorem 5.2. □

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