THE METAMORPHOSIS OF $K_4 \setminus e$ DESIGNS INTO MAXIMUM PACKINGS OF K_n WITH 4-CYCLES

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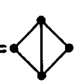
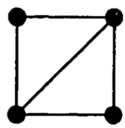
Abstract

Let $K_4 \setminus e =$ . If remove the "diagonal" edge the result is a 4-cycle. Let (X, B) be a $K_4 \setminus e$ design of order n; i.e., an edge disjoint decomposition of K_n into copies of $K_4 \setminus e$. Let $D(B)$ be the collection of "diagonals" removed from the graphs in B and $C_1(B)$ the resulting collection of 4-cycles. If $C_2(B)$ is a reassembly of these edges into 4-cycles and L is the collection of edges in $D(B)$ not used in a 4-cycle of $C_2(B)$, then $(X, (C_1(B) \cup C_2(B)), L)$ is a packing of K_n with 4-cycles and is called a *metamorphosis* of (X, B). We construct for every $n \equiv 0$ or $1 (\bmod 5) \geq 6, n \neq 11$, a $K_4 \setminus e$ design of order n having a metamorphosis into a *maximum packing* of K_n with 4-cycles. There exists a maximum packing of K_{11} with 4-cycles, but it cannot be obtained from a $K_4 \setminus e$ design.

1 Introduction

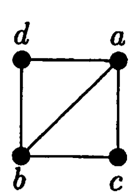
A $K_4 \setminus e$ *design* of order n is a pair (X, B), where B is a collection of edge-disjoint copies of the graph



which partitions the edge set of K_n (the complete undirected graph on n vertices) with vertex set X. It is well-known that the spectrum for $K_4 \setminus e$ designs (= the set of all n such that a $K_4 \setminus e$ design of order n exists) is precisely the set of all $n \equiv 0$ or $1 \pmod 5 \geq 6$. (See [1] for example.) A 4-*cycle system* of order n is a pair (X, C), where C is a collection of edge-disjoint 4-cycles which partitions the edge set of K_n with vertex set X. Again it is well-known [2] that the spectrum for 4-cycle systems is precisely the set of all $n \equiv 1 \pmod 8$. In both of the above definitions if we drop the quantification "partitions" we have the definition of a *partial* $K_4 \setminus e$ design and a *partial* 4-cycle system.

In what follows we will denote the edge with vertices a and b by $\{a, b\}$; the m-cycle with edges $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}, \{x_m, x_1\}$ by any cyclic shift of $(x_1, x_2, x_3, \ldots, x_m)$ or $(x_2, x_1, x_m, x_{m-1}, \ldots, x_3)$; and the graph

$K_4 \setminus e =$



by any one of $[a, b, c, d]$, $[a, b, d, c]$, $[b, a, c, d]$, or $[b, a, d, c]$.

A *packing* of K_n with 4-cycles is a triple (X, C, L), where (X, C) is a *partial* 4-cycle system, and L is the set of edges which do not belong to one of the 4-cycles in C. (So that there is no confusion, $E(K_n) = E(C) \cup L$.) The collection of edges belonging to L is called the *leave* of the packing. If $|C|$ is as large as possible, or equivalently, $|L|$ is as small as possible, the packing (X, C, L) is said to be *maximum*.

It is well-known (see [2] for example) that a maximum packing of K_n with 4-cycles has leave:

(i) a 1-factor if n is even;

(ii) the empty set if $n \equiv 1 \pmod 8$, (the spectrum for 4-cycle systems is precisely the set of all $n \equiv 1 \pmod 8$);

(iii) a 3-cycle if $n \equiv 3 \pmod 8$;

(iv) a graph of even degree with 6 edges (= 2 disjoint 3-cycles, 2 3-cycles with a common vertex (= a bowtie), or a 6-cycle) if $n \equiv 5 \pmod 8$ (only a bowtie is possible for $n = 5$); and

(v) a 5-cycle if $n \equiv 7 \pmod 8$.

Now let (X, B) be a $K_4 \setminus e$ design of order n and let $D(B) = \{\{a, b\} | [a, b, c, d] \in B\}$ and $C_1(B) = \{(a, c, b, d) | [a, b, c, d] \in B\}$. Then $(X, C_1(B))$ is a partial 4-cycle system. If the edges belonging to $D(B)$ can be arranged into a collection of 4-cycles $C_2(B)$ with leave L, then $(X, C_1(B) \cup C_2(B), L)$ is a packing of K_n with 4-cycles, and is said to be a *metamorphosis* of the $K_4 \setminus e$ design (X, B). (The algorithm of going from (X, B) to $(X, C_1(B) \cup C_2(B), L)$ is also called a *metamorphosis*.)

The purpose of this paper is the complete solution of the problem of constructing for each $n \equiv 0$ or $1 \pmod 5 \geq 6, n \neq 11$, a $K_4 \setminus e$ design having a metamorphosis into a maximum packing of K_n with 4-cycles with all possible leaves. (There exists a maximum packing of K_{11} with 4-cycles, but it *cannot* be obtained from a $K_4 \setminus e$ design.)

2 $K_4 \setminus e$ designs of small order.

In this section we give 12 examples which are necessary for the recursive constructions in the next section as well as a proof of the nonexistence of a $K_4 \setminus e$ design of order 11 having a metamorphosis into a maximum packing of K_{11} with 4-cycles.

Example 2.1 (n = 6) Let (X, B) be the $K_4 \setminus e$ design where $B = \{[1,2,5,6],[3,4,1,2],[5,6,3,4]\}$. Then $(X, C_1(B) \cup C_2(B), L)$ is a maximum packing of K_6 with 4-cycles, where $C_2(B) = \emptyset$ and $L = \{\{1,2\},\{3,4\},\{5,6\}\}$. □

Example 2.2 (n = 10) Let (X, B) be the $K_4 \setminus e$ design where $B = \{[1,2,3,4],[3,4,5,6],[5,6,1,2], [7,8,1,2],[9,10,1,2],[7,9,3,4],[8,10,3,4], [7,10,5,6],[8,9,5,6]\}$. Then $(X, C_1(B) \cup C_2(B), L)$ is a maximum packing of K_{10} with 4-cycles where $C_2(B) = \{(7,9,8,10)\}$ and $L = \{\{1,2\},\{3,4\}, \{5,6\},\{7,8\},\{9,10\}\}$. □

Lemma 2.3 *There does not exist a $K_4 \setminus e$ design of order* 11 *having a metamorphosis into a maximum packing of K_{11} with* 4*-cycles.*

Proof: To begin with, there exists a maximum packing of K_{11} with 4-cycles. (See [2] for example.). What we show here is that we *cannot obtain* a maximum packing from a $K_4 \setminus e$ design of order 11. Let (X, B) be a $K_4 \setminus e$ design of order 11, $D(B) = \{\{a,b\} | [a,b,c,d] \in B\}$, and suppose there exists a 4-cycle (x,y,z,w) where the edges belong to $D(B)$. Then B contains 4 graphs of the form $[x,y,\cdot,\cdot]$, $[y,z,\cdot,\cdot]$, $[z,w,\cdot,\cdot]$, and $[w,x,\cdot,\cdot]$. Now, the edge $\{x,z\}$ cannot belong to any of these graphs and so must belong to one of the 7 remaining graphs in B. This requires at least one of these graphs to be of the form $[x,\cdot,\cdot,\cdot]$ or $[z,\cdot,\cdot,\cdot]$. Hence the degree of either x or z restricted to these 3 graphs is 9. Since the degree of each vertex in K_{11} is 10 and since each graph in B has degree 2 or 3, this is impossible. Therefore we cannot assemble even one 4-cycle from the edges in $D(B)$, much less two. □

Example 2.4 (n = 15) Let (X, B) be the $K_4 \setminus e$ design where $X = \{1,2,3,4,5\} \times Z_3$ and $B = \{[(1,2+i),(2,i),(1,i),(3,1+i)],\ [(1,i),(2,2+i),(4,2+i),(5,2+i)],\ [(2,i),(2,1+i),(4,2+i),(5,2+i)],\ [(3,i),(3,1+i),(2,1+i),(4,2+i)],\ [(3,i),(4,i),(1,2+i),(5,2+i)],\ [(3,i),(5,i),(1,i),(5,i+1)],\ [(4,i),(5,1+i),(1,i),(4,1+i)] \mid i \in Z_3\}$. Then $(X, C_1(B) \cup C_2(B), L)$ is a maximum packing of K_{15} with 4-cycles where $C_2(B) = \{((1,0),(2,2),(2,0),(2,1))\} \cup \{((3,i+1),(3,i),(4,i),(5,i+1)) \mid i \in Z_3\}$ and $L = \{((1,1),(2,0),(1,2),(2,1),(2,2))\}$. □

Example 2.5 (n = 15, with a hole H of size 5 having a metamorphosis into a packing of $K_{15} \setminus H$ with 4-cycles with leave a 3-cycle having *exactly one* vertex in the hole H.) Let $B = \{[1,11,2,10],$ $[3,9,1,11], [6,11,5,7], [4,8,6,11], [2,5,12,13], [3,8,12,13], [7,10,12,13],$ $[1,4,12,13], [6,9,12,13], [5,10,14,15], [3,7,14,15], [2,8,14,15], [1,6,14,15],$ $[4,9,14,15], [1,7,5,8], [2,6,3,10], [3,4,5,10], [8,9,5,10], [2,7,4,9]\}$. Then (X, B) is a $K_4 \setminus e$ design of order 15 with *hole* $H = \{11,12,13,14,15\}$; i.e., B is a decomposition of $E(K_{15}) \setminus E(K_5)$ into copies of $K_4 \setminus e$ (where $V(K_5) = \{11,12,13,14,15\}$). Then $(X, C_1(B) \cup C_2(B), L)$ is a packing of $K_{15} \setminus K_5$ with 4-cycles where $C_2(B) = \{(1,4,3,7),(3,8,4,9),(2,5,10,7),(2,6,9,8)\}$ and $L = \{(1,6,11)\}$. *Note* that the leave $(1,6,11)$ has exactly one vertex in H. □

Example 2.6 (n = 16) Let (X_1, B_1) be the $K_4 \setminus e$ design of order 10 in Example 2.2, $F = \{F_1, F_2, F_3, F_4, F_5\}$ a 1-factorization of K_6 with vertex set X_2 *disjoint* with X_1, and $\pi = \{\{x_1,y_1\},\{x_2,y_2\},\{x_3,y_3\},\{x_4,y_4\},\{x_5,y_5\}\}$ a 1-factor of K_{10} with vertex set X_1. Define a collection B of copies of $K_4 \setminus e$ as follows:

(1) $B_1 \subseteq B$; and

(2) $[a, b, x_i, y_i] \in B$ for all $\{a, b\} \in F_i$, $i = 1, 2, 3, 4, 5$.

Then $(X_1 \cup X_2, B)$ is a $K_4 \setminus e$ design of order 16. The metamorphosis is the following. Delete the edges $\{a, b\}, [a, b, x_i, y_i] \in B$, from the type (2) graphs and let (X_2, C_2, L_2) be the maximum packing in Example 2.1. Further, let $(X_1, C_1(B_1) \cup C_2(B_1), L_1)$ be the metamorphosis in Example 2.2. Then $(X_1 \cup X_2, C_1(B) \cup C_2(B), L)$ is a metamorphosis of $(X_1 \cup X_2, B)$ into a maximum packing of K_{16} with 4-cycles where $C_1(B) = C_1(B_1) \cup \{(a, x_i, b, y_i) \mid [a, b, x_i, y_i] \text{ is a type (2) graph}\}$, $C_2(B) = C_2(B_1) \cup C_2$, and $L = L_1 \cup L_2$. □

Example 2.7 (n = 16, with a hole H of size 6 having a metamorphosis into a packing of $K_{16} \setminus H$ with 4-cycles with leave a 1-factor of $V(K_{16} \setminus H)$.) Let H, X_1, X_2 be pairwise disjoint sets where $|H| = |X_1| = 6$ and $|X_2| = 4$. Let $F = \{F_1, F_2, F_3, F_4, F_5\}$ be a 1-factorization of X_1 and $G = \{G_1, G_2, G_3\}$ a 1-factorization of X_2. Let $X = H \cup X_1 \cup X_2$ and define a collection B of copies of $K_4 \setminus e$ as follows:

(1) $[x, y, a_1, b_1] \in B$ for all $\{x, y\} \in F_1 \cup G_1$, $[x, y, a_2, b_2] \in B$ for all $\{x, y\} \in F_2 \cup G_2$, and $[x, y, a_3, b_3] \in B$ for all $\{x, y\} \in F_3 \cup G_3$, where $H = \{a_1, b_1, a_2, b_2, a_3, b_3\}$.

(2) $[x, y, c_1, d_1] \in B$ for all $\{x, y\} \in F_4$, and $[x, y, c_2, d_2] \in B$ for all $\{x, y\} \in F_5$, where $X_2 = \{c_1, d_1, c_2, d_2\}$.

Then (X, B) is a $K_4 \setminus e$ design with hole H. The metamorphosis is the following. Delete all edges from X_1 and X_2 and let (X_1, C_1, L_1) and (X_2, C_2, L_2) be maximum packings of K_6 and K_4 with 4-cycles. ((X_2, C_2, L_2) consists of exactly one 4-cycle with leave a 1-factor.) Then $(X, C_1(B) \cup C_2(B), L_1 \cup L_2)$ is a packing of (X, B) into 4-cycles with hole H, $C_2(B) = C_1 \cup C_2$, with leave the 1-factor $L_1 \cup L_2$ of $V(K_{16} \setminus H)$. □

Example 2.8 (n = 20.) Let (X, B) be the $K_4 \setminus e$ design with $B = \{[2,12,1,11], [3,13,6,16], [4,14,10,20], [5,15,4,14], [6,16,1,11], [7,17,1,11], [8,18,5,15], [9,19,6,16], [10,20,1,11], [1,11,3,13], [2,13,4,14], [2,16,8,18], [4,18,9,19], [5,19,10,20], [6,17,4,14], [7,19,2,12], [8,20,3,13], [12,3,14,4], [12,6,18,8], [14,8,19,9], [15,9,20,10], [16,7,14,4], [17,9,12,2], [18,10,13,3], [13,12,5,15], [16,12,10,20], [18,14,1,11], [19,15,1,11], [17,16,5,15], [19,17,3,13], [20,18,7,17], [2,3,5,15], [2,6,10,20], [4,8,1,11], [5,9,1,11], [6,7,5,15], [7,9,3,13], [8,10,7,17]\}$. Then $(X, C_1(B) \cup C_2(B), L)$ is a maximum packing of K_{20} with 4-cycles where $C_2(B) = \{(2,13,12,16), (7,19,17,16), (2,6,12,3), (5,9,15,19), (8,10,18,14), (6,7,9,17), (4,8,20,18)\}$ and $L = \{\{1,11\}, \{2,12\}, \{3,13\}, \{4,14\}, \{5,15\}, \{6,16\}, \{7,17\}, \{8,18\}, \{9,19\}, \{10,20\}\}$. □

Example 2.9 (n = 21, with all possible leaves). Let $X = Z_5 \times \{1,2,3,4\}$ and let (X, G, B^*) be a TDD $(5,4)$ with groups $G = Z_5 \times \{i\}; i = 1,2,3,4$; and blocks $B^* = \{(i,1),(i,2),(i,3),(i,4)\}$, $\{(i,1),(3+i,2),(4+i,3),(1+i,4)\}$, $\{(i,1),(2+i,2),(1+i,3),(4+i,4)\}$, $\{(3+i,1),(2+i,2),(i,3),(1+i,4)\}$, and $\{(2+i,1),(3+i,2),(i,3),(4+i,4)\}$, $i \in Z_5$. Set $S = \{\infty\} \cup X$ and define a collection B of copies of $K_4 \setminus e$ as follows:

(1) For each $i = 1,2,3,4$, define a $K_4 \setminus e$ design on $\{\infty\} \cup (Z_5 \times \{i\})$ having a metamorphosis into a maximum packing with 4-cycles with leave $\{\{\infty,(0,i)\}, \{(1,i),(2,i)\}, \{(3,i),(4,i)\}\}$ and put the copies of $K_4 \setminus e$ in B.

(2) Remove the edges $\{(i,1),(i,3)\}$, $\{(i,1),(3+i,2)\}$, $\{(i,1),(2+i,2)\}$, $\{(3+i,2),(i,3)\}$, and $\{(2+i,2),(i,3)\}$ from the blocks belonging to B^* and rearrange these edges into the 5 copies of $K_4 \setminus e$ $[(i,1),(i,3),(2+i,2),(3+i,2)]$, $i \in Z_5$. Place these graphs in B as well as $[(i,2),(i,4),(i,1),(i,3)]$, $[(4+i,3),(1+i,4),(i,1),(3+i,2)]$, $[(i+1,3),(4+i,4),(i,1),(2+i,2)]$, $[(1+i,4),(3+i,1),(2+i,2),(i,3)]$, and $[(4+i,4),(2+i,1),(3+i,2),(i,3)]$, for all $i \in Z_5$.

Then (S, B) is a $K_4 \setminus e$ design. The metamorphosis is the following:

Recall that the edges $\{\infty, (0,i)\}, \{(1,i),(2,i)\}$, and $\{(3,i),(4,i)\}$, $i = 1,2,3,4$, are deleted in (1). Now remove the edges $\{(i,1),(i,3)\}$ from the graphs $[(i,1),(i,3),(2+i,2),(3+i,2)]$ in (2) as well as the edges $\{(i,2),(i,4)\}$, $\{(4+i,3),(1+i,4)\}$, $\{(1+i,3),(4+i,4)\}$, $\{(3+i,1),(1+i,4)\}$, and $\{(2+i,1),(4+i,4)\}$ from the graphs $[(i,2),(i,4),(i,1),(i,3)]$, $[(4+i,3),(1+i,4),(i,1),(3+i,2)]$, $[(i+1,3),(4+i,4),(i,1),(2+i,2)]$, $[(1+i,4),(3+i,1),(2+i,2),(i,3)]$, and $[(4+i,4),(2+i,1),(3+i,2),(i,3)]$.

Reassemble these edges including the deleted edges in (1) into a collection of 4-cycles $C_2(B)$ as follows: $((1,1),(2,1),(2,3),(1,3))$, $((3,1),(4,1),(4,3),(3,3))$, $((1,2),(2,2),(2,4),(1,4))$, $((3,2),(4,2),(4,4),(3,4))$, and $((3+i,1),(1+i,4),(3+i,3),(i,4))$, $i \in Z_5$. Then $(S, C_1(B) \cup C_2(B), L)$ is a maximum packing of K_{21} with 4-cycles where L is the *bowtie* $(\infty,(0,1),(0,3))$, $(\infty,(0,2),(0,4))$.

There are three possible leaves for $n = 21$; the other two are 2 disjoint 3-cycles and a 6-cycle. Here is a solution for a pair of disjoint 3-cycles. Since the 4-cycle $((0,1),(3,4),(0,3),(2,4)) \in C_2(B)$ we can use these edges in a rearrangement. Let $T_1 = \{(\infty,(0,1),(0,3)), (\infty,(0,2),(0,4)), ((0,1),(3,4),(0,3),(2,4))\}$ and $T_2 = \{((0,1),(0,3),(3,4)), (\infty,(0,2),(0,4)), (\infty,(0,1),(2,4),(0,3))\}$. Then T_1 and T_2 are balanced (= cover *exactly* the same edges).

Set $C_2(B_2) = (C_2(B) \setminus \{((0,1),(3,4),(0,3),(2,4))\}) \cup \{(\infty,(0,1),(2,4),(0,3))\}$ and $L_2 = \{((0,1),(0,3),(3,4)), (\infty,(0,2),(0,4))\}$. Then $(S, C_1(B) \cup C_2(B_2), L_2)$ is a maximum packing of K_{21} with 4-cycles with leave 2 disjoint 3-cycles.

The following is a solution for a 6-cycle. Let (X, B) be the $(K_4 \setminus e)$ design where $X = \{1,2,3,4,5,6,7\} \times Z_3$ and $B = \{[(3,i),(4,1+i),(1,i),(3,1+i)]$, $[(3,i),(6,1+i),(1,1+i),(7,2+i)]$, $[(4,1+i),(5,i),(1,1+i),(7,i)]$,

$[(5,i),(6,1+i),(1,i),(7,1+i)]$, $[(5,2+i),(7,1+i),(1,1+i),(3,i)]$, $[(7,1+i),(7,2+i),(1,i),(4,1+i)]$, $[(7,2+i),(2,i),(2,1+i),(6,i)]$, $[(2,i),(5,2+i),(3,1+i),(5,1+i)]$, $[(4,1+i),(6,1+i),(5,1+i),(5,2+i)]$, $[(6,1+i),(3,1+i),(4,i),(6,i)]$, $[(3,1+i),(2,1+i),(5,1+i),(7,1+i)]$, $[(2,1+i),(4,1+i),(4,i),(6,i)]$, $[(1,i),(2,2+i),(1,2+i),(3,1+i)]$, $[(2,1+i),(1,i),(4,2+i),(6,2+i)] \mid i \in Z_3\}$. Then $(X, C_1(B) \cup C_2(B), L)$ is a maximum packing of K_{21} with 4-cycles where $C_2(B) = \{((3,i),(4,1+i),(5,i),(6,1+i)), ((5,2+i),(7,1+i),(7,2+i),(2,i)), ((4,1+i),(6,1+i),(3,1+i),(2,1+i)) \mid i \in Z_3\}$ and $L = \{((1,0),(2,2),(1,1),(2,0),(1,2),(2,1))\}$. □

Example 2.10 (n = 21, with a hole H of size 11, having a metamorphosis into a packing of $K_{21} \setminus H$ with 4-cycles with leave a 3-cycle having exactly one vertex in H.) Let $X = Z_5 \times \{1,2,3,4\}$; (X, G, T) a $TD(5,4)$ with groups $\{i\} \times \{1,2,3,4\}, i \in Z_5$; $H = \{\infty\} \cup (Z_5 \times \{1,2\})$; and set $S = \{\infty\} \cup X$. Define a collection of copies of $K_4 \setminus e$ as follows:

(1) For each block $\{(x,1),(y,2),(z,3),(w,4)\} \in T$ put $[(z,3),(w,4),(x,1),(y,2)]$ in B.

(2) For $j = 3$ and 4 define a $K_4 \setminus e$ design of order 6 on $\{\infty\} \cup (Z_5 \times \{j\})$ having a metamorphosis into a maximum packing with 4-cycles with leave $\{\infty,(0,j)\}, \{(1,j),(2,j)\}, \{(3,j),(4,j)\}$ and put these copies of $K_4 \setminus e$ in B.

Then (S, B) is a $K_4 \setminus e$ design of order 21 with hole H of order 11.

The metamorphosis is the following: Remove the edges $\{(z,3),(w,4)\}$ in (1), all $[(z,3),(w,4),(x,1),(y,2)]$ in B as well as the edges in (2). Reassemble these edges into a collection of 4-cycles $C_2(B)$ as follows: $((1,3),(2,3),(2,4),(1,4))$, $((3,3),(4,3),(4,4),(3,4))$, and $((i,3),(1+i,4),(4+i,3),(3+i,4)), i \in Z_5$. Then $(S, C_1(B) \cup C_2(B), L)$ is a metamorphosis into a packing of $K_{21} \setminus H$ with 4-cycles with leave $L = \{\infty,(0,3),(0,4)\}$ having exactly one vertex in H. □

Example 2.11 (n = 25) Let $X = Z_5 \times Z_5$, $b_1 = \{(0,0),(0,1),(1,0),(2,2)\}$, $b_2 = \{(0,0),(0,2),(2,0),(4,4)\}$, and $B^* = \{b_1 + (i,j), b_2 + (i,j) \mid i,j \in Z_5\}$. Then (X, B^*) is a block design of order 25 with block size 5. Now remove the 50 edges $[(0,2+j),(2,j),(3,3+j),(2,4+j)]$ and $[(1,2+j),(4,1+j),(2,1+j),(3,1+j)]$, $j \in Z_5$, from the blocks of B^*. (It is straightforward to check that no two of these edges belongs to the same block of B^*). Let B be the collection of copies of $K_4 \setminus e$ obtained from deleting these edges from the blocks of B^* plus the 10 copies of $K_4 \setminus e$ $[(0,2+j),(2,j),(3,3+j),(2,4+j)]$ and $[(1,2+j),(4,1+j),(2,1+j),(3,1+j)], j \in Z_5$. Then (X, B) is a $K_4 \setminus e$ design of order 25. The metamorphosis is as follows: Delete the 60 "diagonals" from the 60 copies of $K_4 \setminus e$ in B and rearrange them into the 15 4-cycles: $C_2(B) = \{((0,2+j),(2,j),(1,2+j),(4,1+j)), ((4,j),(0,j),(3,j),(4,2+j)), ((1,j),(1,2+j),(0,3+j),(3,2+j)) \mid j \in Z_5\}$. Then $(X, C_1(B) \cup C_2(B))$ is a 4-cycle system of order 25. □

Example 2.12 (n = 26) Let B be the collection of copies of $K_4 \setminus e$ given by: $[2i, 1+2i, 12+2i, 13+2i]$, $[2i, 2+2i, 6+2i, 7+2i]$, $[2i, 3+2i, 10+2i, 11+2i]$, $[1+2i, 2+2i, 10+2i, 11+2i]$, and $[1+2i, 3+2i, 6+2i, 7+2i]$, $i \in Z_{13}$. Then (Z_{26}, B) be a $K_4 \setminus e$ design of order 26, and $(Z_{26}, C_1(B) \cup C_2(B), L)$ is a maximum packing of K_{26} with 4-cycles where $C_2(B) = \{(2i, 2+2i, 1+2i, 3+2i) \mid i \in Z_{13}\}$ and L is the 1-factor consisting of the edges $\{2i, 1+2i\}$, $i \in Z_{13}$. □

Example 2.13 (n = 31) Let $X = \{\infty\} \cup (Z_6 \times Z_5)$ and define a collection B of copies of $K_4 \setminus e$ as follows: (1) For each $i \in Z_6$, let $(\{\infty\} \cup (\{i\} \times Z_5), B_i)$ be the $K_4 \setminus e$ design of order 6 in Example 2.1 and put $B_i \subseteq B$. (2) Place the following 75 graphs in B, where $j \in Z_5$: $\{[(0,j),(1,j),(3,3+j),(4,3+j)]$, $[(0,j),(2,j),(4,4+j),(5,4+j)]$, $[(0,j),(3,j),(5,1+j),(1,3+j)]$, $[(0,j),(4,j),(1,1+j),(2,2+j)]$, $[(0,j),(5,j),(2,4+j),(3,1+j)]$, $[(2,j),(5,j),(0,2+j),(1,2+j)]$, $[(3,j),(1,j),(0,1+j),(2,4+j)]$, $[(4,j),(2,j),(0,4+$

$j),(3,3+j)]$, $[(5,j),(3,j),(0,3+j),(4,4+j)]$, $[(1,j),(4,j),(0,3+j),(5,4+j)]$, $[(3,j),(4,j),(2,3+j),(5,2+j)]$, $[(4,j),(5,j),(1,4+j),(3,2+j)]$, $[(5,j),(1,j),(2,2+j),(4,2+j)]$, $[(1,j),(2,j),(3,4+j),(5,2+j)]$, $[(2,j),(3,j),(1,4+j),(4,1+j)]\}$. Then (X,B) is a $K_4 \setminus e$ design of order 31.

The metamorphosis is the following: In (1) use the metamorphosis with leave $\{\infty,(i,0)\},\{(i,1),(i,2)\}$, $\{(i,3),(i,4)\}$. Delete all edges of the form $\{(x,j),(y,j)\}$, $x,y \in Z_6$, from the type (2) graphs. The edges $\{\infty,(i,0)\}$, $i \in Z_6$, plus the edges $\{(x,0),(y,0)\}$, $x,y \in Z_6$, is a copy of K_7; partition these edges into a maximum packing $(\{\infty\} \cup (Z_6 \times \{0\}), C_1, L)$ of K_7 with 4-cycles where L is a 5-cycle (see [2]). Reassemble the edges $\{(i,1),(i,2)\}$, $\{(i,3),(i,4)\}$, $i \in Z_6$, and $\{(0,j),(1,j)\},\{(2,j),(3,j)\}$, $\{(4,j),(5,j)\}$, $j \in Z_5 \setminus \{0\}$, into the collection of 4-cycles $C_2 = \{((0,1),(1,1),(1,2),(0,2))$, $((2,1),(3,1),(3,2),(2,2))$, $((4,1),(5,1),(5,2),(4,2))$, $((0,3),(1,3),(1,4),(0,4))$, $((2,3),(3,3),(3,4),(2,4))$, $((4,3),(5,3),(5,4),(4,4))\}$.

Finally partition all type (2) edges not already used into the collection of 4-cycles $C_3 = \{((0,j),(2,j),(1,j),(3,j))$, $((0,j),(4,j),(1,j),(5,j))$, $((2,j),(4,j),(3,j),(5,j))\}$, $j \in Z_5 \setminus \{0\}$. Then $(X, C_1(B) \cup C_2(B), L)$ is a maximum packing of K_{31} with 4-cycles where $C_2(B) = C_1 \cup C_2 \cup C_3$ and L is a 5-cycle. □

3 The general constructions

We now give four recursive constructions which, along with the examples in Section 2, give a *complete solution* of the problem of constructing $K_4 \setminus e$ designs having metamorphoses into maximum packings of K_n with 4-cycles.

We will use the following theorem repeatedly in the constructions.

Theorem 3.1 (D. Sotteau [4]) *Necessary and sufficient conditions for the complete bipartite graph $K_{m,n}$ to be partitioned into $2k$-cycles are: (i)* *m and n are even, (ii) $k \geq m$ and n, and (iii) $2k|mn$.* □

In what follows we will be partitioning $K_{m,n}$ into 4-cycles, so the necessary and sufficient conditions of Sotteau's Theorem reduce to simply m and n are even.

The 10k Construction. In view of the examples in Section 2 we can assume $10k \geq 30$. Let $(X, \circ)$ be a commutative quasigroup of order $2k$ with holes $H = \{h_1, h_2, \ldots, h_k\}$ of size 2. (See [3] for example.) Set $P = X \times \{1,2,3,4,5\}$ and define a collection B of copies of $K_4 \setminus e$ as follows:

(1) For each $i = 1, 2, \ldots, k$, let $(h_i \times \{1,2,3,4,5\}, h_i^*)$ be a $K_4 \setminus e$ design of order 10 having a metamorphosis into a maximum packing with 4-cycles with leave a 1-factor L_i (Example 2.2) and put $h_i^* \subseteq B$.

(2) If x and y belong to different holes of H, place the 5 copies of $K_4 \setminus e$ $[(x,1),(y,1),(x \circ y,2),(x \circ y,4)]$, $[(x,2),(y,2),(x \circ y,3),(x \circ y,5)]$, $[(x,3),(y,3),(x \circ y,4),(x \circ y,1)]$, $[(x,4),(y,4),(x \circ y,5),(x \circ y,2)]$, and $[(x,5),(y,5),(x \circ y,1),(x \circ y,3)]$ in B.

Then (P, B) is a $K_4 \setminus e$ design of order $10k$.

The metamorphosis is the following: For each hole belonging to H use the metamorphosis in (1). Delete all edges of the form $\{(x,i),(y,i)\}$ from the type (2) graphs. Rearrange these edges into 4 cycles. Then $(P, C_1(B) \cup C_2(B), L)$ is a maximum packing of K_{10k} with 4-cycles where $L = \cup_{i=1}^{k} L_i$. □

Lemma 3.2 *There exists a $K_4 \setminus e$ design of order n having a metamorphosis into a maximum packing of K_n with 4-cycles for all $n \equiv 0$ (mod 10).* □

Before giving the $10k+1$ Construction we will need some preliminary results.

Two collections of graphs G_1 and G_2 are said to be *balanced* provided they contain *exactly* the same edges. The following collections of graphs are balanced:

$A_1 = \{(\infty_1, x, y), (\infty_2, z, w), (x, z, y, w)\}$ and $A_2 = \{(\infty_1, x, w, \infty_2, z, y), (x, y, w, z)\}$; $D_1 = \{(\infty_1, x, y), (\infty_1, z, w), (\infty_1, u, v), (x, z, y, w), (x, u, y, v), (z, u, w, v)\}$ and $D_2 = \{(\infty_1, x, y, z), (\infty_1, w, u, v), (x, u, y, v), (x, z, v, w), (\infty_1, y, w, z, u)\}$; and $C_1 = \{(\infty_1, \infty_2, \infty_3), (\infty_1, (x,1), (y,1)), (\infty_2, (z,1), (w,1)), (\infty_3, (u,1), (v,1)), ((x,1), (z,1), (y,1), (w,1)), ((x,1), (u,1), (y,1), (v,1)), ((z,1), (u,1), (w,1), (v,1))\}$ and $C_2 = \{(\infty_1, (y,1), (z,1), \infty_2), (\infty_2, (w,1), (u,1), \infty_3), (\infty_1, \infty_3, (v,1), (x,1)), ((x,1), (y,1), (u,1), (z,1)), ((y,1), (w,1), (z,1), (v,1)), ((x,1), (w,1), (v,1), (u,1))\}$.

We will use these collections of balanced pairs in the general constructions which follow.

The 10k + 1 Construction. Since we have examples for $n = 21$ and 31 we will assume $10k + 1 \geq 41$. Write $10k + 1 = 10(k-1) + 11$ and $k - 1 = 3 + 4t + r$, where $0 \leq r \leq 3$. Let $(X, \circ)$ be a commutative quasigroup of order $2(k-1)$ with holes $H = \{h_1, h_2, \ldots, h_{k-1}\}$ of size 2. Set $S = \{\infty_1, \infty_2, \infty_3, \ldots, \infty_{11}\} \cup (X \times \{1,2,3,4,5\})$ and define a collection B of copies of $K_4 \setminus e$ as follows:

(1) Let $(\{\infty_1, \infty_2, \ldots, \infty_{11}\} \cup (h_1 \times \{1,2,3,4,5\}), h_1^*)$ be any one of the $K_4 \setminus e$ designs of order 21 in Example 2.9. (These designs have metamorphoses into maximum packings of K_{21} with 4-cycles with leaves a bowtie, a pair of disjoint 3-cycles, and a 6-cycle.) Put $h_1^* \subseteq B$.

(2) For $i = 2$ and 3, let $(\{\infty_1, \infty_2, \ldots, \infty_{11}\} \cup (h_i \times \{1,2,3,4,5\}), h_i^*)$ be a $K_4 \setminus e$ design of order 21 with hole $\{\infty_1, \infty_2, \ldots, \infty_{11}\}$ having a metamorphosis into a maximum packing of $K_{21} \setminus \{\infty_1, \infty_2, \ldots, \infty_{11}\}$ with 4-cycles with leave $(\infty_i, (x,1), (y,1))$, $\{x, y\} = h_i$. Put $h_i^* \subseteq B$. (See Example 2.10.)

(3) Let $F_i = \{4+4i, 5+4i, 6+4i, 7+4i\}$, $0 \leq i \leq t$, and let $(\{\infty_1, \infty_2, \ldots,$

$\infty_{11}\} \cup (h_j \times \{1,2,3,4,5\}), h_j^*), j \in F_i$, be a $K_4 \setminus e$ design of order 21 with hole $\{\infty_1, \infty_2, \ldots, \infty_{11}\}$ having a metamorphosis into a maximum packing of $K_{21} \setminus \{\infty_1, \infty_2, \ldots, \infty_{11}\}$ with 4-cycles with leave $\{\infty_1, (x,1), (y,1)\}$, where $h_j = \{x, y\}$. Put $h_j^* \subseteq B$.

(4) If $r = 1$, use (3) with h_{k-1} and the leave $\{\infty_1, (x,1), (y,1)\}$. If $r = 2$, use (3) with h_{k-2} and h_{k-1} and the leaves $\{(\infty_1, (x,1), (y,1)), (\infty_1, (z,1), (w,1))\}$ or $\{(\infty_1, (x,1), (y,1)), (\infty_2, (z,1), (w,1)\}$, where $h_{k-2} = \{x, y\}$ and $h_{k-1} = \{z, w\}$. If $r = 3$, use (3) with h_{k-3}, h_{k-2}, and h_{k-1} with leaves $\{\infty_1, (x,1), (y,1)\}$, where $h_{k-i} = \{x, y\}$. In each case put $h_i^* \subseteq B$.

(5) If x and y belong to different holes of H, put the 5 graphs $[(x,1), (y,1), (x \circ y, 2), (x \circ y, 4)]$, $[(x,2), (y,2), (x \circ y, 3), (x \circ y, 5)]$, $[(x,3), (y,3), (x \circ y, 4), (x \circ y, 1)]$, $[(x,4), (y,4), (x \circ y, 5), (x \circ y, 2)]$, and $[(x,5), (y,5), (x \circ y, 1), (x \circ y, 3)]$ in B.

Then (S, B) is a $K_4 \setminus e$ design.

The metamorphosis is the following: In (1) use the metamorphosis with leave $\{(\infty_1, \infty_2, \infty_3), (\infty_1, (x,1), (y,1))\}$, where $h_1 = \{x, y\}$. Use the metamorphosis in (2), (3), and (4). Delete the edges $\{(x,i), (y,i)\}$, x and y in different holes of $H, i = 1,2,3,4,5$. We now reassemble the deleted edges as follows: Combine the leave in (1) with the leaves in (2) plus all edges between the holes h_1, h_2, and h_3. This gives a copy of C_1 (see above). Replace C_1 with C_2. (Note that C_2 consists of 6 4-cycles.) For *each* $0 \leq i \leq t$, and each $j \in F_i$ use the metamorphosis in (3). Then the 4 leaves in (3) plus all edges between the holes $h_j, j \in F_i$, is a copy of K_9. Replace these deleted edges with a 4-cycle system. For $r = 1, 2$ or 3, use (4) along with all edges between the holes. Finally partition all type (5) edges not already used into 4-cycles. If $r = 1$, we have a maximum packing with leave a 3-cycle. If $r = 2$, we have maximum packings with leave a bowtie or 2 disjoint 3-cycles. To obtain a leave of a 6-cycle, use the leave $\{(\infty_1, (x,1), (y,1)), (\infty_2, (z,1), (w,1))$, along with $((x,1), (z,1), (y,1), (w,1))$. This gives a copy of A_1, which we can replace with A_2. If $r = 3$, use (4) along with the 3 4-cycles between the holes. This gives a copy of D_1 which can be replaced with D_2 to give a maximum packing with leave a 5-cycle.

Combining all of the above results gives the following lemma.

Lemma 3.3 *There exists a $K_4 \setminus e$ design of order n having a metamorphosis into a maximum packing of K_n with 4-cycles with all possible leaves for all $n \equiv 1 \pmod{10} \geq 21$. (There does not exist a $K_4 \setminus e$ design having a metamorphosis into a maximum packing of K_{11} with 4-cycles.)* □

The 10k + 5 Construction. Since there does not exist a $K_4 \setminus e$ design of order 5 and we already have examples for $n = 15$ and 25 so we will assume $10k + 5 \geq 35$. Let $(X, \circ)$ be a commutative quasigroup of order $2k$ with holes $H = \{h_1, h_2, \ldots, h_k\}$ of size 2. Set $S = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (X \times \{1, 2, 3, 4, 5\})$ and define a collection B of copies of $K_4 \setminus e$ as follows:

(1) Let $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (h_1 \times \{1, 2, 3, 4, 5\}), h_1^*)$ be a $K_4 \setminus e$ design of order 15 having a metamorphosis into a maximum packing of K_{15} with 4-cycles with leave $C_5 = (\infty_1, (1, 1), (2, 1), (2, 2), (1, 2))$ (Example 2.4), and put $h_1^* \subseteq B$. (We can assume $h_1 = \{1, 2\}$.)

(2) For each $i = 2, 3, \ldots k$, let $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (h_i \times \{1, 2, 3, 4, 5\}), h_i^*)$ be a $K_4 \setminus e$ design of order 15 with hole $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ having a metamorphosis into a collection of 4-cycles with leave $\{\infty_1, (x, 1), (y, 1)\}, h_i = \{x, y\}$, (Example 2.5), and put $h_i^* \subseteq B$.

(3) If x and y belong to different holes the same as (5) in the $10k + 1$ Construction.

Then (S, B) is a $K_4 \setminus e$ design of order $10k + 5$. The metamorphosis is as follows:

Write $k = 2 + 4t + r \geq 3$ (since $10k + 5 \geq 35$). The metamorphosis is the following: In (1) use the metamorphosis with leave the 5-cycle $(\infty_1, (1, 1), (2, 1), (2, 2), (1, 2))$. Use the metamorphosis for the hole

$h_2 = \{3,4\}$ with leave $(\infty_1,(3,1),(4,1))$. For the holes $h_3, h_4, \ldots, h_{4t+2}$ use the metamorphosis with leave $(\infty_1,(x,1),(y,1))$ where $h_i = \{x,y\}$. If $r = 1, 2,$ or 3 use the metamorphosis in the $10k+1$ Construction for the holes $h_{4t+3}, h_{4t+4}, h_{4t+5}$ as the case may be. Delete all type (3) edges and reassemble in 4-cycles. Reassemble the remaining edges as follows: Reassemble the edges belonging to $(\infty_1,(1,1),(2,1),(2,2),(1,2))$, $(\infty_1,(3,1),(4,1))$ and $((1,1),(3,1),(2,1),(4,1))$ into the 3 4-cycles $(\infty_1,(1,2),(1,1),(4,1))$, $(\infty_1,(2,2),(2,1),(3,1))$, and $((1,1),(3,1),(4,1),(2,1))$. Reassemble the leaves and 4-cycles between the holes $h_3, h_4, h_5, \ldots, h_{4t+2}$ as in the $10k+1$ Construction. The result is a maximum packing of K_{10k+5} into 4-cycles.

Lemma 3.4 *There exists a $K_4 \setminus e$ design of order n having a metamorphosis into a maximum packing of K_n with 4-cycles with all possible leaves for all $n \equiv 5 \pmod{10} \geq 15$. (There does not exist a $K_4 \setminus e$ design of order 5.)* □

The $10k+6$ Construction. Let $(X, \circ)$ be a commutative quasigroup of order $2k$ with holes $H = \{h_1, h_2, \ldots, h_k\}$ of size 2. Set $S = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\} \cup (X \times \{1,2,3,4,5\})$ and define a collection B of copies of $K_4 \setminus e$ as follows:

(1) Let $(\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\infty_6\} \cup (h_1 \times \{1,2,3,4,5\}), h_1^*)$ be a $K_4 \setminus e$ design of order 16 having a metamorphosis into a maximum packing of K_{16} with 4-cycles with leave the 1-factor L_1. (Example 2.6.) Put $h_1^* \subseteq B$.

(2) For each $i = 2, 3, \ldots, k$, let $(\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\infty_6\} \cup (h_i \times \{1,2,3,4,5\}), h_i^*)$ be a $K_4 \setminus e$ design of order 16 having a metamorphosis into a collection of 4-cycles with leave the 1-factor L_i on $h_i \times \{1,2,3,4,5\}$. (Example 2.7.) Put $h_i^* \subseteq B$.

(3) If x and y belong to different holes the same as (5) in the $10k+1$ Construction.

Then (S, B) is a $K_4 \setminus e$ design of order $10k+6$. The metamorphosis is the following. Use the metamorphoses in (1) and (2). Delete all edges of the form $\{(x,i),(y,i)\}$ from the type (3) graphs. Rearranging these edges into 4-cycles gives a maximum packing $(S, C_1(B) \cup C_2(B), L)$ of K_{10k+6} where $L = L_1 \cup L_2, \ldots, L_k$.

Lemma 3.5 *There exists a $K_4 \setminus e$ design of order n having a metamorphosis into a maximum packing of K_n with 4-cycles for all $n \equiv 6 \pmod{10}$.*
□

4 Summary

Combining Lemmas 2.3, 3.2, 3.3, 3.4 and 3.5 gives the following theorem.

Theorem 4.1 *There exists a $K_4 \setminus e$ design of every order $n \equiv 0$ or $1 \pmod{5} \geq 6, n \neq 11$, having a metamorphosis into a maximum packing of K_n with 4-cycles with all possible leaves. (There exists a maximum packing of K_{11} with 4-cycles, but it cannot be obtained from a $K_4 \setminus e$ design.)* □.

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