ON THE COMBINATORICS OF MULTI-RESTRICTED NUMBERS

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ABSTRACT. The so-called multi-restricted numbers generalize and extend the role of Stirling numbers and Bessel numbers in various problems of combinatorial enumeration. Multi-restricted numbers of the second kind count set partitions with a given number of parts, none of whose cardinalities may exceed a fixed threshold or "restriction". The numbers are shown to satisfy a three-term recurrence relation. Both analytic and combinatorial proofs for this relation are presented. Multi-restricted numbers of both the first and second kinds provide connections between the orbit decompositions of subsets of powers of a finite group permutation representation, in which the number of occurrences of elements is restricted. An exponential generating function for the number of orbits on such restricted powers is given in terms of powers of partial sums of the exponential function.

1. Introduction.

Let G be a finite group. A *G-set* (Q,G) or *permutation representation* of the group G consists of a set Q, together with a (right) action of G on Q via a homomorphism

$$G \to Q! \; ; \; g \mapsto (q \mapsto qg) \tag{1.1}$$

from G into the group $Q!$ of all permutations of the set Q. A G-set (Q,G) may be construed as an algebra of unary operations on the set Q. The subalgebra $Q^{[n]}$ of Q^n consisting of all n-tuples of distinct elements of Q is called the n-th *irredundant power* of the G-set (Q,G), and denoted by

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$(Q,G)^{[n]}$. The subalgebra $Q^{[[n]]}$ of Q^n consisting of all n-tuples in which no element is repeated more than once is called the n-th *bi-restricted power* of the G-set (Q,G), and denoted by $(Q,G)^{[[n]]}$. Orbit decompositions of direct powers $(Q,G)^n$ were discussed in [S1]. In [CS], the orbit decompositions of irredundant powers and bi-restricted powers were presented. The decompositions of the irredundant powers are related to the decompositions of the direct powers by the Stirling numbers of the first and second kinds [CS, (4.2)]. The decompositions of the bi-restricted powers are related to the decompositions of the irredundant powers by means of the so-called Bessel numbers, reparametrized coefficients of Bessel polynomials [Br; Gr; KF; CS, §4].

The exponential generating functions for the numbers of orbits in the various irredundant powers and bi-restricted powers are

$$(1.2)\qquad \frac{1}{|G|}\sum_{g\in G}(1+t)^{\pi(g)} \quad\text{and}\quad \frac{1}{|G|}\sum_{g\in G}\left(1+t+\frac{t^2}{2!}\right)^{\pi(g)}$$

respectively, where $\pi(g)$ is the number of points of Q fixed by an element g of G [CS, Th.5.2 and Th.5.3]. These generating functions may be considered as drastic truncations of the exponential generating function $|G|^{-1}\sum_{g\in G}e^{t\pi(g)}$ for the numbers of orbits in the direct power G-sets, since

$$(1.3)\qquad \frac{1}{|G|}\sum_{g\in G}e^{t\pi(g)}=\frac{1}{|G|}\sum_{g\in G}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots\right)^{\pi(g)}$$

[S2, (5.1)].

In this paper, we consider the general case of such truncations, i.e. for each positive integer m, we consider

$$(1.4)\qquad \frac{1}{|G|}\sum_{g\in G}\left(1+t+\frac{t^2}{2!}+\dots+\frac{t^m}{m!}\right)^{\pi(g)}.$$

In Section 6, an appropriate G-subset of (Q^n,G) is defined, the so-called *m-restricted power G-set* $(Q,G)^{[n,m]}$, so that (1.4) becomes the exponential generating function for the number of orbits in it (Theorem 6.5). For each positive integer m, the subset $Q^{[n,m]}$ of Q^n consists of all n-tuples in which no element appears more than m times. Then orbit decompositions of the m-restricted powers $(Q,G)^{[n,m]}$ are related to the orbit decompositions of the irredundant powers $(Q,G)^{[n]}$ via the so-called *m-restricted numbers of the second kind*, as introduced analytically in Definition 5.1.

General properties of the m-restricted numbers of the second kind are investigated in Section 5. Proposition 5.2 shows how they approximate the Stirling numbers of the second kind. Proposition 5.3 gives one explicit computation of the numbers, as well as a combinatorial interpretation in terms of set partitions. Theorem 5.4 states the key property of the multi-restricted numbers of the second kind, the three-term recurrence relation that they satisfy. This relation is the analogue of the two-term recurrence relation (2.5) satisfied by the Bessel numbers, or the well-known two-term recurrence relation satisfied by the Stirling numbers of the second kind [Ai, 3.29(ii)]. The multi-restricted numbers may then be computed by the three-term recurrence relation, subject to the boundary conditions specified by Proposition 5.2. The paper offers two alternative derivations of the three-term recurrence relation in Theorem 5.4. The first is purely analytic. The second is combinatorial, based on the interpretation of the multi-restricted numbers given by Proposition 5.3.

As a dual to the m-restricted numbers of the second kind, Section 7 defines the *m-restricted number of the first kind* $M_1^m(n,k)$ to be the (n,k)-entry of the inverse of the matrix whose (p,j)-entry for each p,j is the m-restricted number of the second kind $M_2^m(p,j)$. This provides an inverse relation between the m-restricted powers and the irredundant powers. The paper concludes with Theorem 7.6 using both kinds of multi-restricted numbers to describe the general relationship between the orbit decompositions of any pair of m_1- and m_2-restricted power G-sets.

Introductory sections briefly recall the Stirling numbers and Bessel polynomials (Section 2), the duality between direct powers and irredundant powers (Section 3), and the previously studied bi-restricted powers (Section 4).

2. Stirling numbers and Bessel numbers.

For each positive integer n, the product $X(X-1)(X-2)\dots(X-n+1)$ in the integral polynomial ring $\mathbb{Z}[X]$ over an indeterminate X is denoted by $[X]_n$. Since $\{X^n \mid n \in \mathbb{N}\}$ and $\{[X]_n \mid n \in \mathbb{N}\}$ are free generating sets for $\mathbb{Z}[X]$ as a $\mathbb{Z}$-module, each can be uniquely expressed as a linear combination of the others.

Definition 2.1. The *Stirling numbers of the first kind* $S_1(n,k)$ and the *Stirling numbers of the second kind* $S_2(n,k)$ are given by

$$(2.1)\qquad X^n = \sum_{k=0}^{n} S_2(n,k)[X]_k \qquad \text{and} \qquad [X]_n = \sum_{k=0}^{n} S_1(n,k)X^k. \quad \square$$

Proposition 2.2. (Cf. [Ai, 3.14].) *The Stirling number of the second kind $S_2(n,k)$ is the number of partitions of an n-set into exactly k many nonempty subsets.* $\square$

For each natural number n, the *Bessel polynomial* $y_n(x)$ is defined to be the (unique) polynomial of degree n with unit constant term

$$(2.2)\qquad y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!}\left(\frac{x}{2}\right)^k$$

which satisfies the differential equation $x^2y'' + (2x+2)y' = n(n+1)y$ [Br, Gr, KF]. Then for each positive integer n, the n-th Bessel polynomial may be written in the form

$$(2.3)\qquad P_n(x) = \sum_{k=0}^{\infty} B_{n,k}x^{n-k},$$

where the *Bessel coefficient* $B_{n,k}$ is given by

$$B_{n,k} = \frac{(2n-k)!}{2^{n-k}(k)!(n-k)!} \tag{2.4}$$

for $n \geq k$, and by $B_{n,k} = 0$ for $n < k$. For non-negative integers n and k, the (n,k)-th *Bessel number* $B(n,k)$ is defined to be the Bessel coefficient $B_{k,2k-n}$. The combinatorial significance of the Bessel numbers is given by the following.

Proposition 2.3. *For any positive n and k, the Bessel number $B(n,k)$ is the number of partitions of an n-set into k nonempty subsets, each of size at most* 2. □

Proposition 2.4. *The Bessel numbers satisfy the recursion*

$$B(n,k) = B(n-1,k-1) + (n-1)B(n-2,k-1) \tag{2.5}$$

for $n \geq k$. □

Theorem 2.5 [CS, Th. 2.1]. *For an indeterminate X, let $f(t) = \left(1 + t + t^2/2!\right)^X$. Then*

$$f^{(n)}(0) = \sum_{k=1}^{n} B(n,k)[X]_k, \tag{2.6}$$

where $f^{(n)}(0)$ is the n-th derivative of f with respect to t at $t = 0$. □

3. Direct powers and irredundant powers.

For a finite group G, let $\underline{\underline{G}}$ be the variety of G-sets, construed as a category with homomorphisms (G-equivariant maps) as morphisms. For an object Q of $\underline{\underline{G}}$, let $[Q]$ denote the isomorphism class of Q in $\underline{\underline{G}}$. Let $A^+(G)$ be the set of isomorphism classes of finite G-sets. This set becomes a commutative, unital semiring $(A^+(G), +, \cdot, 0, 1)$ under $[P] + [Q] = [P + Q]$, $[P] \cdot [Q] = [P \times Q]$, $0 = [\varnothing]$, and $1 = [1]$. It embeds canonically into a commutative ring, the *integral Burnside algebra* of the group G [TD, §1.2].

For each positive integer n, the *irredundant power* G-set $Q^{[n]}$ is defined to be the complement in the direct power Q^n of the subset consisting of all n-tuples comprising at most $n-1$ distinct elements of Q (cf. [Hu,II.1.10]). The following proposition shows that the irredundant power G-sets $(Q,G)^{[n]}$ are dual to the direct power G-sets $(Q,G)^n$ via the Stirling numbers of the first and second kinds.

Proposition 3.1 [CS, Prop. 4.2].

$$(3.1)\quad [Q^n] = \sum_{k=1}^{n} S_2(n,k)[Q^{[k]}] \qquad \textit{and} \qquad [Q^{[n]}] = \sum_{k=1}^{n} S_1(n,k)[Q^k]\ . \quad \square$$

For a G-set (Q,G), let $\pi(g)$ be the number of points of Q fixed by an element g of G. By Burnside's Lemma[Hu, V.20.4], the average number of fixed points

$$(3.2)\qquad \frac{1}{|G|}\sum_{g\in G}\pi(g)^n$$

is the number of orbits of G on the n-th direct power Q^n. By Proposition 3.1,

$$(3.3)\qquad \frac{1}{|G|}\sum_{g\in G}[\pi(g)]_n$$

is the number of orbits of G on the n-th irredundant power $Q^{[n]}$[C1, Lemma 6.3].

Recall that the exponential generating function for a sequence $(a_n)_{n=0}^{\infty}$ is $\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$.

Theorem 3.2[S1,(5.1); CS,Th. 5.2]. *The exponential generating functions for the numbers of orbits on the direct power G-sets $(Q,G)^n$ and the irredundant power G-sets $(Q,G)^{[n]}$ are respectively*

$$(3.4)\qquad \frac{1}{|G|}\sum_{g\in G}(e^t)^{\pi(g)} \qquad \textit{and} \qquad \frac{1}{|G|}\sum_{g\in G}(1+t)^{\pi(g)},$$

where $\pi(g)$ is the number of points of Q fixed by an element g of G. $\square$

4. Bi-restricted powers.

Consider the n-th direct power Q^n as the set of functions from the n-set $\{1,2,3,\ldots,n\}$ to Q. Then the n-th irredundant power $Q^{[n]}$ is the subset consisting of injective functions from the n-set into Q. For each positive integer n, the n-th *bi-restricted power set* $Q^{[[n]]}$ of the set Q is defined to be

$$Q^{[[n]]} = \{f : \{1,2,3,\ldots,n\} \to Q \mid \forall q \in Q, |f^{-1}\{q\}| \leq 2\}. \tag{4.1}$$

For a G-set (Q,G), the restriction of the direct power action of G on Q^n to $Q^{[[n]]}$ is called the n-th *bi-restricted power* of (Q,G), denoted by $(Q,G)^{[[n]]}$ [CS, Def. 4.3]. Thus $Q^{[n]} \subseteq Q^{[[n]]} \subseteq Q^n$.

The following proposition shows how the Bessel numbers yield a dual relation between the irredundant powers and the bi-restricted powers .

Proposition 4.1 [CS, Props. 4.4, 5].

(1) $[Q^{[[n]]}] = \sum_{k=1}^{n} B(n,k)[Q^{[k]}]$;

(2) $[Q^{[n]}] = \sum_{k=1}^{n}(-1)^{n-k}B(2n-k-1,n-1)[Q^{[[k]]}]$. □

Bringing in the Stirling numbers, one obtains relations with the direct powers.

Proposition 4.2 [CS, Cor. 4.6].

(1) $[Q^{[[n]]}] = \sum_{k=1}^{n}\sum_{m=k}^{n} B(n,m)S_1(m,k)[Q^k]$;

(2) $[Q^n] = \sum_{k=1}^{n}\sum_{m=k}^{n}(-1)^{m-k}S_2(n,m)B(2m-k-1,m-1)[Q^{[[k]]}]$. □

The following theorem shows that the exponential generating function for the number of orbits on the bi-restricted powers is an intermediate function between the exponential generating functions in (3.4).

Theorem 4.3 [CS, Th. 5.3]. *The exponential generating function for the number of orbits on the n-th bi-restricted powers* $(Q,G)^{[[n]]}$ *is*

$$f(t) = \frac{1}{|G|}\sum_{g\in G}\left(1+t+\frac{t^2}{2!}\right)^{\pi(g)} \tag{4.2}$$

where $\pi(g)$ is the number of points of Q fixed by an element g of G. □

5. Multi-restricted numbers.

The multi-restricted numbers are defined analytically by an extension of the relation (2.5) satisfied by the Bessel numbers.

Definition 5.1. For a given positive integer m and for an indeterminate X, set

$$f(t) = [g(t)]^X \text{ with } g(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^m}{m!}. \tag{5.1}$$

Then the *multi-restricted numbers* $M_2^m(n,k)$ *of the second kind* are given by

$$f^{(n)}(0) = \sum_{k=1}^{n} M_2^m(n,k)[X]_k, \tag{5.2}$$

where $f^{(n)}(0)$ is the n-th derivative of f with respect to t at $t = 0$.

Proposition 5.2. *For any positive integers m and n,*

(1) $M_2^m(n,k) = 0$ *if* $k < \lceil \frac{n}{m} \rceil$;

(2) $M_2^m(n,n) = 1$;

(3) $M_2^m(n,k) = S_2(n,k)$ *if* $m > n - k$.

Proof. Use the notation of (5.1). If $n \le m$, the n-th derivative of g with respect to t is exactly the same as the n-th derivative of e^t at $t = 0$, since $g(0) = 1 = g'(0) = \cdots = g^{(m)}(0)$. So the n-th derivative of f at $t = 0$ is the same as the n-th derivative of e^{tX} at $t = 0$ if $n \le m$. By (2.1),

$$\left.\frac{d^n}{dt^n}e^{tX}\right|_{t=0} = \sum_{k=1}^{n} S_2(n,k)[X]_k. \tag{5.3}$$

Hence $M_2^m(n,k) = S_2(n,k)$ if $n \le m$. Also, the first m terms in $f^{(n)}(t)$, which contains $[X]_k$ for all $k > n - m$, do not have any higher derivative of g beyond the m-th derivative. So the n-th derivative of f has the same first m derivatives as e^{tX} at $t = 0$, whence (3) holds. By (3), $M_2^m(n,n) = S_2(n,n)$. So (2) holds. Since $g^{(n)}(t) = 0$ if $n > m$, (1) readily follows by induction. □

Table 1 shows the general scheme for the first few m-restricted numbers of the second kind. Each S_2 and M_2^m entry is to be replaced respectively by $S_2(n,k)$ and $M_2^m(n,k)$ for the appropriate n and k. The empty cells are to be filled with 0's. As an illustration of this general scheme, Table 2 exhibits the case $m = 3$.

The next result gives a combinatorial interpretation of the multi-restricted numbers.

Proposition 5.3. *For any positive integers m and n,*

$$(5.4)\quad M_2^m(n,k) = \sum_{\substack{k_1+k_2+\cdots+k_m=k\\ k_1+2k_2+\cdots+mk_m=n}} \frac{n!}{(1!)^{k_1}(2!)^{k_2}\ldots(m!)^{k_m}k_1!k_2!\ldots k_m!},$$

i.e. for any positive integers m and n, $M_2^m(n,k)$ is the total number of k-partitions of an n-set of type $1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m}(m+1)^0\ldots n^0$.

Proof. Use the notation of (5.1). The terms with $[X]_k$ in $f^{(n)}(t)$ take the form

$$(5.5)\qquad \sum_{\substack{k_1+k_2+\cdots+k_m=k\\ k_1+2k_2+\cdots+mk_m=n}} [X]_k(g(t))^{X-k}(g'(t))^{k_1}(g''(t))^{k_2}\ldots(g^{(m)}(t))^{k_m}.$$

For all nonnegative $\lambda_1, \lambda_2, \ldots \lambda_n$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = k$ and $\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n$, the number of k-partitions of an n-set of type $1^{\lambda_1}2^{\lambda_2}\ldots n^{\lambda_n}$ is

$$(5.6)\qquad \frac{n!}{(1!)^{\lambda_1}(2!)^{\lambda_2}\ldots(n!)^{\lambda_n}(\lambda_1)!(\lambda_2)!\ldots(\lambda_n)!}.$$

Hence the number of terms of the form

$$[X]_k(g(t))^{X-k}(g'(t))^{k_1}(g''(t))^{k_2}\ldots(g^{(m)}(t))^{k_m}$$

in $f^{(n)}(t)$ is

$$\frac{n!}{(1!)^{k_1}(2!)^{k_2}\dots(m!)^{k_m}k_1!k_2!\dots k_m!}.$$

Since $g(0) = g'(0) = g''(0) = \cdots = g^{(m)}(0) = 1$, one then has

(5.7)

$$f^{(n)}(0) = \sum_{k=1}^{n}\left(\sum_{\substack{k_1+k_2+\cdots+k_m=k\\ k_1+2k_2+\cdots+mk_m=n}}\frac{n!}{(1!)^{k_1}(2!)^{k_2}\dots(m!)^{k_m}k_1!k_2!\dots k_m!}\right)[X]_k,$$

as required. □

The following theorem presents the three-term recurrence relation for the multi-restricted numbers of the second kind. The immediate proof given is analytical, and depends on Lemma 5.5 which appears below the rest of the proof. Following Lemma 5.5, an alternative combinatorial derivation of the recurrence relation is given, depending on Proposition 5.3.

Theorem 5.4. *For any positive integers $n > m$, one has*

(5.8)

$$M_2^m(n,k) = M_2^m(n-1,k-1)+kM_2^m(n-1,k)-\binom{n-1}{m}M_2^m(n-m-1,k-1).$$

Proof. Use the notation of (5.1). Just two types of terms in $f^{(n-1)}(t)$ yield

$$[X]_k(g(t))^{X-k}(g'(t))^{k_1}(g''(t))^{k_2}\dots(g^{(m)}(t))^{k_m}$$

in $f^{(n)}(t)$. The first type is

$$[X]_{k-1}(g(t))^{X-k+1}(g'(t))^{l_1}(g''(t))^{l_2}\dots(g^{(m)}(t))^{l_m}, \tag{5.9}$$

where $l_1 + l_2 + \cdots + l_m = k - 1$ and $l_1 + 2l_2 + \cdots + ml_m = n - 1$. The second type is

$$[X]_k(g(t))^{X-k}(g'(t))^{h_1}(g''(t))^{h_2}\dots(g^{(m)}(t))^{h_m}, \tag{5.10}$$

where $h_1 + h_2 + \cdots + h_m = k$ and $h_1 + 2h_2 + \cdots + mh_m = n - 1$.

The derivative of the first type (5.9) yields exactly one $[X]_k$-term, namely

$$[X]_k(g(t))^{X-k}(g'(t))^{l_1+1}(g''(t))^{l_2}\dots(g^{(m)}(t))^{l_m}.$$

This gives the contribution $M_2^m(n-1,k-1)$ to $M_2^m(n,k)$ on the right hand side of (5.8). The derivative of the second type (5.10) produces m terms with $[X]_k$, namely:

(5.11)

$$\begin{aligned}&[X]_k(g(t))^{X-k}h_1(g'(t))^{h_1-1}(g''(t))^{h_2+1}(g^{(3)}(t))^{h_3}\dots(g^{(m)}(t))^{h_m}\\ &+[X]_k(g(t))^{X-k}(g'(t))^{h_1}h_2(g''(t))^{h_2-1}(g^{(3)}(t))^{h_3+1}\dots(g^{(m)}(t))^{h_m}\\ &+\dots\\ &+[X]_k(g(t))^{X-k}(g'(t))^{h_1}(g''(t))^{h_2}(g^{(3)}(t))^{h_3}\dots h_m(g^{(m)}(t))^{h_m-1}g^{(m+1)}(t).\end{aligned}$$

Since $g(0)=g'(0)=g''(0)=\dots=g^{(m)}(0)=1$ and $g^{(m+1)}(t)=0$, (5.11) at $t=0$ becomes $(h_1+h_2+\dots+h_{m-1})[X]_k$. Since $h_1+h_2+\dots+h_{m-1}=k-h_m$, one obtains the contribution of k times $M_2^m(n-1,k)$ to the right hand side of (5.8), corrected by the subtraction of h_m times the number of occurrences of (5.10) in $f^{(n-1)}(t)$. Therefore

(5.12)

$$\begin{aligned}M_2^m(n,k) &= M_2^m(n-1,k-1) + kM_2^m(n-1,k)\\ &- \sum_{\substack{h_1+h_2+\dots+h_m=k\\ h_1+2h_2+\dots+mh_m=n-1}} \frac{h_m(n-1)!}{(1!)^{h_1}(2!)^{h_2}\dots(m!)^{h_m}h_1!h_2!\dots h_m!}.\end{aligned}$$

The analytical proof of Theorem 5.4 is completed by the following lemma. □

Lemma 5.5. *For positive integers $n>m$, one has*

$$\sum_{\substack{h_1+h_2+\dots+h_m=k\\ h_1+2h_2+\dots+mh_m=n-1}} \frac{h_m(n-1)!}{(1!)^{h_1}(2!)^{h_2}\dots(m!)^{h_m}h_1!h_2!\dots h_m!}=\binom{n-1}{m}M_2^m(n-m-1,k-1).$$

Proof. The left-hand side is equal to

(5.13)

$$\sum_{\substack{h_1+h_2+\cdots+h_m=k \\ h_1+2h_2+\cdots+mh_m=n-1 \\ h_m\geq 1}} \frac{h_m(n-1)!}{(1!)^{h_1}(2!)^{h_2}\dots(m!)^{h_m}h_1!h_2!\dots h_m!}$$

$$= \sum_{\substack{h_1+h_2+\cdots+(h_m-1)=k-1 \\ h_1+2h_2+\cdots+m(h_m-1)=n-m-1 \\ h_m-1\geq 0}} \frac{(n-1)(n-2)\dots(n-m)(n-m-1)!}{(1!)^{h_1}(2!)^{h_2}\dots(m!)^{h_m-1}m!h_1!h_2!\dots(h_m-1)!}.$$

Let $p_i = h_i$ for all $i \leq m-1$, and let $p_m = h_m - 1$. Then (5.13) is equal to

(5.14)

$$\binom{n-1}{m} \sum_{\substack{p_1+p_2+\cdots+p_m=k-1 \\ p_1+2p_2+\cdots+mp_m=n-m-1}} \frac{(n-1)!}{(1!)^{p_1}(2!)^{p_2}\dots(m!)^{p_m}p_1!p_2!\dots p_m!},$$

which by Proposition 5.3 corresponds exactly to the $\binom{n-1}{m}M_2^m(n-m-1,k-1)$ term on the right hand side. □

The combinatorial derivation of the recurrence relation (5.8) will now be given. By Proposition 5.3, the left hand side of (5.8), namely $M_2^m(n,k)$, counts the number of partitions of an n-set that have exactly k parts, none with more than m elements. Call such a set partition an "m-restricted k-partition." The right hand side of (5.8) represents the two distinct methods of building up such a partition from a partition of an $(n-1)$-set by the addition of one extra element. One method is to start with an m-restricted $(k-1)$-partition of the $(n-1)$-set, and to put the extra element in a part on its own. The number of ways to do this is $M_2^m(n-1,k-1)$, the first term on the right hand side of (5.8). The second method is to add the extra element to any one of the k parts of a given m-restricted k-partition of the $(n-1)$-set. The number of ways to do this is $kM_2^m(n-1,k)$, the second term on the right hand side of (5.8). However, in certain cases this method will yield a k-partition of the n-set that has one part with $m+1$ elements, exceeding the imposed restriction. One must thus subtract the number of such cases from the sum of the first two terms of the right hand side of (5.8). Each such case arises when the extra element is added to a part that already has m elements. There are $\binom{n-1}{m}$ such subsets, and each such subset combines with $M_2^m(n-m-1,k-1)$ different m-restricted $(k-1)$-partitions of the $(n-m-1)$-set of remaining elements to produce the m-restricted k-partition of the $(n-1)$-set. The sum of the first two terms on the right hand side of (5.8) is thus corrected by the subtraction of the third term $\binom{n-1}{m}M_2^m(n-m-1,k-1)$.

6. Multi-restricted power set actions.

Definition 6.1. For a set Q and given positive integers m and n, the n-th *m-restricted power set* is

$$Q^{[n,m]} = \{f : \{1,2,3,\dots,n\} \to Q \mid \forall q \in Q, |f^{-1}\{q\}| \leq m\}. \tag{6.1}$$

For a G-set (Q,G), the restriction of the diagonal action of G on Q^n to $Q^{[n,m]}$ is called the n-th *m-restricted power* of (Q,G), and denoted by $(Q,G)^{[n,m]}$.

Lemma 6.2. *For any positive integer n,*

$$Q^{[n,1]} = Q^{[n]} \subseteq Q^{[n,2]} = Q^{[[n]]} \subseteq Q^{[n,3]} \subseteq \cdots \subseteq Q^{[n,n]} = Q^n. \quad \square \tag{6.2}$$

Multi-restricted powers are related to irredundant powers via the multi-restricted numbers of the second kind.

Proposition 6.3. *For positive integers m and n, one has*

$$[Q^{[n,m]}] = \sum_{k=1}^{n} M_2^m(n,k)[Q^{[k]}]. \tag{6.3}$$

Proof. Let $Q_k^n = \{f \in Q^n \mid k = |\mathrm{Im}(f)|\}$ and

$$A_k^n = \{f \in Q^n \mid k = |\mathrm{Im}(f)| \text{ and } \forall q \in Q, |f^{-1}(q)| \leq m\}.$$

Then $A_k^n = Q_k^n \cap Q^{[n,m]}$, and $Q^{[n,m]}$ is the disjoint union of the A_k^n. For any partition π of the n-set $\{1,2,3,\ldots,n\}$ of type $1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m}(m+1)^0\ldots n^0$, let $Q_\pi = \{f \in Q^n \mid \pi = \ker(f)\}$. Then Q_π is in A_k^n, and is G-isomorphic to $Q^{[k]}$. Since the number of partitions of an n-set of type $1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m}(m+1)^0\ldots n^0$ is $M_2^m(n,k)$, the G-set A_k^n is G-isomorphic to the disjoint union of $M_2^m(n,k)$ copies of $Q^{[k]}$. Thus

$$Q^{[n,m]} = \bigcup_{k=1}^{n} A_k^n \cong \bigcup_{k=1}^{n} M_2^m(n,k) Q^{[k]}. \tag{6.4}$$

Considering the isomorphism classes from (6.4), the desired result (6.3) is obtained. □

By Propositions 3.1 and 6.3, the multi-restricted powers can be expressed in terms of the direct powers as follows.

Corollary 6.4. *For positive integers m and n, one has*

$$[Q^{[n,m]}] = \sum_{k=1}^{n}\sum_{p=k}^{n} M_2^m(n,p) S_1(p,k)[Q^k]. \quad \square \tag{6.5}$$

One concludes that the truncation (1.4) of $|G|^{-1}\sum_{g\in G} e^{t\pi(g)}$ generates the numbers of orbits in the multi-restricted powers of a G-set (Q,G).

Theorem 6.5. *The exponential generating function for the number of orbits on the n-th m-restricted power G-set $(Q,G)^{[n,m]}$ is*

$$f(t) = \frac{1}{|G|}\sum_{g\in G}\left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^m}{m!}\right)^{\pi(g)}, \tag{6.6}$$

where $\pi(g)$ is the number of points of Q fixed by an element g of G.

Proof. By Definition 5.1, the n-th derivative of f with respect to t at $t=0$ is

$$\begin{aligned} f^{(n)}(0) &= \frac{1}{|G|}\sum_{g\in G}\left(\sum_{k=1}^{n} M_2^m(n,k)[\pi(g)]_k\right) \\ &= \sum_{k=1}^{n} M_2^m(n,k)\left(\frac{1}{|G|}\sum_{g\in G}[\pi(g)]_k\right). \end{aligned} \tag{6.7}$$

By (3.3) and Proposition 6.3, it then follows that $f^{(n)}(0)$ gives the number of orbits of G on the n-th m-restricted power $Q^{[n,m]}$. □

7. Inverse relationships.

Proposition 5.2 shows that multi-restricted numbers of the second kind approximate Stirling numbers of the second kind. To obtain corresponding approximations to Stirling numbers of the first kind, consider the matrix M_2^m whose (n,k)-th entry is $M_2^m(n,k)$ for each n, k. By Proposition 5.2(2), M_2^m is a lower triangular matrix whose diagonal elements are all 1. One may thus consider the matrix M_1^m inverse to M_2^m.

Definition 7.1. The *m-restricted number of the first kind* $M_1^m(n,k)$ is defined to be the (n,k)-entry of the matrix M_1^m.

Proposition 7.2. *For positive integers* m *and* $n > 1$,

$$M_1^m(n,m) + M_1^m(n,m-1) + \cdots + M_1^m(n,1) = 0. \tag{7.1}$$

Proof. Since M_2^m and M_1^m are mutually inverse, $M_1^m \cdot M_2^m = I$, where I is the identity matrix. Thus the (n,p)-th entry of $M_1^m \cdot M_2^m$ is

$$\sum_{k=p}^{n} M_1^m(n,k)M_2^m(k,p) = \begin{cases} 1 & \text{if } n = p \\ 0 & \text{otherwise.} \end{cases} \tag{7.2}$$

By Proposition 5.2(3), $M_2^m(n,1) = S_2(n,1)$ if $m > n-1$, i.e. $M_2^m(n,1) = 1$ if $n \le m$, and 0 otherwise. Thus if $p = 1$, one has

$$\sum_{k=1}^{n} M_1^m(n,k)M_2^m(k,1) = \sum_{k=1}^{m} M_1^m(n,k). \tag{7.3}$$

Combining (7.2) and (7.3), (7.1) is obtained. □

Since the matrices $[M_2^m(n,k)]$ and $[M_1^m(n,k)]$ are mutually inverse, one obtains an inverse to the formula of Proposition 6.3 as follows.

Proposition 7.3. *For positive integers m and n, one has*

$$[Q^{[n]}] = \sum_{k=1}^{n} M_1^m(n,k)[Q^{[k,m]}]. \quad \square \tag{7.4}$$

Proposition 7.3 yields a counterpart to Proposition 5.2.

Corollary 7.4.

(1) $M_1^m(n,n) = 1$.

(2) $M_1^m(n,k) = S_1(n,k) \quad \textit{if} \quad m > n-k$.

Proof. Since M_2^m is a lower triangular matrix whose diagonal entries are all 1's, the eigenvalues of M_2^m are all 1's, and the inverse matrix is also a lower triangular matrix whose diagonal entries are all 1's. Hence, (*1*) holds, and (*2*) for $n-k \leq 0$ also holds. The rest of (*2*) i.e. the case of $n-k>0$ is done by induction on n. The base case $n=1$ follows from (*1*). Suppose that for all $p<n$, $M_1^m(p,k) = S_1(p,k)$ if $p-k<m$. Let $\vec{r}$ be the n-th row of the matrix M_2^m, and let $\vec{c}$ be the k-th column of the matrix M_1^m. Since $M_2^m(n,k) = S_2(n,k)$ for $m > n-k$ by Proposition 5.2, $\vec{r}$ and the transpose of $\vec{c}$ are as follows:

$$\vec{r} = (\overbrace{*,\cdots,*}^{n-m}, S_2(n,n-m+1),\cdots,S_2(n,n-1),S_2(n,n),0,\cdots);$$
$$\vec{c}^{\,t} = (\underbrace{0,\cdots,0}_{n-m}, S_1(n-m+1,k),\cdots,S_1(n-1,k),M_1(n,k),*,\cdots),$$

for some number $*$. Since the matrices M_2^m and M_1^m are mutually inverse and $n \neq k$,

$$\begin{aligned} 0 &= \vec{r}\cdot\vec{c} \\ &= \sum_{p=n-m+1}^{n-1} S_2(n,p)\cdot S_1(p,k) + S_2(n,n)M_1^m(n,k). \end{aligned} \tag{7.5}$$

Similarly, the n-th row of the matrix $[S_2(n,k)]$ and the k-th column of the matrix $[S_1(n,k)]$ for $n \neq k$ yield the following:

$$(7.6) \qquad 0 = \sum_{p=n-m+1}^{n} S_2(n,p) \cdot S_1(p,k)$$
$$= \sum_{p=n-m+1}^{n-1} S_2(n,p) \cdot S_1(p,k) + S_2(n,n)S_1(n,k).$$

Comparing (7.5) and (7.6), $M_1^m(n,k) = S_1(n,k)$. □

By Propositions 3.1 and 7.3, direct powers can be expressed in terms of m-restricted powers, yielding the following inverse to the formula of Corollary 6.4.

Corollary 7.5.

$$(7.7) \qquad [Q^n] = \sum_{k=1}^{n} \sum_{p=k}^{n} S_2(n,p) M_1^m(p,k)[Q^{[k,m]}]. \quad \square$$

Multi-restricted power sets are related to one another. The following result, which is easily proved by combining Propositions 6.3 and 7.3, exhibits the relation between general m_1-restricted powers and m_2-restricted powers. This relation subsumes all the relations introduced earlier in Sections 3 and 4.

Theorem 7.6. *For positive integers n, m_1 and m_2, one has*

$$(7.8) \qquad [Q^{[n,m_1]}] = \sum_{k=1}^{n} \sum_{p=k}^{n} M_2^{m_1}(n,p) M_1^{m_2}(p,k)[Q^{[k,m_2]}]. \quad \square$$

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$M_2^m(n,k)$	$k=1$	2	3	$\cdots$	m	$m+1$	$m+2$	$m+3$	$\cdots$	$2m$
$n=1$	1									
2	1	1								
3	1	S_2	1							
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$						
m	$1=S_2$	S_2	S_2	$\cdots$	1					
$m+1$		S_2	S_2	$\cdots$	S_2	1				
$m+2$		M_2^m	S_2	$\cdots$	S_2	S_2	1			
$m+3$		M_2^m	M_2^m	$\cdot^{\cdot^{\cdot}}$	S_2	S_2	S_2	1		
$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
$2m$		M_2^m	M_2^m	$\cdots$	M_2^m	S_2	S_2	S_2	S_2	1
$2m+1$			M_2^m	$\cdots$	M_2^m	M_2^m	S_2	S_2	S_2	S_2
$2m+2$			M_2^m	$\cdots$	M_2^m	M_2^m	M_2^m	S_2	S_2	S_2

TABLE 1: The m-restricted numbers of the second kind

$M_2^3(n,k)$	$k=1$	2	3	4	5	6	7	8	9
$n=1$	1								
2	1	1							
3	1	3	1						
4		7	6	1					
5		10	25	10	1				
6		10	75	65	15	1			
7			175	315	140	21	1		
8			280	1225	980	266	28	1	
9			280	3780	5565	2520	462	36	1

TABLE 2: The 3-restricted numbers of the second kind

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