Minimum Degree and the Orientation of a Graph

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Abstract

Let G be a simple graph such that $\delta(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor + k$, where k is a non-negative integer, and let $f : V(G) \to Z^+$ be a function having the following properties: (i) $\frac{d_G(x)}{2} - \frac{k+1}{2} \leq f(x) \leq \frac{d_G(x)}{2} + \frac{k+1}{2}$ for every $x \in V(G)$, (ii) $\sum_{x \in V(G)} f(x) = |E(G)|$. Then G has an orientation D such that $d_D^+(x) = f(x)$, for every $x \in V(G)$.

All graphs considered are assumed to be simple and finite. We refer the reader to [1] for standard graph theoretic terms not defined in this paper.

Let G be a graph. The degree $d_G(u)$ of a vertex u in G is the number of edges of G incident with u. The minimum degree of a vertex in G is denoted by $\delta(G)$. For any set S of vertices of G, we define the neighbour set of S in G to be the set of all vertices adjacent to vertices in S; this set is denoted by $N_G(S)$. If S and T are disjoint sets of vertices of G, we write $E_G(S,T)$ and $e_G(S,T)$ for the set and the number respectively of the edges of G joining S to T. If we replace the edges of G by arcs, we will get a directed graph D, which is called an orientation of G. An edge e of G is said to be subdivided when it is deleted and replaced by a path of length two connecting its ends, the internal vertex of this path being a new vertex.

Let $f(x)$ be an integer valued function defined on the set $V(G)$ such that $0 \leq f(x) \leq d_G(x)$ for each $x \in V(G)$. A spanning subgraph F is called an f-factor of G if $d_F(x) = f(x)$ for each vertex $x \in V(G)$.

Let D be a directed graph. The indegree $d_D^-(u)$ of a vertex u in D is the number of arcs with head u, the outdegree $d_D^+(u)$ of u is the number of arcs with tail u.

The following theorem is the main result of this paper.

Theorem 1: Let G be a graph such that $\delta(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor + k$, where k is a non-negative integer, and let $f : V(G) \to Z^+$ be a function having the following properties: (i) $\frac{d_G(x)}{2} - \frac{k+1}{2} \leq f(x) \leq \frac{d_G(x)}{2} + \frac{k+1}{2}$ for every $x \in V(G)$, (ii) $\sum_{x \in V(G)} f(x) = |E(G)|$.
Then G has an orientation D such that $d_D^+(x) = f(x)$, for every $x \in V(G)$.

For the proof of Theorem 1, we will use the following lemmas.

Lemma 1 [2]: Let G be a graph and let function $f : V(G) \to Z^+$. We subdivide every edge of G and define f to be 1 for the new vertices. The resulting graph G^* will have an f-factor if and only if G has an orientation D such that $d_D^+(x) = f(x)$ for every $x \in V(D)$.

Proof: Suppose first that G^* has an f-factor F. Clearly every edge of G^* has an end-vertex in $V(G)$ and the other in $V(G^*) - V(G)$. Define S to be the set of edges belonging to F and $S' = E(G^*) - E(F)$. We orient the elements of S in the following way: the tail of every arc to belong to $V(G)$ and the head to belong to $V(G^*) - V(G)$. We also orient the elements of S' as follows: the tail of every arc to belong to $V(G^*) - V(G)$ and the head to belong to $V(G)$. By considering such an orientation of G^*, we get a directed graph D^* such that $d_{D^*}^+(x) = f(x)$, for every $x \in V(D^*)$.

Now we apply to every element of $V(G^*) - V(G)$ the following procedure: For $u \in V(G^*) - V(G)$, let a_1 be the arc of D^* having u as a tail and let a_2 be the arc having u as a head. Let v_1 also be the tail of a_2 and v_2 the head of a_1. We delete u, a_1, a_2 from D^* and we add an arc having v_1 as a tail and v_2 as a head. The resulting directed graph D is an orientation of G satisfying $f(x) = d^+_{D^*}(x) = d^+_D(x)$ for every $x \in V(D)$.

By reversing the arguments we can prove easily that if G has an orientation D such that $f(x) = d^+_D(x)$ for every $x \in V(D)$, then G^* has an f-factor.

□

Lemma 2 [3]: Let G be a bipartite graph with bipartition (A, B) and let $f : V(G) \to Z^+$ be a function. Then G has an f-factor if and only if

(i) $\sum_{x\in A} f(x) = \sum_{x \in B} f(x)$

and

(ii) $\sum_{x \in X} f(x) \leq \sum_{x\in B} \min\{f(x), e_G(x, X)\}$ for every $X \subseteq A$.

Lemma 3: Let G be a graph and let function $f : V(G) \to Z^+$. Then G has an orientation D such that $d^+_D(x) = f(x)$ for every $x \in V(D)$ if and only if (i) $\sum_{x \in V(G)} f(x) = |E(G)|$ and (ii) $\sum_{x\in X} f(x) \leq e_G(X, X) + e_G(X, V(G) - X)$ for every $X \subseteq V(G)$.

Proof: We subdivide every edge of G and let G^* be the resulting graph. Clearly G^* is a bipartite graph with bipartition $(V(G^*) - V(G), V(G))$. We define f to be 1 for the new vertices (for the elements of $V(G^*) - V(G)$). According to Lemma 1, G has an orientation D if and only if G^* has an f-factor. But from Lemma 2, the bipartite graph G^* has an f-factor if and only if $\sum_{x \in V(G)} f(x) = \sum_{x \in V(G^*) - V(G)} f(x) = |E(G)|$ and $\sum_{x \in X} f(x) \leq \sum_{x \in V(G^*) - V(G)} \min\{f(x), e_{G^*}(x, X)\}$ for every $X \subseteq V(G)$. Now $\sum_{x \in V(G^*) - V(G)} \min\{f(x), e_{G^*}(x, X)\} = \sum_{x \in V(G^*) - V(G)} \min\{1, e_{G^*}(x, X)\} = e_G(X, X) + e_G(X, V(G) - X)$, since if $e_{G^*}(x, X) = 1$, this will mean that there exists an edge in G having exactly one end-vertex in X and if $e_{G^*}(x, X) = 2$, this will mean that there exists an edge in G having both end-vertices in X.

Therefore G has an orientation D if and only if (i) $\sum_{x \in V(G)} f(x) = |E(G)|$ and (ii) $\sum_{x \in X} f(x) \leq e_G(X, X) + e_G(X, V(G) - X)$ for every $X \subseteq V(G)$.

□

Proof of Theorem 1:

Suppose that G does not have such an orientation D. Then from Lemma 3, there exists $X \subseteq V(G)$ such that

$$\sum_{x \in X} f(x) > e_G(X, X) + e_G(X, V(G) - X).$$

Thus if we let $X = T$ and $V(G) - X = S$, we have

$$\sum_{x \in T} f(x) > e_G(T, T) + e_G(T, S) \tag{1}$$

For $x \in V(G)$, define $d_T(x) = |N_G(x) \cap T|$ and $d_S(x) = |N_G(x) \cap S|$. We note that

$$e_G(T, T) + e_G(T, S) = \sum_{x \in T} \frac{d_T(x)}{2} + \sum_{x \in T} d_S(x).$$

So (1) implies

$$\sum_{x \in T} f(x) > \sum_{x \in T} \frac{d_T(x)}{2} + \sum_{x \in T} d_S(x) \tag{2}$$

and hence by condition (i) of the theorem

$$\sum_{x \in T} \left(\frac{d_G(x)}{2} + \frac{k+1}{2} \right) > \sum_{x \in T} \frac{d_T(x)}{2} + \sum_{x \in T} d_S(x) \tag{3}$$

At this point we consider the following two cases.

Case 1: $|T| \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$

We note first that

$$d_S(x) \geq k + 1 \tag{4}$$

for every $x \in T$, since $\delta(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor + k$.

We also have from (3),

$$\sum_{x\in T}\left(\frac{d_T(x)}{2}+\frac{d_S(x)}{2}+\frac{k+1}{2}\right)>\sum_{x\in T}\frac{d_T(x)}{2}+\sum_{x\in T}d_S(x)$$

which implies $\left(\frac{k+1}{2}\right)|T|>\sum_{x\in T}\frac{d_S(x)}{2}$. Thus using (4), $\left(\frac{k+1}{2}\right)|T|>\left(\frac{k+1}{2}\right)|T|$ which is a contradiction.

Case 2: $|T|\geq\left\lfloor\frac{|V(G)|}{2}\right\rfloor+1$

Suppose that $|T|=\left\lfloor\frac{|V(G)|}{2}\right\rfloor+l$, where l is a positive integer. This implies obviously that $|S|=\left\lceil\frac{|V(G)|}{2}\right\rceil-l$. We also have, from condition (ii) of the Theorem, $\sum_{x\in V(G)}f(x)=|E(G)|=\frac{\sum_{x\in V(G)}d_G(x)}{2}$.

Thus

$$\sum_{x\in T}f(x)+\sum_{x\in S}f(x)=\sum_{x\in T}\frac{d_G(x)}{2}+\sum_{x\in S}\frac{d_G(x)}{2}$$

and so

$$\sum_{x\in T}f(x)=\sum_{x\in T}\frac{d_G(x)}{2}+\sum_{x\in S}\frac{d_G(x)}{2}-\sum_{x\in S}f(x) \tag{5}$$

But using condition (i) of the theorem, (5) implies,

$$\sum_{x\in T}f(x)\leq\sum_{x\in T}\frac{d_G(x)}{2}+\sum_{x\in S}\frac{d_G(x)}{2}-\sum_{x\in S}\left(\frac{d_G(x)}{2}-\frac{k+1}{2}\right)$$

and hence

$$\sum_{x\in T}f(x)\leq\sum_{x\in T}\frac{d_G(x)}{2}+\left(\frac{k+1}{2}\right)|S| \tag{6}$$

Now we have from (2) and (6),

$$\sum_{x\in T}\frac{d_G(x)}{2}+\left(\frac{k+1}{2}\right)|S|>\sum_{x\in T}\frac{d_T(x)}{2}+\sum_{x\in T}d_S(x)$$

The above can be written as

$$\sum_{x\in T}\frac{d_G(x)}{2}+\left(\frac{k+1}{2}\right)|S|>\sum_{x\in T}\frac{d_G(x)}{2}+\sum_{x\in T}\frac{d_S(x)}{2}$$

and hence

$$\left(\frac{k+1}{2}\right)|S| > \sum_{x\in T} \frac{d_S(x)}{2} \tag{7}$$

But $|S| = \left\lceil \frac{|V(G)|}{2} \right\rceil - l$ and $\delta(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor + k$. So $e_G(x,T) \geq k+l$ for every $x \in S$, which implies $(k+1)|S| \leq (k+l)|S| \leq e_G(S,T) = \sum_{x\in T} d_S(x)$. Therefore we have from (7), $\left(\frac{k+1}{2}\right)|S| > \left(\frac{k+1}{2}\right)|S|$ which is again a contradiction.

□

The conditions of Theorem 1 are in some sense best possible. We will describe first a family of graphs G having slightly lower minimum degree and not having the properties implied by the theorem. We construct such graphs G as follows. We start from a k−regular bipartite graph with bipartition (A, B) where $|A| = |B| = m \geq k+1$, m is an even number and $A = \{u_1, u_2, \ldots, u_m\}$. We add edges to this graph having both end-vertices in A, so that A induces a graph in which the vertices $u_1, u_2, \ldots, u_{m-2}$ have degree $m-2$ and the vertices u_{m-1}, u_m have degree $m-1$. We also add edges having both end-vertices in B, so that B induces a graph in which all vertices have degree $m-2$. Finally we add a new vertex u which is joined to $u_1, u_2, \ldots, u_{m-2}$ and to all the elements of B. For the resulting graph G, we have $\delta(G) = (m-1) + k = \left\lfloor \frac{2m+1}{2} \right\rfloor + k - 1 = \left\lfloor \frac{|V(G)|}{2} \right\rfloor + k - 1$ since $|V(G)| = |A| + |B| + 1 = 2m+1$.

Now let function $f : V(G) \to Z^+$ such that $f(x) = \frac{d_G(x)}{2} + \frac{k+1}{2} = \frac{k+m-1}{2} + \frac{k+1}{2} = \frac{2k+m}{2}$ when $x \in A$, $f(x) = \frac{d_G(x)}{2} - \frac{k+1}{2} = \frac{k+m-1}{2} - \frac{k+1}{2} = \frac{m-2}{2}$ when $x \in B$ and $f(u) = \frac{d_G(u)}{2} = \frac{2m-2}{2}$.

Clearly

$$\sum_{x\in V(G)} f(x) = \sum_{x\in A} f(x) + \sum_{x\in B} f(x) + f(u) =$$

$$= \sum_{x\in A}\left(\frac{d_G(x)}{2}+\frac{k+1}{2}\right) + \sum_{x\in B}\left(\frac{d_G(x)}{2}-\frac{k+1}{2}\right) + \frac{d_G(u)}{2} =$$

$$= \sum_{x\in A}\frac{d_G(x)}{2} + \sum_{x\in B}\frac{d_G(x)}{2} + \frac{d_G(u)}{2} = \frac{1}{2}\sum_{x\in V(G)} d_G(x) = |E(G)|.$$

On the other hand G, by Lemma 3, does not have an orientation D such that $d_D^+(x) = f(x)$ for every $x \in V(G)$, because

$$\sum_{x\in A} f(x) > e_G(A,A) + e_G(A, V(G) - A)$$

since

$$\sum_{x\in A} f(x) = \sum_{x\in A}\left(\frac{d_G(x)}{2}+\frac{k+1}{2}\right) = \left(\frac{2k+m}{2}\right)m,$$

$$e_G(A, V(G) - A) = km + m - 2$$

and

$$e_G(A,A) = \frac{(m-2)(m-2)+2(m-1)}{2}.$$

We will show next that the bounds in condition (i) are also best possible. For this purpose we construct a family of graphs G as follows. We start from a k−regular bipartite graph with bipartition (A, B) where $|A| = |B| = m$ and m is an even number. We join every element of A to all the other elements of A and every element of B to all the other elements of B. Finally we add a new vertex u, which is joined to each element of $A \cup B$. For the resulting graph G, we have

$$\delta(G) = (m-1)+k+1 = m+k = \left\lfloor\frac{2m+1}{2}\right\rfloor + k = \left\lfloor\frac{|V(G)|}{2}\right\rfloor + k$$

Now let function $f : V(G) \to Z^+$ such that $f(x) = \left\lceil\frac{d_G(x)}{2}+\frac{k+1}{2}\right\rceil = \left\lceil\frac{m+2k+1}{2}\right\rceil$ when $x \in A$, $f(x) = \left\lfloor\frac{d_G(x)}{2}-\frac{k+1}{2}\right\rfloor = \left\lfloor\frac{m-1}{2}\right\rfloor$ when $x \in B$ and $f(u) = \frac{d_G(u)}{2} = m$.

Clearly

$$\begin{aligned}
\sum_{x\in V(G)} f(x) &= \sum_{x\in A} f(x) + \sum_{x\in B} f(x) + f(u) \\
&= m\left\lceil \frac{m+2k+1}{2} \right\rceil + m\left\lfloor \frac{m-1}{2} \right\rfloor + m \\
&= m\left(\frac{m+2k+2}{2}\right) + m\left(\frac{m-2}{2}\right) + m \\
&= \sum_{x\in A}\left(\frac{d_G(x)}{2} + \frac{k+1}{2}\right) + \sum_{x\in B}\left(\frac{d_G(x)}{2} - \frac{k+1}{2}\right) + \frac{d_G(u)}{2} \\
&= \sum_{x\in A}\frac{d_G(x)}{2} + \sum_{x\in B}\frac{d_G(x)}{2} + \frac{d_G(u)}{2} \\
&= \frac{1}{2}\sum_{x\in V(G)} d_G(x) \\
&= |E(G)|
\end{aligned}$$

On the other hand by Lemma 3 G does not have an orientation D such that $d_D^+(x) = f(x)$, for every $x \in V(G)$ because

$$\sum_{x\in A} f(x) > e_G(A,A) + e_G(A, V(G) - A)$$

since

$$\begin{aligned}
\sum_{x\in A} f(x) &= \sum_{x\in A}\left(\left\lceil \frac{d_G(x)}{2} + \frac{k+1}{2} \right\rceil\right) \\
&= \sum_{x\in A}\left\lceil \frac{m+2k+1}{2} \right\rceil \\
&= \frac{m(m+2k+2)}{2},
\end{aligned}$$

$$e_G(A,A) = \frac{m(m-1)}{2} \quad \text{and} \quad e_G(A, V(G) - A) = km + m.$$

References

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[3] O. Ore, Theory of Graphs, American Mathematics Society College Publishers, 1962.