Bi-slim Flag-Transitive Geometries of Gonality 3: Construction and Classification

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Abstract

We consider point-line geometries having three points on every line, having three lines through every point (*bi-slim geometries*), and containing triangles. We give some (new) constructions and we prove that every flag-transitive such geometry either belongs to a certain infinite class described by Coxeter a long time ago, or is one of three well defined sporadic ones, namely, the Möbius-Kantor geometry on 8 points, the Desargues geometry on 10 points, or a unique infinite example related to the tiling of the real Euclidean plane in regular hexagons. We also classify the possible groups.

1 Introduction

In incidence geometry the classification of certain types of geometries (i.e., geometries satisfying common axioms) is a central problem. In most cases, however, one needs additional assumptions, and often some transitive action is hypothesized, because the standard examples usually have a large group of collineations. One of the most popular hypotheses is without doubt the assumption of *flag transitivity*. One of the reasons is that a geometry can be reconstructed in a canonical way using a flag-transitive group and the various stabilizers of the elements of a fixed flag. Many results thus characterizing classical and sporadic simple groups are available, see [4] for examples. Note, though, that in many cases flag transitivity is not (yet) enough to classify. A good example is the class of finite projective planes, where a full classification of the flag-transitive ones is so far only possible if one hypothesizes a nontrivial flag stabilizer. In the present paper we are concerned with point-line geometries of small order and gonality. A point-line geometry is a system $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ consisting of a point set $\mathcal{P}$, a line set $\mathcal{L}$ and a symmetric incidence relation I between $\mathcal{P}$ and $\mathcal{L}$ expressing precisely when a point is incident with a line. Usually we think of a line as the set of points incident with it and we accordingly use phrases like "a point is on a line", "a line goes through a point", etc. If all lines of Γ carry the same number $s+1$ of points and all points are incident with the same number $t+1$ of lines, then we say that Γ has *order* (s,t). If $s=1$, then the geometry is usually called *thin*, while if both s and t are at least 2, the geometry is *thick*. If $s=2$, then we call Γ *slim*. If also the dual of Γ is slim, i.e., if also $t=2$, then we call Γ *bi-slim*. Note that the *dual* of Γ is obtained from Γ by interchanging the point set with the line set. Note that finiteness of s and t does not imply finiteness of Γ. In particular, there are lots of bi-slim infinite geometries. The *incidence graph* $\mathcal{I}(\Gamma)$ or *Levi graph* of Γ is the graph with vertex set $\mathcal{P}\cup\mathcal{L}$ and adjacency relation I. The *gonality* of Γ is half of the girth of $\mathcal{I}(\Gamma)$, where the girth is the length of the smallest cycle in $\mathcal{I}(\Gamma)$. The girth is indeed an even number since $\mathcal{I}(\Gamma)$ is obviously bipartite. If the gonality is at least 3, then lines are determined by their point sets. A *flag* is an incident point-line pair. The *distance* of elements in Γ is measured in the Levi graph. If $x\in\mathcal{P}\cup\mathcal{L}$, then $\Gamma_i(x)$ denotes the set of elements of Γ at distance i from x. In this paper, our primary aim is to classify all (not necessarily finite) bi-slim flag-transitive point-line geometries of gonality 3. But first, we want to give the examples. A secondary aim is to provide unexpected geometric constructions for some of these. It will turn out that all examples must be found in an infinite class of finite geometries depending on two natural parameters already described by Coxeter in [3], except for three notable exceptions, two of which are finite and the other the unique infinite example. Without going into detail yet about the examples, we can already state our main result.

Theorem 1.1 *Let Γ be a (not necessarily finite) bi-slim flag-transitive point-line geometry of gonality 3 with a flag stabilizer H. Then one of the following possibilities occurs.*

(*i*) *Γ is the unique infinite example related to the tiling of the Euclidean plane in regular hexagons, and $|H|\in\{1,2\}$;*

(*ii*) *Γ is isomorphic to the Möbius-Kantor geometry, and again $|H|\in\{1,2\}$;*

(*iii*) *Γ is isomorphic to the Desargues geometry, and H is elementary abelian and of order 2 or 4;*

(*iv*) *Γ is finite and belongs to a well defined infinite class depending on two natural parameters $r\geq s$, with $r+s\geq 3$. In this case $|H|\in\{1,2\}$ for $s=0$ and for $r=s$, except if $(r,s)=(3,0)$. In the latter case Γ*

is the Pappus geometry and H is elementary abelian of order 1, 2 *or* 4. *If* $(r, s) = (2, 1)$, *then* Γ *is the Fano geometry and either* $|H| = 1$ *or* $|H| = 8$ *and* H *is dihedral. In all the other cases* $|H| = 1$.

For more details on the structure of the groups and a discussion of the multiple possibilities for H in most cases, see below in the construction section.

If we call *duality* an incidence preserving permutation interchanging the point set with the line set, then we remark that all the examples are self dual. They are even *self polar*, i.e., there is a duality of order 2. This just comes out of the classification.

We define an *absolute point* of a polarity as a point incident with its image under the polarity.

We mention the following consequence of Theorem 1.1.

Corollary 1.2 *Let* $\mathcal{G} = (X, E)$ *be a trivalent graph of girth* > 4 *containing a* 6-*cycle. If* $\mathcal{G}$ *admits an automorphism group acting transitively on the ordered edges (hence on ordered pairs* $(e, f) \in X \times X$, *with* $\{e, f\} \in E$), *then* $\mathcal{G}$ *is either the incidence graph of one of the geometries mentioned in* Theorem 1.1, *or it is the Petersen graph. In all cases, the groups can be explicitly described.*

We will prove this in Section 5.

2 Construction of some specific examples

2.1 The Fano geometry

The Fano geometry is the projective plane of order (2, 2). The point set is the set of 7 nonzero vectors of a 3-dimensional vector space over the field $\mathbf{GF}(2)$ of 2 elements, the lines are the 7 vector planes in this vector space, with natural incidence relation. There is a Frobenius group of order 21 acting sharply flag-transitively, but every collineation group acting flag-transitively and having nontrivial flag stabilizer coincides with the full group of collineations, which is $\mathbf{PGL}_3(2)$. Since its order is $168 = 8.21$, the stabilizer H of a flag has order 8 in the full collineation group.

Another well known construction of the Fano geometry is to take as point set the integers modulo 7, and the lines are the translates of the 3-set $\{0, 1, 3\}$.

2.2 The Möbius-Kantor geometry

The points of the Möbius-Kantor geometry are the 8 nonzero vectors of a 2-dimensional vector space over the field $\mathbf{GF}(3)$ of 3 elements; the lines are the 8 proper translates of the vector lines. So the vector lines themselves are no lines of the geometry. Clearly $\mathbf{GL}_2(3)$ is a collineation group and it turns out that it is the full collineation group. This is easy to see. It has order 48, which is exactly twice the number of flags. Since $\mathbf{GL}_2(3)$ clearly acts flag-transitively, we have that the size of the stabilizer of any flag equals 2. The subgroup $\mathbf{SL}_2(3)$ acts sharply transitively on the set of flags.

Remark that the Fano geometry and the Möbius-Kantor geometry are the unique bi-slim geometries with 7 and 8 points, respectively.

There is a cyclic collineation group of order 8 acting regularly on the point set and on the line set. This can be seen with the following construction, which is similar to the second construction of the Fano geometry above. Take as point set the set of integers modulo 8, and the lines are the translates of the 3-set $\{0, 1, 3\}$.

The two constructions mentioned in this section can be found in [3].

2.3 The Pappus geometry

Here the points are the 9 vectors of a 2-dimensional vector space over the field $\mathbf{GF}(3)$; the lines are the non-vertical vector lines and their translates, where a vertical vector line is just an arbitrarily chosen vector line as the Y-axis. There is a unique 3-group of order 27 acting sharply flag-transitively. The stabilizer in the full collineation group of the origin together with the X-axis, viewed as a flag, is isomorphic to the group of diagonal matrices in $\mathbf{GL}_2(3)$ and hence is isomorphic to Klein's four group. Consequently the full collineation group has size 108, has a normal Sylow 3-subgroup and admits 3 different subgroups of index 2. So there are three different collineation groups with $|H| = 2$ and one with $|H| = 4$.

Of course there is also the classical construction of the Pappus geometry (better known as the *Pappus configuration*) in the real projective plane as follows. Consider three collinear points a_1, a_2, a_3, and three collinear points b_1, b_2, b_3 such that the intersection point of the corresponding lines is not amongst $a_1, \ldots, b_3$. Then the lines $a_i b_j$ and $a_j b_i$ meet in some point c_k, with $\{i, j, k\} = \{1, 2, 3\}$, and c_1, c_2, c_3 are automatically collinear. The points and lines just mentioned form the Pappus geometry. Remark that the Fano and the Möbius-Kantor geometry do not admit realizations in the real projective plane with points and lines. and hence no such construction is available for these geometries. Although the Möbius-Kantor geometry admits a realization in the complex plane.

In the next section we will see that the Pappus geometry belongs to an infinite family of flag-transitive bi-slim geometries and this will provide yet another construction of it.

2.4 The Desargues geometry

Consider a non-degenerate conic in the projective plane $\mathbf{PG}(2,5)$. An *internal point* is a point not incident with any tangent line to the conic. An *external line* is a line not meeting the conic. Then the points of the Desargues geometry are the 10 internal points of the conic and the lines are the 10 external lines, while incidence is natural. From this construction it is clear that $\mathbf{PGL}_2(5)$ is a collineation group, and that the Desargues geometry admits a polarity without absolute points

A more combinatorial construction goes as follows. The points are the pairs of the 5-set $\{1,2,3,4,5\}$; the lines are the triples of that 5-set and incidence is natural. Here clearly the symmetric group $\mathbf{S}_5$ is — once again — a collineation group (and isomorphic to $\mathbf{PGL}_2(5)$). In fact it is easy to show that it is the full collineation group, acting transitively on the set of flags of the Desargues geometry. So here $|H| = 120/30 = 4$ when choosing the full collineation group. If restricting to the unique subgroup of index 2 (being the alternating group $\mathbf{A}_5$) then using the second construction, we see that we still have a flag-transitive group, this time with $|H| = 2$. Since $\mathbf{S}_5$ has no group of index 4. there is no sharply flag-transitive collineation group.

Again there is the classical construction in the real projective plane with two triangles in perspective from a point, implying that the corresponding sides of the triangles meet on a common line. Hence the triangles are also in dual perspective from that line.

Alternatively, one can take as point set the set of points of the affine plane $\mathbf{AG}(2,4)$ not lying on a fixed affine hyperoval. The lines are the 10 secant lines to the hyperoval.

2.5 The Coxeter geometry

This geometry is discovered by Coxeter [3]. We give two independent and new constructions. First, consider the 2-dimensional vector space over

$\mathbf{GF}(4)$ and choose a *vertical* Y-axis. The points of the Coxeter geometry are the 12 vectors not on the Y-axis; the lines are the proper translates of the vector lines distinct from the Y-axis; incidence is natural. Clearly, a collineation group is the semilinear group of permutations of the vector space with a lower triangular matrix. This group acts flag-transitively and contains 72 elements. So $|H| = 2$. It will follow from our classification that the full collineation group is not bigger. Restriction to the linear transformations shows that we also have a sharply transitive group of collineations of the flag space. But another rather exceptional phenomenon occurs here. The restriction of the full collineation group to the linear transformations of the vector space fixing the vectors of the Y-axis pointwise (a group abstractly isomorphic to $\mathbf{A}_4$) acts sharply transitively on the point set of the Coxeter geometry. Also, the restriction to $\mathbf{SL}_2(4)$ of the full collineation group is isomorphic to $\mathbf{A}_4$, it is not conjugate to the subgroup in the previous sentence, and it acts sharply transitively on the point set. On top of that, the direct product of the subgroup consisting of the (linear) collineations corresponding to scalar matrices, with the subgroup of transvections leaving all points of the Y-axis invariant, also acts sharply transitively on the point set. This time, the subgroup is isomorphic to the abelian group $2 \times 2 \times 3$ ("atlas" notation [2]. Hence there are 3 pairwise non-conjugate normal regular subgroups (viewed as permutation representation on the point set of the Coxeter geometry) of the full collineation group. There is also an additional normal regular subgroup on the set of flags.

The second construction is more combinatorial in nature, and does not induce the full collineation group of the geometry. Both the point and line set are copies of the vertex set of a truncated tetrahedron. A point is incident with a line if the corresponding vertices are adjacent. This gives us a self polar geometry with 12 points and 12 lines, and one can check by exhibiting an explicit isomorphism by hand that it is isomorphic to the Coxeter geometry introduced in the previous paragraph. The automorphism group inherited from the tetrahedron is $\mathbf{S}_4$ and its unique subgroup of index 2 is one of the three normal subgroups of the full collineation group of the Coxeter geometry acting regularly on the point set of the Coxeter geometry. The map sending a vertex viewed as point (line) onto the same vertex viewed as line (point) is a polarity without absolute points.

The self polarity can also be seen in the first construction by phrasing it in terms of the projective plane $\mathbf{PG}(2, 4)$, where we choose two lines L_1, L_2 (meeting in x_1) and a point x_2 on one of these lines, say L_1, distinct from x_1. The points of the Coxeter geometry are then the points of $\mathbf{PG}(2, 4)$ not on L_1, L_2, and the lines are the lines of $\mathbf{PG}(2, 4)$ not through x_1, x_2, incidence being natural. If we call a polarity *linear* when it has trivial adjoining field automorphism, then there are precisely three linear polarities of $\mathbf{PG}(2,4)$ taking x_2 to L_2 and x_1 to L_1. These polarities preserve automatically the Coxeter geometry, and have all their absolute points on L_2. Consequently these polarities have no absolute points in the Coxeter geometry, just as the polarity in the truncated tetrahedron model. In fact for each polarity without absolute points, there is a tetrahedron model where this polarity is the natural one described above.

Following Coxeter [3], the incidence graphs of the Möbius-Kantor geometry, the Desargues geometry and the Coxeter geometry can be written as $\{8\}+\{8/3\}$, $\{10\}+\{10/3\}$ and $\{12\}+\{12/5\}$, respectively, where the notation $\{n\}+\{n/m\}$ refers to the graph with vertex set $\{1,2,\ldots,n,1',2',\ldots,n'\}$ and adjacency relation $\sim$ defined by $i\sim j$ (respectively $i'\sim j'$ and $i\sim j'$) if and only if $i-j\equiv\pm1 \bmod n$ (respectively $i-j\equiv\pm m \bmod n$ and $i=j$). For instance, the Petersen graph is $\{5\}+\{5/2\}$ in this notation. Remark that the Desargues geometry can be constructed from the Petersen graph in exactly the same way as the Coxeter geometry is constructed from the truncated tetrahedron. A geometric description of the unique polarity of the Desargues geometry now follows: it maps any point to the unique line at maximal distance in the incidence graph.

2.6 A geometry in PG(2,5)

In $\mathbf{PG}(2,5)$ we consider a proper triangle with vertices x_1,x_2,x_3. The points of Γ are the points of $\mathbf{PG}(2,5)$ not on one of the lines x_ix_j, $1\le i<j\le 3$, and the lines are the lines of $\mathbf{PG}(2,5)$ not through any of x_1,x_2,x_3. Incidence is natural. Alternatively, one can take the same point set, but now take as lines the non-degenerate conics through x_1,x_2,x_3. To see that these two definitions are equivalent, coordinatize such that x_1,x_2,x_3 have coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, respectively, and just map the line with equation $aX+bY+cZ=0$ birationally to the irreducible conic with equation $aYZ+bXZ+cXY=0$. Since line pencils are clearly mapped onto conic pencils, this induces a (birational) permutation of the points and hence an isomorphism of geometries.

In the previous description, the diagonal matrices of $\mathbf{GL}_2(5)$ act regularly on the point set. We can extend this group with the symmetric action of $\mathbf{S}_3$ acting on the vertices of the triangle. Hence this gives us a group of order 96 for which $|H|=2$. It will follow from our classification that the full collineation group is not larger. There is a subgroup of index 2 obtained from the unique subgroup of order 3 of $\mathbf{S}_3$ acting sharply flag-transitively.

For further reference, we will call this geometry the *birational geometry*.

2.7 The infinite example

A good geometric description of the infinite example satisfying the assumptions of Theorem 1.1 is by its incidence graph, which is the (bipartite) graph obtained from the tiling of the real Euclidean plane into regular hexagons.

A more explicit description runs as follows. The points are the ordered pairs (i, j) of integers i, j. The lines are the triples $\{(i, j), (i, j+1), (i+1, j)\}$, with i, j any integers. The free abelian group of rank 2 acts as an obvious collineation group regularly on the set of points and on the set of lines. The permutations $\pi_1 : (i, j) \mapsto (j, i)$ and $\pi_2 : (i, j) \mapsto (i, -i-j)$ generate a group isomorphic to $\mathbf{S}_3$ in the stabilizer of $(0, 0)$ in the full collineation group and hence we obtain a flag-transitive group with flag stabilizer of size 2. Our proof of Theorem 1.1 will imply that this actually describes the full collineation group. Extending the regular group of the point set with the subgroup of order 3 of $\mathbf{S}_3$, we obtain a sharply flag-transitive group of collineations. This subgroup of the full collineation group can also be described as the set of collineations belonging to $\mathbf{SL}_2(\mathbb{R})$.

For further reference we will call this infinite example the *honeycomb geometry*.

3 A construction of an infinite class

All members of the infinite class we will describe are quotients of the honeycomb geometry. We give an explicit construction based on the incidence graph. Note that these geometries are described in [3], and our description is just a bit more detailed, because we want to recognize each of them in our proof later on.

3.1 The geometry $\mathcal{G}_{(r,s)}$

Let $\mathcal{G}$ be the incidence graph of the infinite example, which is the (bipartite) graph obtained from the tiling of the real Euclidean plane into regular hexagons. The (incidence graph of the) members of the infinite class will be described as quotients of this graph.

The parameters r and s in $\mathcal{G}_{(r,s)}$ are nonnegative integers with $r \geq s$ and $r + s \geq 3$.

We define a coordinate system for the real Euclidean plane as follows. We choose an arbitrary vertex of $\mathcal{G}$ as the origin $(0, 0)$. The unit vectors $\vec{e_1}$ and

$\vec{e_2}$ are chosen in such a way that they form an angle of sixty degrees and the end points are vertices of $\mathcal{G}$ at graph-theoretical distance 2 from $(0,0)$ contained in a common hexagon through the origin.

The points of $\mathcal{G}_{(r,s)}$ are the ordered pairs (i,j), with i,j integers and with identification of all pairs $(i,j) + k(r,s) + l(-s,r+s) = (i+kr-ls, j+ks+lr+ls)$ with k,l integers. The lines of the geometry are the 3-sets $\{(i,j),(i+1,j),(i+1,j-1)\}$ consisting of the three points incident with the line, and where for each point the above identification rule holds. The above line can be identified with the vertex with coordinates $(i+2/3, j-1/3)$.

Let m be the Euclidean distance between the origin and the vertex with coordinates (r,s). By applying the cosine rule in the triangle $(0,0)(r,0)(r,s)$ we find that $m^2 = r^2+rs+s^2$. It is easy to see that the quadrangle formed by the vertices $(0,0)(r,s)(-s,r+s)(-r-s,r)$ is a rhombus with length of the sides equal to m. Every point can be represented by a pair (i,j) with coordinates i,j in the rhombus without the line segments $[(r,s)(-s,r+s)]$ and $[(-s,r+s)(-r-s,r)]$. We will note this domain as $\mathcal{D}$. Now, if there were two representatives for a point in $\mathcal{D}$, then one would be on a line through the other parallel to $(0,0)(r,s)$, to $(0,0)(-s,r+s)$ or to $(0,0)(-r-s,r)$ at distance m from each other. Since this is impossible, every point of the geometry has a unique representation (i,j) in the domain $\mathcal{D}$ which is therefore called a *fundamental domain*.

In order to count the number of points in the geometry, we have to count the number of vertices corresponding to points in the fundamental domain $\mathcal{D}$. The area of $\mathcal{D}$ is equal to $\frac{\sqrt{3}}{2}m^2$. The area of one hexagon is equal to $\frac{\sqrt{3}}{2}$. The number of hexagons in $\mathcal{D}$ is m^2. We can assume that every hexagon contributes one vertex representing a point of the geometry and one vertex representing a line. Hence, the geometry $\mathcal{G}_{(r,s)}$ contains $m^2 = r^2+rs+s^2$ vertices and also r^2+rs+s^2 lines.

Remark that rotations over $+120$ and -120 degrees with center a vertex corresponding to a point of the geometry and translations from a vertex corresponding to a point to another vertex corresponding to a point are automorphisms of the graph $\mathcal{G}$ preserving the identification. Consequently the geometry $\mathcal{G}_{(r,s)}$ has a flag-transitive collineation group induced by these rotations and translations.

3.2 A square geometry

This is a special case of the previous, setting the parameter s equal to zero.

An alternative description of this geometry is possible by taking a new Y-axis: the line at -120 degrees from the X-axis. Then we can define a point by a tuple (i,j) with i,j integers modulo r. The number of points in $\mathcal{G}_{(r,0)}$ is therefore equal to r^2. Every line can be represented by a 3-set $\{(i,j),(i,j+1),(i+1,j+1)\}$ with i,j integers modulo r. Hence there are also r^2 lines.

Remark that in this case also reflections through axes corresponding to edges of the graph $\mathcal{G}$ are automorphisms of the graph preserving the identification, hence inducing collineations of the geometry.

If $r = 4$, then we leave it to the reader to check that we obtain the birational geometry.

3.3 A triple square geometry

This geometry is also a special case of $\mathcal{G}_{(r,s)}$ with r equal to s.

There exists an alternative description for this geometry, by taking another coordinate system for the real Euclidean plane. As positive X-axis we take the line from $(0,0)$ to $(-2r,r)$. The positive Y-axis forms an angle of $+150$ degrees with the new X-axis. The unit vector $\vec{e_1}$ is the vector between $(0,0)$ and a vertex on the positive X-axis at graph-theoretical distance 4 from $(0,0)$. The unit vector $\vec{e_2}$ on the Y-axis is the vector between $(0,0)$ and a vertex at graph-theoretical distance 2 from the origin. A point of the geometry is then represented by a tuple (i,j) with i an integer modulo r and j an integer modulo $3r$. We can conclude that the geometry $\mathcal{G}_{(r,r)}$ contains $3r^2$ points. Every line can be represented by a 3-set $\{(i,j),(i+1,j+1),(i+1,j+2)\}$ with i an integer modulo r and j an integer modulo $3r$. Hence there are also $3r^2$ lines in the geometry.

Remark that, as for the square geometry, reflections through axes corresponding to edges of the graph $\mathcal{G}$ are automorphisms of the graph preserving the identification, hence inducing collineations of the geometry.

4 The classification

We now prove our main result. In order to do so, it is convenient to sometimes distinguish between the sharply transitive case and the case where there is a nontrivial flag stabilizer. More precisely, we structure our proof as follows.

Let Γ be a flag-transitive bi-slim geometry of gonality 3. Let x be any point of Γ and L any line incident with x. Let x_1, x_2 be the two other points incident with L, and let L_1, L_2 be the two other lines incident with

x. The first case we consider is the case where at least two points different from x on L_1 and L_2 are collinear with x_1. This can be obtained without loss of generality by renaming x_2 as x_1, if necessary. We subdivide that case into the sharply flag-transitive case and the case of a nontrivial flag stabilizer. Here, the small examples arise. The second case, where exactly one of the four points on L_1 or L_2 different from x is collinear with x_1, is the generic case and gives rise to the infinite class and the infinite example. There, we do not have to distinguish between trivial and nontrivial flag stabilizer. Note that we always assume that the flag stabilizer is finite. The case of an infinite flag stabilizer is treated separately.

So, throughout this section, $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a bi-slim (connected) geometry having n points and admitting a flag-transitive collineation group G of order $|G| \geq 3n$. Let x, x_1, x_2, L, L_1, L_2 be as above. Suppose that x_i, $i = 1, 2$, is collinear with ℓ_i points on L_1 and L_2 different from x. So $1 \leq \ell_i \leq 4$, as the gonality of Γ is 3 and there is a flag-transitive group. We may assume without loss of generality that $\ell_1 \geq \ell_2$.

We introduce some more notation. The points on L_i, $i = 1, 2$, different from x will be denoted by y_i and z_i.

Case I: The flag stabilizer is infinite

It is easy to check that in this case, up to duality, there is a collineation $\theta \in G$ fixing y_1, z_1, x, L, L_2 and interchanging x_1 with x_2. By conjugation, there is also a collineation $\theta' \subset G$ fixing x_1, x_2, x, L_1, L_2 and interchanging y_1 with z_1. Since the gonality of Γ is equal to 3, there must be some triangle, which can be chosen without loss of generality to contain the points x_1, y_1. Note that this implies that x_1y_1 is a line. Applying θ, we see that z_1 is collinear with all points of the line L. By flag transitivity, we may assume that x is collinear with all points of x_1y_1, and hence also with all points of the line x_1z_1. Applying θ', we infer that x_2y_1 and x_2z_1 are lines of Γ all points of which are collinear with x. Now every point collinear with x is incident with three lines all points of which are collinear with x, and so Γ only contains seven points, a contradiction.

Hence from now on, we may assume that the flag stabilizer is finite. We will not repeat this assumption over and over in our subdivisions.

Case II: $\ell_1 \geq 2$

Case IIa: There is a nontrivial flag stabilizer

Let $\theta \in G$ be a nontrivial element fixing the flag $\{x, L\}$ of Γ. Since we assume that $G_{x,L}$ is finite, θ has finite order t and clearly t must be even. For we can always find elements $u, w \in \mathcal{P} \cup \mathcal{L}$ with $u\mathrm{I}w$ fixed by θ and such that $\Gamma_1(w)$ is not fixed pointwise by θ. If t were odd, then θ^t would still act nontrivially on $\Gamma_1(w)$, fixing u and interchanging the two other elements, a contradiction. Hence $G_{x,L}$ is a 2-group. Put $\Gamma_1(x) = \{L, L_1, L_2\}$ and $\Gamma_1(L) = \{x, x_1, x_2\}$. Using the flag-transitive action of G, we easily see that there is an involution σ in $G_{x,L}$ interchanging either x_1 and x_2, or L_1 and L_2, or both. We now first consider the latter case. So we subdivide our proof.

4.1 Case $x_1^\sigma = x_2$ and $L_1^\sigma = L_2$

4.1.1 Case where there is a line through x_1 meeting all of L, L_1, L_2 nontrivially

In this case, we may assume that $x_1y_1y_2$ is a line. Applying σ and noting that, if $y_1^\sigma = y_2$, then there are two lines containing the two points y_1, y_2, we see that also $x_2z_1z_2$ is a line. By assumption, there is a collineation $\theta \in G$ fixing L and mapping x to x_2. Without loss of generality, and applying σ^θ if necessary, we may assume that θ maps L_1 to $x_2z_1z_2$, and x_2 to x. Hence x_1 is fixed. The line $x_2z_1z_2$ is mapped onto one of the lines L_1, L_2, and consequently the image M of the line $x_1y_1y_2$ is incident with x_1, meets L_1 or L_2 in either z_1 or z_2. Remark that the line M is not fixed since y_1 goes to either z_1 or z_2, since $L_1^\theta = x_2z_1z_2$. Also, M meets the third line through x_2 (different from L and $x_2z_1z_2$), say in the point a. So we may assume that x_1z_1a is a line. Now we consider $\theta' \in G$ mapping x to x_1. The inverse image of L_1 meets the three lines L, L_1, L_2, and hence must be one of $x_1y_1y_2$ or $x_2z_1z_2$, the other one having as image a line incident with a, y_2, x_2. Conjugating σ with a collineation fixing x and mapping L onto L_1, we obtain an involution σ' interchanging z_1, z_2 and y_2 with y_1, x_1 and x_2, respectively. Hence x_2y_2a is preserved and a is fixed. Consequently the line x_1z_1a goes to y_1z_2a. Now all points and lines thus far found are incident with exactly 3 elements, and so by connectivity, Γ must consist of $\mathcal{P} = \{x, x_1, x_2, y_1, y_2, z_1, z_2, a\}$ with line set $\mathcal{L} = \{xx_1x_2, xy_1z_1, xy_2z_2, x_1y_1y_2, x_2z_1z_2, x_1z_1a, x_2y_2a, y_1z_2a\}$. This is the Möbius-Kantor geometry.

A similar reasoning holds if there is a line through x_2 meeting L, L_1 and L_2 nontrivially.

So from now on (by flag transitivity), we may assume that no point y of Γ is collinear with all points of any line M of Γ, with $y \not I M$.

4.1.2 Case where every line through x_1 meets L_1 or L_2

Without loss of generality we may assume that $x_1y_1a_1$ and $x_1y_2a_2$ are lines, with $a_1, a_2 \in \mathcal{P}$ distinct and not incident with L, L_1, L_2. Note first that y_1 cannot be collinear with x_2, for otherwise y_1 is collinear with all points of L. Applying σ, we obtain lines $x_2z_1b_1$ and $x_2z_2b_2$, with $b_1, b_2 \in \mathcal{P}$ distinct and not incident with L, L_1, L_2, but not necessarily distinct from a_1, a_2. In fact we will show that $\{a_1, a_2\} = \{b_1, b_2\}$. We conclude that, by flag transitivity, whenever there is a triangle $\{u, v, w\}$, then there is also a triangle $\{u, v', w'\}$, where uvv' and uww' are lines, with $v \neq v'$ and $w \neq w'$. So, since we have the triangle $\{x_1, x, y_1\}$, we also have the triangle $\{x_1, x_2, a_1\}$. So $a_1 \in \Gamma_2(x_2)$ and consequently $a_1 \in \{b_1, b_2\}$. Similarly, $a_2 \in \{b_1, b_2\}$ and so $\{a_1, a_2\} = \{b_1, b_2\}$ as promised. Obviously, two possibilities occur.

$a_1 = b_1$ and $a_2 = b_2$. With the above notation, we then have that vw and $v'w'$ intersect. We claim that y_1 is collinear with y_2. Indeed, if not, then y_1 is collinear with z_2, and we have the hexagon $(x_1, y_1, z_2, x_2, z_1, y_2, x_1)$, *centered at x*. By transitivity, there is also a hexagon centered at x_1, and since y_1 and y_2 are not collinear, y_1 must be collinear with a_2. This implies that $y_1z_2a_2$ is a line, clearly a contradiction. The claim follows. So $y_1y_2a_3$ is a line and also $z_1z_2a_3$ is a line, with a_3 a new point. Hence we have now two triangles *centered at a point* (with self explaining terminology). Applying this to x_1, we see that a_1 is collinear with a_2; similarly with a_3 and it is now trivial to see that $a_1a_2a_3$ is a line. Since all points and lines introduced so far are incident with three elements, there are no further points and lines. We now recognize the Desargues' geometry.

$a_1 = b_2$ and $a_2 = b_1$. Suppose that y_1 and y_2 are collinear. Then σ' (see above) maps x_2z_2 onto y_2x_1 and hence a_1 onto a_2. On the other hand, z_1x_2 is mapped onto y_1y_2 and so a_2 is mapped onto the "third" point of y_1y_2. Since σ' is involutive, $y_1y_2a_1$ is a line and so $x_1 = y_2$, a contradiction. Hence y_1 is collinear with z_2, and y_2 is collinear with z_1. The argument with σ' now shows that both a_1 and a_2 are fixed, but belong to respectively y_2z_1 and y_1z_2. We now obtained the nine points and nine lines of the Pappus geometry.

4.2 Case $x_1^\sigma = x_2$ and $L_i^\sigma = L_i$, $i = 1, 2$

We consider similar subcases as before.

4.2.1 Case where there is a line through x_1 meeting all of L, L_1, L_2 nontrivially

Without loss of generality, we may assume that $x_1y_1y_2$ is a line. There are two possibilities for σ. If σ fixes, again without loss of generality, y_1 and z_1, then σ does not fix y_2 otherwise $x_2y_1y_2$ is a line, a contradiction. Hence σ interchanges y_1 with z_1 and y_2 with z_2. Then $x_2z_1z_2$ is also a line, and the same argument as in 4.1.1 shows that there is a point a with x_1z_1a and x_2y_2a lines (without loss of generality). Applying σ now proves that x_1 is collinear with z_2 (since $x_2 \in \Gamma_2(y_2)$). But we already have six distinct points in $\Gamma_2(x_1) = \{y_1, y_2, x, x_2, z_1, a\}$. Since clearly $z_2 \neq a$, this is a contradiction. In the former case we remark that some conjugate of σ fixes y_2, z_2 and interchanges x_1 with x_2 and y_1 with z_1. Applying σ and this conjugate a number of times, it is easy to see that one obtains the Fano geometry.

From now on, we may again assume that no point y of Γ is collinear with all points of any line M of Γ, with $y \not{I} M$.

4.2.2 Case where every line through x_1 meets L_1 or L_2

We may assume that x_1 is collinear with both y_1 and y_2, but x_1y_1 and x_1y_2 are different lines. Because of our assumption stemming from the previous case, we necessarily have that σ does not fix any of x_1, y_1, y_2. So also x_2z_1 and x_2z_2 are (distinct) lines of Γ. As above, one shows that, if $a_1 \mathrm{I} x_1y_1$, $a_2 \mathrm{I} x_1y_2$, $b_1 \mathrm{I} x_2z_1$ and $b_2 \mathrm{I} x_2z_2$, with $a_1, a_2, b_1, b_2 \notin \Gamma_2(x)$, then $\{a_1, a_2\} = \{b_1, b_2\}$. The case $a_1 = b_1$ leads, as above, to the Desargues geometry. And one can check that, similarly, the case $a_1 = b_2$ leads again to the Pappus geometry.

So we have come to the Case IIb.

Case IIb: The flag stabilizer is trivial

Here, G is a sharply flag-transitive group. So there is a unique collineation $g \in G$ mapping the flag $\{x, L\}$ onto the flag $\{x, L_1\}$. Consequently, the image of L_1 is equal to L_2, otherwise g fixes the flag $\{x, L_2\}$. Similarly, the collineation mapping $\{x, L\}$ onto $\{x_1, L\}$ maps x_1 onto x_2 and x_2 onto x. We will use this in our proof below without further notice.

We again make some distinctions.

4.3.1 Case where there is a line through x_1 meeting all of L, L_1, L_2 nontrivially

In this case, we may assume that $x_1y_1y_2$ is a line. By sharp flag transitivity, there is a collineation θ in G mapping the flag $\{x, L\}$ onto the flag $\{x_1, L\}$, and θ maps x_1 onto x_2. Hence, the image under θ of $x_1y_1y_2$ is a line through x_2 meeting all of the three lines through x_1. Without loss of generality, we may assume that x_2y_1a is a line, with a the intersection point of x_2y_1 and the third line through x_1, different from L and $x_1y_1y_2$. Remark that a may be equal to z_2, but not to one of the points of the set $\{x, x_1, x_2, y_1, z_1, y_2\}$. We can distinguish four possibilities for the images of the points x_1, x_2, y_1, z_1, y_2 and z_2 under the above mentioned collineation g.

In the first case, the collineation g maps the point x_1 onto the point y_1 and y_1 onto y_2. Hence, y_2 is mapped onto x_1. If a is equal to z_2 it is easy to see that we find the Fano geometry. Now, let's look what happens when a is not equal to z_2. Applying g to the line x_2y_1a gives the line z_1y_2, which in turn is mapped onto the line z_2x_1. Since there are only three lines through x_1, we see that $x_1z_2 = x_1a$. The image of z_2x_1a under g is equal to x_2y_1a, which implies that the point a is fixed under g and z_1y_2a is a line. There are now two distinct non-intersecting lines each intersecting all the lines through the point x_1. Looking at the inverse of the collineation θ there also have to be two lines — one through x_1 and one through x_2 — intersecting all of the lines through x. This easily implies that the points x_2, z_1 and z_2 are collinear. Since all points and lines introduced so far are incident with three elements, there are no further points and lines. We now obtained the eight points and lines of the Möbius-Kantor geometry.

Next, we consider the second possibility where g maps x_1 onto y_1 and y_1 onto z_2. Then, the line $x_1y_1y_2$ is mapped onto $x_2y_1z_2$ which in turn has image $z_1z_2x_1$ (hence $a = z_2$). The collineation θ maps the three lines through x_1 onto three lines through x_2 and permutes the points y_1, z_1, y_2 and z_2 amongst themselves. Indeed, these are the points collinear to x and to x_1 not on L, and θ maps x to x_1. It is now easy to see that $x_2z_1y_2$ is a line and we recognize the Fano geometry.

The third possibility assumes that g maps x_1 onto z_1 and z_1 onto y_2. We have that $x_1z_1z_2$ and $x_2y_2z_1$ are lines. Hence, the point a is here equal to z_2. Hence, we have that also $x_2y_1z_2$ is a line. We obtained the seven points and seven lines of the Fano geometry.

In the last case, g maps the point x_1 onto z_1 and z_1 onto z_2. This is similar to the previous two cases and we again obtain the Fano geometry.

4.3.2 Case where every line through x_1 meets L_1 or L_2

Recall that the Desargues geometry does not have a sharply flag-transitive collineation group. Nevertheless, since the collineation group A_5 actually contains collineations g and θ with the properties we use (both see above), we will see that in the following proof the Desargues geometry arises several times. This just means that the situation under consideration does not occur.

Without loss of generality we may assume that $x_1y_1a_1$ and $x_1y_2a_2$ are lines, with $a_1, a_2 \in \mathcal{P}$ distinct and not incident with L, L_1, L_2. The line $x_1y_1a_1$, respectively $x_1y_2a_2$ meets the line L_1, respectively L_2. Hence, θ (see above) maps those two lines onto two lines through x_2, one meeting the line $x_1y_1a_1$, the other meeting the line $x_1y_2a_2$. The only possible images are a line through x_2 and a_1 and a line through x_2 and a_2. Indeed, otherwise we are in Case 4.3.1, with y_1 or y_2 in the role of x. Remark that $\{z_1, z_2\} \subseteq x_2a_1 \cup x_2a_2$, but we will not use this observation, because it leads to a superfluous case distinction. We can distinguish again the same four possibilities for the action of the collineation g on the points x_1, x_2, y_1, z_1, y_2 and z_2 as in the previous case.

In the first case, where g maps x_1 onto y_1 and y_1 onto y_2, the line $x_1y_2a_2$ is mapped onto $y_1x_1a_1$. Hence, the image of a_2 under g is equal to the point a_1. Also, the line $x_1y_1a_1$ is mapped onto a line through y_1 and y_2. The third point on that line, say a_3, must be different from the nine points we already have. The image under θ of y_1 is one of a_1, a_2, otherwise y_1 is collinear with x_2 and we are in Case 4.3.1 again. Similarly $y_2^\theta \in \{a_1, a_2\}$. This implies that a_1 and a_2 are collinear, and, applying g, also a_1a_3 and a_2a_3 are lines. Applying θ^{-1} and then g and g^2, we see that z_1z_2, x_2z_2 and x_2z_1 are lines. Hence, if $a_1a_3 \neq a_1a_2$, then $a_3 \in \{z_1, z_2\}$, as a_1 is on one of x_2z_1, x_2z_2, a contradiction. We conclude that $a_1a_2a_3$ is a line. One now sees that $a_1 \mathrm{I} x_2z_2$ leads to a contradiction considering the action of g. So $a_1 \mathrm{I} x_2z_1$ and $a_2 \mathrm{I} x_2z_2$, and $a_3 \mathrm{I} z_1z_2$. We obtain the Desargues geometry, a contradiction as mentioned before.

The second possibility for g consists of mapping the point x_1 onto y_1 and y_1 onto z_2. Then y_1z_2 and x_2y_1 are lines. Hence L meets the three lines through y_1 nontrivially and we are in Case 4.3.1.

Thirdly, we consider g mapping x_1 onto z_1 and z_1 onto y_2. Here, x_1 and z_1 are collinear which reduces to Case 4.3.1 once again.

The last case is the case where the image of x_1 under g is equal to z_1 and the image of z_1 is equal to z_2. The line $x_1y_1a_1$ is mapped onto the line z_1y_2 and this line is mapped onto x_2z_2. The line $x_1y_2a_2$ has image z_1x_2, which in turn has image z_2y_1. We now have two possibilities: Firstly, $x_2a_1z_2$ and $x_2z_1a_2$ are lines, and secondly, $x_2z_2a_2$ and $x_2a_1z_1$ are lines. In the first case, the line $x_2z_2a_1$ is mapped onto $x_1y_1a_1$, hence the point a_1 is fixed under the collineation g. The image of the line $x_1y_1a_1$ is then $z_1y_2a_1$. Similarly $y_1z_2a_2$ is a line. Now all points and lines thus far found are incident with exactly three elements, and so by connectivity, $\mathcal{P} = \{x, x_1, x_2, y_1, z_1, y_2, z_2, a_1, a_2\}$. We have the Pappus geometry.

Now, let's consider the second possibility. The collineation g maps the line $x_2z_2a_2$ onto the line $y_1x_1a_1$, and this line onto $y_2z_1a_1^g$. The image of the line $x_2z_1a_1$ is equal to $y_1z_2a_1^g$, which in turn has image $y_2x_1a_2$. Hence, the lines y_2z_1 and y_1z_2 have a point $a_1^g = a_3$ in common. It is clear that this point is different from the points we already have. Since the line $y_1z_2a_3$ meets the lines L_1 and L_2, its image under the collineation θ meets the lines $x_1y_2a_2$ and $x_1y_1a_1$ in two different points. Hence, there must be a line joining the points a_1 and a_2. It is then easy to see that this line also contains the point a_3. But now the geometry induced on the points collinear with x is an ordinary hexagon, while the one induced on the points collinear with x_2 contains the ordinary triangle $\{x_1, a_1, a_2\}$. This contradicts the point transitivity of G.

Case III: $\ell_1 = \ell_2 = 1$

Without loss of generality, we here have that x_1y_1, x_2y_2 and z_1z_2 are lines containing no further points of $\{x, x_1, x_2, y_1, y_2, z_1, z_2\}$, and no other line than the six we already have joins two points of the aforementioned set of seven points.

We want to show that in this case we have a geometry $\mathcal{G}_{r,s}$ with $r \geq s$ and $r + s \geq 4$. Hence we have to reconstruct the honeycomb geometry out of Γ as a kind of universal cover. This will be done with standard homotopy arguments using the idea of defining a rank 3 geometry.

Let $\mathcal{X}$ be the set of paths γ starting at the vertex x in the incidence graph $\mathcal{I}(\Gamma)$ of the geometry. In the sequel, we will use obvious notation for the juxtaposition of paths, or of paths and vertices. E.g., if γ is a path and y is a vertex adjacent to the end vertex of γ, then γy is the path obtained from γ by adding y at the end.

For $\gamma_1, \gamma_2 \subset \mathcal{X}$, we say that γ_1 and γ_2 are *elementary homotopic*, in symbols $\gamma_1 \simeq^e \gamma_2$, if $\gamma_1 = \gamma_2$ or if one of the two following cases occur:

(EH1) $\gamma_1 = (x = v_0, v_1, \ldots, v_i, \ldots, v_n)$ and $\gamma_2 = (x = v_0, v_1, \ldots, v_i, w_1, \ldots, w_j, w_{j-1}, \ldots, w_1, v_i, \ldots, v_n)$ or vice versa, with $0 \leq i \leq n$.

(EH2) $\gamma_1 = (x = v_0, \ldots, v_i = w_1, v_{i+1} = w_2, v_{i+2} = w_3, v_{i+3} = w_4, v_{i+4}, \ldots, v_n)$ and $\gamma_2 = (x = v_0, \ldots, v_i = w_1, w_6, w_5, v_{i+3} = w_4, v_{i+4}, \ldots, v_n)$ with $w_1 w_2 w_3 w_4 w_5 w_6 w_1$ a hexagon in $\mathcal{I}(\Gamma)$ and $0 \leq i \leq n-3$.

Remark that the hexagon $w_1 w_2 w_3 w_4 w_5 w_6 w_1$ in the incidence graph corresponds to a triangle in the geometry. Also, $\gamma_1 = \gamma_2$ is a special case of (EH1) for $(w_1, \ldots, w_j)$ the empty path.

As usual, we now define γ and γ' in $\mathcal{X}$ to be *homotopic*, in symbols $\gamma \simeq \gamma'$, if there exist $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathcal{X}$ such that $\gamma = \gamma_1 \simeq^e \gamma_2 \simeq^e \gamma_3 \simeq^e \ldots \simeq^e \gamma_n = \gamma'$. The homotopy class of a path $\gamma \in \mathcal{X}$ will be denoted by $[\gamma]$.

We now define a new graph $\mathcal{G}$ with vertex set the set of homotopy classes of paths in $\mathcal{X}$. The set of edges is defined in the following obvious way. Two distinct vertices $[\gamma_1]$ and $[\gamma_2]$ of $\mathcal{G}$ are *adjacent* if there exists a vertex y in $\mathcal{I}(\Gamma)$ such that $\gamma_1 y \simeq \gamma_2$.

Remark that this definition *implies* that the end vertices of γ_1 and γ_2 are incident in the geometry Γ.

Standard arguments using the cases (EH1) and (EH2) of elementary homotopy show that, if $\gamma_1 y \simeq^e \gamma_2$, then there exists a vertex z such that $\gamma_2 z \simeq \gamma_1$. Note that z is the end vertex of γ_1. So the concept of adjacency is symmetric and the graph $\mathcal{G}$ is well-defined.

We now arrive at a technical point of the proof, and we will not define all notions that we use. We refer to the literature, e.g. [1], [6], [4] and [5].

Note that $\mathcal{G}$ is bipartite and hence it is the incidence graph of a geometry $\Gamma(\mathcal{G})$. We show that $\Gamma(\mathcal{G})$ is the honeycomb geometry. Remarking that the hexagonal tiling of the Euclidean plane is in fact the chamber graph of the unique thin simply connected rank three geometry of type $\tilde{A}_2$, we could try to define a rank 3 geometry where the vertices of $\mathcal{G}$ turn out to be the chambers. This is a rather complicated job, and we will go around this by exhibiting another rank 3 geometry and then delete one type of elements. Indeed, consider the following geometry $\Omega = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, *)$, where $\mathcal{S}_1$ (respectively $\mathcal{S}_2$) is the set of points (respectively lines) of $\Gamma(\mathcal{G})$, $\mathcal{S}_3$ is the set of ordinary triangles in $\Gamma(\mathcal{G})$, and $*$ is the natural and obvious incidence relation. We claim that Ω is the flag complex of a building of type $\tilde{A}_2$. It is easy to check that Ω is a thin geometry of type $\tilde{A}_2$. By Theorem 3(i) of [6], the claim follows if we show simply connectivity, and by Corollary 2.3 of [5], this is equivalent to showing that every closed path in the incidence graph is contractible. It is easily seen that, with the definition of homotopic paths in [5], each path in the incidence graph of Ω is homotopic to a path only using vertices of $S_1 \cup S_2$. Such a path can be identified with a path in $\mathcal{G}$. Furthermore, it is straightforward to check that the notions of homotopy as defined here and defined in [5] coincide over the set of such paths. Also, a standard homotopy argument shows that each closed path in $\mathcal{G}$ is homotopic equivalent to a trivial path. The claim follows. Hence the incidence graph of Ω can be represented as the tiling of the Euclidean plane in regular triangles. Deleting the vertices of S_3, we obtain at one hand $\mathcal{G}$ and at the other hand the regular hexagonal tiling. Hence $\Gamma(\mathcal{G})$ is (isomorphic to) the honeycomb geometry.

We define the function π which projects every vertex $[\gamma]$ in the graph $\mathcal{G}$ on the end vertex of γ. A standard argument shows that this is an epimorphic local isomorphism, i.e., the restriction to the neighbors of any vertex is a bijection onto the image.

Also, every collineation $\alpha \in G$ *lifts* to an automorphism $\tilde{\alpha}$ of the graph $\mathcal{G}$, and hence to a collineation of the honeycomb geometry $\Gamma(\mathcal{G})$. This follows from standard homotopy theory; in fact for every element a of Γ, for every element $\tilde{a}$ of the fiber of a, and for every element $\widetilde{a^\alpha}$ of the fiber of a^α, $\tilde{\alpha}$ can be chosen such that it maps $\tilde{a}$ to $\widetilde{a^\alpha}$. Now note that the stabilizer of a flag in $\Gamma(\mathcal{G})$ has order 2 in the full collineation group. This can easily be seen using the fact that a thin building of type $\tilde{A}_2$ is a Coxeter complex with full automorphism group the associated Coxeter group. Together with the *deck transformations*, G lifts to a flag-transitive group $\tilde{G}$ of $\Gamma(\mathcal{G})$. The latter contains the sharply flag-transitive group $\tilde{G}_0$ induced by all collineations which represent isometries with determinant 1 in the Euclidean plane.

Now let $\mathcal{V}_x$ be the set of all vertices in the graph $\mathcal{G}$ corresponding to the point x of the geometry Γ. So $\mathcal{V}_x$ is the *fiber* determined by x. We consider two possibilities.

$\mathcal{V}_x$ is a singleton It is clear (by flag transitivity) that every fiber is trivial and that Γ is isomorphic to $\Gamma(\mathcal{G})$. Hence Γ itself is the honeycomb geometry.

$\mathcal{V}_x$ is not a singleton Let $\tilde{x}$ be a reference vertex in $\mathcal{V}_x$.

We choose a coordinate system for the real Euclidean plane. The vertex $\tilde{x}$ is chosen as the origin. The unit vector on the X-axis is chosen to be the vector $\overrightarrow{\tilde{x}\widetilde{x_1}}$ with $\widetilde{x_1}$ a vertex corresponding to the point x_1 at graph-theoretical distance 2 from $\tilde{x}$. We may assume that x_1 and y_1 are collinear in Γ. Then the unit vector on the Y-axis is chosen to be the vector $\overrightarrow{\tilde{x}\widetilde{y_1}}$ with $\widetilde{y_1}$ a vertex in $\mathcal{V}_{y_1}$ at graph-theoretical distance 2 from both $\tilde{x}$ and $\widetilde{x_1}$.

Let $\tilde{x}'$ be a vertex in $\mathcal{V}_x$ at minimal Euclidean distance m from the reference vertex $\tilde{x}$.

Rotations over $+120$ degrees and -120 degrees with center $\tilde{x}$ belong to $\tilde{G}_0$ and hence are lifts of collineations in G fixing the point x. Applying these rotations to $\tilde{x}'$ we get two new vertices $\tilde{x}'^{r_{120^+}}$ and $\tilde{x}'^{r_{120^-}}$ in $\mathcal{V}_x$ at the same distance m from $\tilde{x}$. The translation with vector $\overrightarrow{\tilde{x}\tilde{x}'}$ is also the lifting of an automorphism in G fixing the point x. Applying this to the vertices $\tilde{x}'^{r_{120^+}}$ and $\tilde{x}'^{r_{120^-}}$ gives us two new vertices $\tilde{x}''$ and $\tilde{x}'''$ in $\mathcal{V}_x$. Because of the metric properties of a rhombus, the angle between $\tilde{x}\tilde{x}'$ and $\tilde{x}\tilde{x}''$ and between $\tilde{x}\tilde{x}''$ and $\tilde{x}\tilde{x}'^{r_{120^+}}$ is equal to sixty degrees. The same holds for the angle between $\tilde{x}\tilde{x}'$ and $\tilde{x}\tilde{x}'''$ and between $\tilde{x}\tilde{x}'''$ and $\tilde{x}\tilde{x}'^{r_{120^-}}$. The two rotations r_{120^+} and r_{120^-} map the vertices $\tilde{x}''$ and $\tilde{x}'''$ onto each other and onto one other vertex in $\mathcal{V}_x$. We conclude that we get six vertices of $\mathcal{V}_x$ on a regular hexagon around $\tilde{x}$ at distance m from the reference vertex and from each other. From now on, we will suppose that $\tilde{x}'$ is lying between the positive X- and the positive Y-axis. The coordinates of $\tilde{x}'$ are given by the tuple (r, s) with $r, s \geq 0$. Without loss of generality we may assume that $r \geq s$ (interchanging X- and Y-axis if necessary).

Applying successively rotations of 60 degrees, a tedious calculation shows that the coordinates of the six vertices in $\mathcal{V}_x$ on the regular hexagon around $\tilde{x}$ are (r, s), $(-s, r+s)$, $-(r, s)+(-s, r+s) = (-r-s, r)$, $-(r, s) = (-r, -s)$, $-(-s, r+s) = (s, -r-s)$ and $-(-s, r+s)+(r, s) = (r+s, -r)$. Remark that the first two vectors generate the others by taking sums. In fact, by the minimality of m, all elements of $\mathcal{V}_x$ are generated by (r, s) and $(-s, r+s)$ by taking sums. Hence a generic element of $\mathcal{V}_x$ has coordinates $k(r, s) + l(-s, r+s) = (kr - ls, ks + lr + ls)$ with k and l integers.

Consider an arbitrary vertex $\tilde{z}$ in $V(\mathcal{I}(\Gamma))$ corresponding to a point z of the geometry Γ. The translation with vector $\overrightarrow{\tilde{x}\tilde{z}}$ is the lifting of an automorphism in the collineation group G mapping the point x onto the point z. Hence we see that the vertices of $\mathcal{V}_z$ are parameterized by $(i, j) + k(r, s) + l(-s, r+s) = (i + kr - ls, j + ks + lr + ls)$ with k and l integers, and with (i, j) the coordinates of $\tilde{z}$.

A line of the geometry can be described by a 3-set of coordinates of the three points incident with that line: a possible representation is given by $\{(i, j), (i+1, j), (i+1, j-1)\}$.

We thus recognize the geometry $\mathcal{G}_{(r,s)}$.

If the flag stabilizer of Γ, and hence also of $\Gamma(\mathcal{G})$, is not trivial, then $\mathcal{V}_x$ is invariant under the symmetry about the bisector of the X-axis and the Y-axis, and it is easy to see that in this case there are exactly two possibilities. Firstly, this bisector contains an element of $\mathcal{V}_x$, and we obtain a triple square geometry. Secondly, the X-axis contains an element of V_x, and we obtain a square geometry.

This completes the proof of Theorem 1.1.

Remark. Roughly speaking, our main result says that a flag-transitive bi-slim geometry of gonality 3 is (1) either a standard quotient of the honeycomb geometry, (2) or the unique proper quotient of a "dodecahedron geometry" (see below), (3) or a nonstandard quotient of the honeycomb geometry — the Möbius-Kantor geometry. In case (1) relatively larger groups appear in the small examples because of the richer "local structure" — by which we mean the geometry of points collinear to a given point.

Indeed, one can check that the Möbius-Kantor geometry can be obtained from the honeycomb geometry by considering a fundamental domain consisting of 8 hexagons as follows. Let H_1 and H_2 be two hexagons of the hexagonal tiling of the Euclidean plane sharing an edge e. Then consider all hexagons of the tiling sharing an edge f with one of H_1 or H_2, where f is not opposite the edge e in H_1 or H_2. This construction can be generalized by taking more hexagons, but only in this small case the collineation group of the resulting geometry is flag-transitive.

With regard to the Desargues geometry, it can also be constructed as follows. Consider the geometry Γ arising from the dodecahedron by taking as point set the vertices of the dodecahedron and as lines a second copy of the vertices. Incidence is adjacency. Then we obtain a geometry of gonality 4. Now identify elements which come from opposite vertices of the dodecahedron. This produces the Desargues geometry.

In conclusion, all geometries appearing in our main result are quotients of geometries related to regular tilings of either the Euclidean plane, or the 2-sphere.

5 Proof of Corollary 1.2

Under the given assumptions, there are two possibilities. The first one is that the graph is the incidence graph of a flag-transitive bi-slim geometry with gonality 6. But then the result follows easily. The second possibility is that the geometry obtained from the graph by taking as point and line set two copies of X, and declaring a point incident with a line if the corresponding vertices in $\mathcal{G}$ are adjacent, is a flag-transitive bi-slim geometry with gonality 6. It is easy to check (see for instance [7]) that such graphs arise only from polarities without absolute points. Now it is an exercise to check that only the Desargues geometry has a polarity without absolute points which gives rise to an edge-transitive graph, namely, the Petersen graph. In the geometries $\mathcal{G}r.s$. one checks that polarities without absolute points are induced by point symmetries of the hexagonal tiling of the Euclidian plane. But the resulting graph has edges which lie on triangles, and edges which are not contained in any triangle. Hence these graphs do not satisfy our assumptions and the corollary is proved.

References

[1] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. I. Données radicielles valuées. *Inst. Hautes Études Sci. Publ. Math.* **41** (1972), 5 – 252.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*. Clarendon Press, Oxford (1985).

[3] H. S. M. Coxeter. Self-dual configurations and regular graphs. *Bull. Amer. Math. Soc.* **56** (1950), 413 – 455.

[4] A. Pasini. *Diagram Geometries*. Clarendon Press, Oxford, 1994.

[5] M. A. Ronan. Coverings and automorphisms of chamber systems. *European J. Combin.* **1** (1980), 259 – 269.

[6] J. Tits. A local approach to buildings, in *The Geometric Vein. The Coxeter Festschrift* (ed. D. Chandler *et al.*), Springer-Verlag (1981), 519 – 547.

[7] H. Van Maldeghem. Slim and bi-slim geometries, to appear.

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