Recursive constructions on large sets of some balanced incomplete block designs

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Abstract

Larges sets of balanced incomplete block (BIB) designs and resolvable BIB designs are discussed. Some recursive constructions of such large sets are given. Some existence results in particular for practical k are reviewed.

1 Introduction

Let v, k and λ be three positive integers such that $v \geq k \geq 2$. We denote the set of all i-subsets of a set X by $\mathcal{P}_i(X)$.

A balanced incomplete block (BIB) design, denoted by $\mathrm{B}(v, k, \lambda)$, is a pair $(X, \mathcal{B})$ in which X is a finite set with cardinality v and $\mathcal{B}$ is a subset of $\mathcal{P}_k(X)$ such that every element of $\mathcal{P}_2(X)$ appears exactly λ times in $\mathcal{B}$. In this case each $\mathcal{P}_1(X)$ appears a constant, r (say), times in $\mathcal{B}$. This is called a replication number of the design. Further let b be the cardinality of $\mathcal{B}$.

A large set of disjoint $\mathrm{B}(v, k, \lambda)$, denoted by $\mathrm{LB}(v, k, \lambda)$, is a partition of $\mathcal{P}_k(X)$ for a v-set X into $\mathrm{B}(v, k, \lambda)$ without repeated k-subsets. We denote by $s(v, k, \lambda)$ the maximum number of mutually disjoint $\mathrm{B}(v, k, \lambda)$ on the v-set X. Obviously, it holds that $s(v, k, \lambda) \leq \binom{v-2}{k-2}/\lambda$ with equality occuring if and only if there is an $\mathrm{LB}(v, k, \lambda)$. Of course, any non-existence result for $\mathrm{B}(v, k, \lambda)$ provides a non-existence result for $\mathrm{LB}(v, k, \lambda)$.

A $\mathrm{B}(v, k, \lambda)$ is said to be α-resolvable of the b k-subsets are separated into t classes, called resolution classes, of β k-subsets each such that in each class every point of X appears α times. Here $b = \beta t$ and $r = \alpha t$. Furthermore, an α-resolvable $\mathrm{B}(v, k, \lambda)$ is said to be affine α-resolvable if any two distinct k-subsets from the same resolution class include q_1 points

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in common, while any two k-subsets from different resolution classes include q_2 points in common. Here it holds (see Raghavarao [16]) that $q_1 = k(\alpha - 1)/(\beta - 1) = k + \lambda - r$ and $q_2 = k^2/v$. An (affine) 1-resolvable design is simply called an (affine) resolvable design, and necessarily $\alpha = 1$, $t = r$, $\beta = v/k$ ($= n$, say, in Section 3), $q_1 = 0$. These designs are denoted by RB(v, k, λ) (ARB(v, k, λ)).

A large set of disjoint RB(v, k, λ) is denoted by LRB(v, k, λ), while a large set of disjoint ARB(v, k, λ) is denoted by LARB(v, k, λ).

A necessary and sufficient condition for the existence of LB$(v, 3, 1)$ is that $v \equiv 1, 3 \pmod 6$ and $v \neq 7$ (see Lu [13, 14], Sharry and Street [17]). A necessary condition for the existence of LRB$(v, 3, 1)$ is obvious to be $v \equiv 3 \pmod 6$. A sufficient condition for the existence of LRB$(v, 3, 1)$ is that $v \equiv 3 \pmod 6$ and $v = 3^s$ for a positive integer s (see Denniston [7]). Other sufficient conditions for the existence are known for LRB$(v, 3, 1)$ (see Chang and Ge [4]). However, there are no rich results on the existence of LB(v, k, λ) with $\lambda = 1$ and $k \geq 4$. There are several observations in literature. A good reference on this topic is Kang [10] for triple systems.

In this paper, some recursive constructions of such large sets, with the existence of new LRB(v, k, λ) for $k \geq 4$, will be discussed.

2 Recursive constructions

Some recursive constructions are provided for LB(v, k, λ) and LRB(v, k, λ). Let $s(v, k, \lambda) = \binom{v-2}{k-2}/\lambda$ ($= s$, say) in the large set. First note that $s = \binom{v}{k}/b = \binom{v-1}{k-1}/r = \binom{v-2}{k-2}/\lambda$ in LB(v, k, λ).

The following can be easily shown by the structure of large sets.

Lemma 2.1. The existence of an LB(v, k, λ) with the stated s implies the existence of an LB$(v, k, p\lambda)$ for s/p being an integer.

Example 2.1. An existing LB$(13, 6, 55)$ with $s = 6$ (see Example 2.4 later) implies the existence of LB$(13, 6, 55 \times 2)$, an LB$(13, 6, 55 \times 3)$ and an LB$(13, 6, 55 \times 6)$.

Usually, we are interested in LB$(v, k, \min\lambda)$ (similarly, LRB$(v, k, \min\lambda)$), where $\min\lambda$ denotes the minimum value of λ among admissible parameters v, k, λ for given v and k.

Theorem 2.1. The existence of an LB(v, k, λ), with a replication number r, and an LB$(v, k+1, r-\lambda)$ implies the existence of an LB$(v+1, k+1, r)$.

Proof. It is obvious that $s = \binom{v-2}{k-2}/\lambda = s(v, k, \lambda) = s(v, k+1, r-\lambda)$. Then a juxtaposition of a B(v, k, λ) in the LB(v, k, λ) and a B$(v, k+1, r-\lambda)$ in the LB$(v, k+1, r-\lambda)$ can yield a B$(v+1, k+1, r)$, after an addition of a new point to all the k-subsets in the B(v, k, λ). Hence there are such s B$(v+1, k+1, r)$ which constitute the required LB$(v+1, k+1, r)$, because all the k-subsets are disjoint and $s(v, k, \lambda) = s(v, k+1, r-\lambda) = s(v+1, k+1, r)$. □

Example 2.2. An LB(13,4,1) with $s = 55$ (see Chouinard [6]) by Lemma 2.1 yields an LB(13,4,5) with $s = 11$, which, together with an LB(13,3,1) with $s = 11$ and $r = 6$ (see Denniston [8]) and Theorem 2.1, produces an LB(14,4,6) with $s = 11$, which is new with $\min\lambda$ for given $v = 14$ and $k = 4$.

Example 2.3. By Lemma 2.1, an LB(12,4,3) with $s = 15$ and $r = 11$ (see Kramer, Magliveras and Stinson [12]) yields an LB(12,4,15) with $s = 3$ and $r = 55$, while an LB(12,5,20) with $s = 6$ (see Kramer, Magliveras and Stinson [12]) yields an LB(12,4,40) with $s = 3$. Hence by Theorem 2.1 the last two large sets together produce an LB(13,5,55) with $s = 3$. However, for $\min\lambda = 5$ the existence of an LB(13,5,5) with $s = 33$ is unknown.

Example 2.4. An LB(12,6,5) with $s = 42$ (see Kramer, Magliveras and Stinson [12]) by Lemma 2.1 yields an LB(12,6,35) with $s = 6$, which, together with an LB(12,5,20) with $s = 6$ (see Kramer, Magliveras and Stinson [12]) and Theorem 2.1, produces an LB(13,6,55) with $s = 6$. However, for $\min\lambda = 5$ the existence of an LB(13,6,5) with $s = 66$ is unknown.

Corollary 2.1 The existence of an LB(v, k, λ) with b k-subsets and a replication number r, an LB$(v, k+1, r-\lambda)$ and an LB$(v, k+2, b-2r+\lambda)$ implies the existence of an LB$(v+2, k+2, b)$.

Proof. The same procedure as the proof of Theorem 2.1 can be taken. At first a combination of a B(v, k, λ) and a B$(v, k+1, r-\lambda)$ yields a B$(v+1, k+1, r)$, while a combination of a B$(v, k+1, r-\lambda)$ and a B$(v, k+2, b-2r+\lambda)$ yields a B$(v+1, k+2, b-r)$. Hence the resulting two designs can produce a B$(v+2, k+2, b)$ by Theorem 2.1. This procedure should be repeated s $(= \binom{v-2}{k-2})/\lambda)$ times. Then the required LB$(v+2, k+2, b)$ can be obtained. □

Example 2.5. An LB(12,4,15) (see Example 2.3), an LB(12,5,40) (see Example 2.3) and an LB(12,6,35) (see Example 2.4), by Corollary 2.1, yields an LB(14,6,165). However, for $\min\lambda = 15$ the existence of an LB(14,6,15) with $s = 33$ is unknown.

Lemma 2.2. The existence of a B$(2k+1, k, \lambda)$, with a replication number r, is equivalent to the existence of an RB$(2k+2, k+1, r)$.

Proof. The necessity is obvious by taking a juxtaposition of a B$(2k+1, k, \lambda)$, with a new point added to all k-subsets, and its complement B$(2k+1, k+1, r/k+\lambda)$. The sufficiency is shown as follows. Since $v = 2(k+1)$ in the RB$(2k+2, k+1, r)$, each resolution class consists of two $(k+1)$-subsets that must be self-complementary to each other. Hence it can be shown that all $(k+1)$-subsets containing a particular point yield a B$(2k+1, k, \lambda)$ with a replication number r, after deletion of the particular point. □

Theorem 2.2. The existence of an LB$(2k+1, k, \lambda)$, with a replication number r, is equivalent to the existence of an LRB$(2k+2, k+1, r)$.

Proof. First note that $s(2k+1, k, \lambda) = \binom{2k-1}{k-2}/\lambda$, and $s(2k+2, k+1, 2k\lambda/(k-1)) = \binom{2k}{k-1}/[2k\lambda/(k-9)] = \binom{2k-1}{k-2}/\lambda$. Hence $s(2k+1, k, \lambda) = s(2k+2, k+$

$1, r)$. Therefore, by Lemma 2.2 the equivalence on the existence of two large sets can be shown, because all the subsets are disjoint. □

Theorem 2.2 can present the following.

Corollary 2.2. The existence of an $\mathrm{LRB}(2k, k, \lambda)$ is equivalent to the existence of an $\mathrm{LB}(2k-1, k-1, \lambda(k-2)/[2(k-1)])$.

Corollary 2.3. A necessary condition for the existence of an $\mathrm{LRB}(2k, k, \lambda)$ is that $\lambda(k-2)/[2(k-1)]$ is a positive integer.

Corollary 2.3 shows that in an $\mathrm{LRB}(2k, k, \lambda)$, λ is divisible by $k-1$. Hence the parameters of an $\mathrm{RB}(2k, k, \lambda)$ in the $\mathrm{LRB}(2k, k, \lambda)$ are expressed by $v = 2k$, $b = 2\ell(2k-1)$, $r = \ell(2k-1)$, k, $\lambda = \ell(k-1)$ for a positive integer ℓ.

Example 2.6. An LB(9,4,3) with $r = 8$ (see Kramer, Magliveras and Stinson [12]) with Theorem 2.2 yields an LRB(10,5,8). Now Corollary 2.3 shows the non-existence of an LRB(10,5,4) with $s = 14$. Hence the new LRB(10,5,8) also has $\min \lambda$ for given $v = 10$ and $k = 5$.

Theorem 2.3. The existence of an $\mathrm{LB}(2k+2, k, \lambda)$, with b k-subsets and a replication number r, and an $\mathrm{LB}(2k+2, k+1, r-\lambda)$ implies the existence of an $\mathrm{LRB}(2k+4, k+2, b)$.

Proof. By Theorem 2.1 we have an $\mathrm{LB}(2k+3, k+1, r)$. Furthermore, by Theorem 2.2 an $\mathrm{LRB}(2k+4, k+2, 2r(k+1)/k)$ can be obtained. Here it holds that $2r(k+1)/k = b$. □

Example 2.7. An LB(12,5,20) with $r = 55$, and $b = 132$ (see Kramer, Magliveras and Stinson [12]) and an LB(12,6,35) (see Example 2.4), by Theorem 2.3, yield an LRB(14,7,132). On the other hand, Corollary 2.3 shows the non-existence of an LRB(14,7,6) with $s = 132$. However, for $\min \lambda = 12$ the existence of an LRB(14,7,12) with $s = 66$ is unknown. Incidently, the existence of an LB(14,7,6) with $s = 132$ is also unknown.

Example 2.8. An LB(12,4,15) with $r = 55$ and $b = 165$ (see Example 2.3), an LB(12,5,40) with $r = 110$ and $b = 264$ (see Example 2.3), an LB(12,6,70) (see Kramer, Magliveras and Stinson [12]) and an LB(12,7,84) (being the complement of the LB(12,5,40)), by Corollary 2.1, yield an LB(14,6,165) with $r = 429$ and $b = 1001$ (using the first three LB) and an LB(14,7,264) (using the three LB from the second). The last two designs can produce an LRB(16,8,1001) with $s = 3$, by Theorem 2.3. Incidentally, for $\min \lambda = 7$ the existence of an LB(16,8,7) with $s = 429$ is unknown.

3 $\mathrm{LRB}(v, k, \lambda)$ or $\mathrm{LARB}(v, k, \lambda)$

It is known (see Raghavarao [16]) that a necessary and sufficient condition for an $\mathrm{RB}(v = nk, k, \lambda)$ with b k-subsets and a replication number r to be affine resolvable is $b = v + r - 1$. In this case it holds that $\lambda = (k-1)/(n-1)$ and $q_2 = k/n$. Hence the parameters of an $\mathrm{ARB}(v, k, \lambda)$ can be expressed as

$$v = nk = n^2[(n-1)t+1], k = n[(n-1)t+1], \lambda = nt+1 \tag{3.1}$$

for a non-negative integer t.

The following two lemmas can be derived also from the integrality of λ and q_2 in ARB($v = nk, k, \lambda$) with $n \geq 2$.

Lemma 3.1. When k is a prime, a necessary condition for the existence of an LARB($v = nk, k, \lambda$) is that $n = k$.

Lemma 3.2. When $k-1$ is a prime, a necessary condition for the existence of an LARB($v = nk, k, \lambda$) is that $n = 2$ or k.

Now LARB(nk, k, λ) with parameters (3.1) may be classified into four classes: (1) $t = 0$ (iff $n = k$), in this case it is an LARB($k^2, k, 1$); (2) $t = 1$ (iff $n = \sqrt{k}$), in this case it is an LARB($n^3, n^2, n+1$); (3) $t \geq 2$ and $n = 2$, in this case it is an LARB($2k, k, k-1$); (4) $t \geq 2$ and $n(\geq 3)$ ($\neq k, \sqrt{k}$). Kimura [11] gives a list on the existence status of LARB(v, k, λ) for $k \leq 20$ and min λ according to the above classification.

We can find many large sets belonging to cases (1) and (3) in literature. Hence LARB($k^2, k, 1$) and LARB($2k, k, k-1$) are further considered in this section.

By Lemma 3.1, when k is a prime, a possible LARB(v, k, λ) is an LARB($k^2, k, 1$). By Lemma 3.2, when $k-1$ is a prime, a possible LARB(v, k, λ) is an LARB($2k, k, k-1$) or LARB($k^2, k, 1$).

In particular, when $\lambda = k-1$, it is obvious that an RB($2k, k, \lambda$) is affine resolvable. Hence by Corollary 2.2, the following can be obtained.

Lemma 3.3. The existence of an LARB($2k, k, k-1$) is equivalent to the existence of an LB($2k-1, k-1, k/2-1$).

Lemma 3.3 implies that in LARB($2k, k, k-1$) k must be even, and then shows the non-existence of an LARB(10,5,4), while note that there exists an LB(10,5,4) (see Kramer, Magliveras and Stinson [12]).

Theorem 3.1. The existence of an LARB($k^2, k, 1$) implies the existence of an LRB($k^2-1, k-1, \binom{k^2-3}{k-3}$) and a k-LRB($k^2-1, k, \binom{k^2-3}{k-2}$), where k-LRB denotes a large set of k-resolvable BIB designs.

Proof. Note that $s(k^2, k, 1) = \binom{k^2-2}{k-2}$ ($= s$, say), $s(k^2-1, k-1, \binom{k^2-3}{k-3}) = 1$, and $s(k^2-1, k, \binom{k^2-3}{k-2}) = 1$. Also the ARB($k^2, k, 1$) has the parameters $b = k(k+1)$ and $r = k+1$. In s ARB($k^2, k, 1$), $(X, \mathcal{B})$, in the LARB($k^2, k, 1$), let x be any point in X, and $\mathcal{B}_i$ be a collection of k-subsets in $\mathcal{B}$ including x, and $\mathcal{B}_i^*$ be a collection of k-subsets in $\mathcal{B}$ not including x, for $i = 1, 2, \ldots, s$. Further let $X' = X - \{x\}$ and $\mathcal{B}_i'$ be a collection of $(k-1)$-subsets in $\mathcal{B}_i$ with deletion of x. Then take $\mathcal{B}' = \cup_{i=1}^s \mathcal{B}_i'$ and $\mathcal{B}^* = \cup_{i=1}^s \mathcal{B}_i^*$. Hence it can be shown that the pair $(X', \mathcal{B}')$ and $(X', \mathcal{B}^*)$ are an LRB($k^2-1, k-1, \binom{k^2-3}{k-3}$) and a k-LRB($k^2-1, k, \binom{k^2-3}{k-2}$), respectively. □

Now we review a case $k = 4$. In this case, it is known (see Anderson [1]) that (i) a necessary and sufficient condition for the existence of a $B(v, 4, 1)$ is $v \equiv 1$, 4 (mod 12), and (ii) a necessary and sufficient condition for the existence of an $RB(v, 4, 1)$ is $v \equiv 4$, (mod 12). Hence for their large sets it is obvious that a necessary condition for the existence of an $LB(v, 4, 1)$ (or $LRB(v, 4, 1)$) is given by $v \equiv 1$, 4 (mod 12) (or $v \equiv 4$ (mod 12)).

As far as the authors are aware of (see Beth, Jungnickel and Lenz [2]), for $k \geq 4$, the existence of $LB(v, k, 1)$ is known only for an LB(13,4,1) (see Chouinard [6]) and an LRB(16,4,1) (see Mathon [15]). Furthermore, within the range of $v \leq 13$, an LRB(12,4,3) (with $s = 15$) is the only unknown case on the exstence for min λ (though $\lambda = 9$ and 15 are still unknown) among $LRB(v, k, \lambda)$ with $k \geq 4$. Note (see Kageyama [9]) that there exists an RB(12,4,3), i.e., [(0,1,3,7), (2,4,9,10), (5,6,8,∞)] mod 11. Recently, Kimura [11] constructed an LRB(12,4,45) (with $s = 1$ and $r = 165$) by giving 165 resolutions classes. The reader can get a solution of an LRB(12,4,45), on request to the second author.

$LARB(v, 4, \lambda)$ are further considered. Since $k - 1 = 3$ being a prime, by Lemma 3.2 and $b = v + r - 1$, it is an LARB(8,4,3) or LARB(16,4,1). The existence of an LARB(16,4,1) is known (see Mathon [15]), while the existence of an LARB(8,4,3) can be disproved by the non-existence of an LB(7,3,1) (see Cayley [3]) and Theorem 2.2 (see also Sharry and Street [18]). Note that Kimura [11] has shown the non-existence of an LARB(8,4,3) directly. Thus, $LARB(v, 4, \lambda)$ is the only LARB(16,4,1). As far as the authors are aware of, for $k \geq 4$, the existence of $LARB(v, k, \lambda)$ are known only for an LARB(12,6,5) and LARB(16,4,1).

Example 3.1. An LARB(16,4,1), by Theorem 3.1, yield an LRB(15,3,13) and a 4-LRB(15,4,$\binom{13}{2}$)).

Kimura [11] also presented 6 disjoint RB(12,4,3), but not for an LRB(12, 4,3). A big list on the existence status of $LB(v, k, \lambda)$ or $LRB(v, k, \lambda)$ for 1105 parameters' sets with the scope of $8 \leq v \leq 28$, $4 \leq k \leq 10$ and $\lambda \leq \binom{v-2}{k-2}$ has been provided. In fact, among $LB(v, k, \lambda)$, 315 designs exist, while among $LRB(v, k, \lambda)$, 21 designs including 2 $LARB(v, k, \lambda)$ exist. A similar list has been provided by Chee, Colbourn and Kreher [5].

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