The Basis Number of the Direct Product of a Theta Graph and a Path

M.M.M. Jaradat
Department of Mathematics
Yarmouk University
Irbid-Jordan
mmjst4@yu.edu.jo

Abstract

The basis number $b(G)$ of a graph G is defined to be the least integer d such that G has a d-fold basis for its cycle space. In this paper we investigate the basis number of the direct product of theta graphs and paths.

1 Introduction.

Unless otherwise specified, all graphs considered here are finite, undirected and simple. For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$. Given a graph G, let $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of its edges. Then a subset S of $E(G)$ corresponds to a $(0,1)$-vector $(b_1, b_2, \ldots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$, and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$-dimensional vector space, denoted by $(Z_2)^{|E(G)|}$, over the field of integer numbers modulo 2. The vectors in $(Z_2)^{|E(G)|}$ which corresponds to the cycles in G generate a subspace called the cycle space of G and denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is well-known that

$$\dim \mathcal{C}(G) = \gamma(G) = |E(G)| - |V(G)| + r \tag{1}$$

where $\gamma(G)$ is the cyclomatic number and r is the number of connected components.

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a d-fold if each edge of G occurs in at most d of the cycles in the basis $\mathcal{B}$. The basis number $b(G)$ of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The fold of an edge e in a set $B \subset C(G)$, denoted by $f_B(e)$, is the number of cycles in B containing e.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The direct product $G = G_1 \wedge G_2$ is the graph with the vertex set $V(G) = V_1 \times V_2$ and the edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E_1 \text{ and } u_2v_2 \in E_2\}$. From the definition above, it is clear that (i) $d_{G_1 \wedge G_2}(x, y) = d_{G_1}(x)d_{G_2}(y)$ and (ii) $|E(G_1 \wedge G_2)| = 2|E_1||E_2|$ where $d_G(v)$ is the degree of the vertex v in the graph G.

The first result concerning the basis number of a graph was obtained in 1937 by MacLane who proved the following:

Theorem 1.1 *(MacLane) A graph G is planar if and only if $b(G) \leq 2$.*

The following theorem due to Schmeichel, which proves the existence of graphs that have arbitrary large basis number.

Theorem 1.2 *(Schmeichel) For any positive integer r, there exists a graph G with $b(G) \geq r$.*

In 1981, Schmeichel proved that for $n \geq 5$, we have $b(K_n) = 3$ where K_n is the complete graph of n vertices, and for $m, n \geq 5$, we have $b(K_{n,m}) = 4$ except possibly for $K_{6,10}, K_{5,n}$ and $K_{6,n} (n = 5, 6, 7, 8)$ where $K_{n,m}$ is the complete bipartite graph of n and m vertices. In 1982, Banks and Schmeichel proved that $b(Q_n) = 4$ where Q_n is the n-cube.

In 1989, Ali investigated the basis number of the direct product of some special graphs. In fact he proved that $b(C \wedge P) \leq 2$, $b(P \wedge P^*) \leq 2$, and for all $|V(C)|$ and $|V(C^*)| \geq 3$, we have $b(C \wedge C^*) = 3$. In 1996, Al-Rhayyel and Jaradat proved the following results concerning the basis number of the direct product of some special graphs: (i) $b(P \wedge S) = 2$, if $|V(S)| \geq 4$ and $|V(P)| \geq 3$ (ii) $b(C \wedge S) = 2$, if $|V(C)| \geq 4$ and $|V(S)| \geq 3$ (iii) $b(\theta \wedge S) = 3$, if $|V(\theta)| \geq 4$ and $|V(S)| \geq 4$ where θ is the theta graph, (iv) $b(S \wedge S^*) \leq 4$, and the equality holds for each $|V(S)| \geq 6$ and $|V(S^*)| \geq 6$ except possibly $|V(S)| = 6$ and $|V(S^*)| = 6, 7, 8, 9$ and $|V(S)| = 7$ and $|V(S^*)| = 6, 7, 8, 9, 11$ where C and C^* stand to cycles, P and P^* stand to paths and S and S^* stand to stars.

We remark that knowing the number of components in a graph is very important to find the dimension of the cycle space as in (1), so we need the following result.

Theorem 1.3 *([7]) Let G and H be two connected graphs. Then $G \wedge H$ is connected if and only if at least one of them contains an odd cycle. Moreover, If both of them are bipartite graphs then $G \wedge H$ consists of two components.*

In this paper we investigate the basis number of the direct product of a theta graph with a path. In the rest of this work, let n and m stand to the number of vertices of $|V(\theta)|$ and $|V(P)|$, respectively.

2 Main Results.

Throughout this section we label the vertices of θ and P by $\{1,2,\ldots,n\}$ and $\{1,2,\ldots,m\}$, respectively. A theta graph θ is defined to be a cycle C with n vertices to which we add a new edge that joins two non-adjacent vertices of C. We may assume 1 and δ are the two vertices of θ of degree 3. A tree T consisting of k equal order paths $\{P^{(1)}, P^{(2)}, \ldots, P^{(k)}\}$ is called a k-special star if there is a vertex, say v_1, such that v_1 is an end vertex for each path in $\{P^{(1)}, P^{(2)}, \ldots, P^{(k)}\}$ and $V(P^{(i)}) \cap V(P^{(j)}) = \{v_1\}$ for each $i \neq j$. In this section we determine the basis number of the direct product of theta graphs and paths.

Lemma 2.1 *([7]) Let P be a path of order greater than or equal to 5 and T be a tree contains a subgraph isomorphic to a 3-special star of order 7. Then $T \wedge P$ is a non-planar graph.*

Let $\mathcal{B}$ be a 2-fold basis of $\theta \wedge P$. Then, the girth of $\theta \wedge P \geq 4$, so

$$\begin{aligned} \frac{2(|E(\theta \wedge P)|)}{4} &\geq \dim \mathcal{C}(\theta \wedge P) \\ nm - n + m - 1 &\geq nm + 2m - 2n - s. \end{aligned}$$

Thus

$$n \geq m + (1 - s) \text{ where } s = \begin{cases} 0, & \text{if } \theta \wedge P \text{ is disconnected }, \\ 1, & \text{if } \theta \wedge P \text{ is connected.} \end{cases}$$

Proposition 2.1 *Let θ be a theta graph and P be a path of order greater than or equal to 5. Then $b(\theta \wedge P) \geq 3$.*

Proof. We prove the theorem according to the girth of θ. From the last inequality and Lemma 2.1, we can rule out the cases where (i) the girth of $\theta = 3$ and $n = 4$ and (ii) the girth of $\theta \geq 4$ and $n \geq 7$. To this end, we need to consider the following two cases:

Case 1. Girth of $\theta = 3$. Then we may assume $\delta = 3$. Consider the following two sets of vertices: $A = \{(1,2),(1,3),(1,4)\}$ and $B = \{(2,3),(3,3),(n,3)\}$. we shall split this case into two subcases:

Case 1a. n is an odd greater than or equal to 5. Consider the subgraph H_1 of $\theta \wedge P$ whose vertex set $V(H_1) = A \cup B \cup \{(2,4)(3,4),(2,2),(3,1),(4,2),$

$(5,1),\ldots,(n-3,2),(n-2,1),(n-1,2)\}$ and edge set consists of the following nine paths: $P_1=(1,2)(2,3), P_2=(1,2)\ (3,3),\ P_3=(1,2)(n,3), P_4$ $=(1,4)(2,3),\ P_5=(1,4)(3,3),\ P_6=(1,4)(n,3),\ P_7=(1,3)(3,4)(2,3), P_8=$ $(1,3)(2,4)(3,3),\ P_9=(1,3)(2,2)(3,1)(4,2)(5,1)\ldots(n-3,2)(n-2,1)(n-1,2)(n,3)$. Then H_1 is homeomorphic to $K_{3,3}$. Therefore $b(\theta\wedge P)\geq 3$.

Case 1b. n is an even greater than or equal to 6. Consider the subgraph H_2 of $\theta\wedge P$ whose vertex set $V(H_2)=A\cup B\cup\{(2,4),(3,4),(2,2),(1,1),(3,2),$ $(4,1),\ldots,(n-3,2),(n-2,1),(n-1,2)\}$ and edge set consists of the following nine paths: $P_1,P_2,P_3,P_4,P_5,P_6,P_7$ and P_8 are as in Case 1a and $P_9=(1,3)(2,2)(1,1)(3,2)(4,1)\ldots(n-3,2)(n-2,1)\ (n-1,2)(n,3)$. Then H_2 is homeomorphic to $K_{3,3}$. Therefore $b(\theta\wedge P)\geq 3$.

Case 2. Girth of $\theta=4$ and $n=6$. Then $\delta=4$. Consider the subgraph H_3 of $\theta\wedge P$ whose vertex set $V(H_3)=\{(1,2),(1,4),(5,2),(2,3),(4,3),(6,3),(3,2),$ $(4,1)\}$ and edge set consists of the following nine paths: P_1 and P_4 are as in Case 1a and $P_2=(1,2)(4,3),\ P_3=(1,2)(6,3), P_5=(1,4)(4,3), P_6$ $=(1,4)(6,3), P_7=(5,2)(4,3), P_8=(5,2)(6,3), P_9=(5,2)(4,1)(3,2)(2,3)$. Then H_3 is homeomorphic to $K_{3,3}$. Therefore $b(\theta\wedge P)\geq 3$.

Theorem 2.1 *For any graph θ of order ≥ 4 and path P of order ≥ 2, we have $b(\theta\wedge P)\leq 3$. Moreover, the equality holds if the order of P is greater than or equal to 5.*

Proof. To prove the theorem, it is sufficient to exhibit a 3-fold basis $\mathcal{B}$. We now consider the following two cases.

Case 1. $\theta\wedge P$ is connected. It follows that at least one of n and δ is odd. We now split this case into two subcases:

Case 1a. n is even and δ is odd. Then for each $j=1,2,\ldots,m-2$ consider the following sets of cycles:

$A_1^{(j)}=\{(i,j)(i+1,j+1)(i,j+2)(i-1,j+1)(i,j)|i=2,3,\ldots,n-1\}\cup$ $\{(1,j)\ (2,j+1)(1,j+2)(n,j+1)(1,j)\cup\{(n,j)(n-1,j+1)(n,j+2)(1,j+1)$ $(n,j)\}$,

$A_2^{(j)}=\{(1,j)(2,j+1)(1,j+2)(\delta,j+1)(1,j)\}$,

$A_3^{(j)}=\{(\delta,j)(\delta-1,j+1)(\delta,j+2)(1,j+1)(\delta,j)\}$.

In addition, the following cycles:

$c_1=(1,1)(2,2)(3,1)\ldots(n,2)(1,1)$,

$c_2=(1,2)(2,1)(3,2)\ldots(n,1)(1,2)$

$c_3=(1,m)(2,m-1)(3,m)\ldots(\delta,m)(1,m-1)(2,m)\ldots(\delta,m-1)(1,m)$.

Note that, the cycles of $A_1^{(j)}$ are edge pairwise disjoint for each $j=1,2,\ldots,$ $m-2$. Thus, $A_1^{(j)}$ is linearly independent and of 1-fold. Let $A_1=\bigcup_{j=1}^{(m-2)}$ $A_1^{(j)}$. Note that, $A_1^{(i)}\cap A_1^{(j)}=\phi$ if $i\neq j$ and $E(A_1^{(i)})\cap E(A_1^{(j)})=\phi$ if $|i-j|>$ 1. Also, each cycle of $A_1^{(j)}$ contains an edge of the form $(i+1,j+1)(i,j+2)$ or $(n-1,j+1)(n,j+2)$ which is not in $A_1^{(j-1)}$. In addition, each cycle of

$A_1^{(j-1)}$ contains an edge of the form $(i,j-1)(i+1,j)$ or $(n,j)(n-1,j+1)$ which is not in $A_1^{(j)}$. Therefore, A_1 is linearly independent. Let $V_1' = \{(i,j)|i+j = \text{ even}\}$, and $V_2' = \{(i,j)|i+j = \text{ odd}\}$. Let H_i be the induced subgraph of V_i' where $i = 1,2$. For each $j = 1,2\cdots,m-2$, set.

$B_1^{(j)} = \{(i,j)(i+1,j+1)(i,j+2)(i-1,j+1)(i,j)|i = 2,3,\ldots,n-1$ and $i+j = \text{even}\} \bigcup\{(1,j)(2,j+1)(1,j+2)(n,j+1)(1,j)|1+j = \text{even}\}$ $\bigcup\{(n,j)(n-1,j+1)(n,j+2)(1,j+1)(n,j)|n+j = \text{even}\}$,

$B_2^{(j)} = \{(i,j)(i+1,j+1)(i,j+2)(i-1,j+1)(i,j)|i = 2,3,\ldots,n-1$ and $i+j = \text{odd}\}\bigcup\{(1,j)(2,j+1)(1,j+2)(n,j+1)(1,j)|1+j = \text{odd}\}$ $\bigcup\{(n,j)(n-1,j+1)(n,j+2)(1,j+1)(n,j)|n+j = \text{odd}\}$.

Let $F^{(i)} = \bigcup_{j=1}^{m-2} B_i^{(j)}$ where $i = 1,2$. We now prove that c_i is independent from the cycles of $F^{(i)}$. Let $E_j^{(i)} = E(C\wedge j(j+1))\cap E(H_i)$ where C is the cycle obtained by deleting the edge 1δ from θ. Then it is an easy matter to verify that $\left\{E_1^{(i)}, E_2^{(i)},\ldots,E_{m-1}^{(i)}\right\}$ is a partition of $E(C\wedge P)\cap E(H_i)$. Moreover, it is clear that $E_1^{(i)} = E(c_i)$ and $E_1^{(i)}\cup E_2^{(i)} = E(B_i^{(1)})$. Thus, if c_i is a sum modulo 2 of some cycles of $F^{(i)}$, say $\{k_1,k_2,\ldots,k_r\}$, then $B_i^{(1)} \subset \{k_1,k_2,\ldots,k_r\}$. Since no edge of $E_2^{(i)}$ belong to $E(c_i)$ and $E_2^{(i)}\cup E_3^{(i)} = E(B_i^{(2)})$, $B_i^{(2)} \subset \{k_1,k_2,\ldots,k_r\}$. By continuing in this way, it implies that $B_i^{(m-2)} \subset \{k_1,k_2,\ldots,k_r\}$. Note that $E_{m-2}^{(i)}\cup E_{m-1}^{(i)} = E(B_i^{(m-2)})$ and each edge of $E_{m-1}^{(i)}$ appears in one and only one cycle of $F^{(i)}$. It follows that $E_{m-1}^{(i)} \subset E(c_i)$. This is a contradiction. Therefore, $F^{(i)}\cup\{c_i\}$ is linearly independent for $i = 1,2$. And since $E(F^{(1)}\cup\{c_1\})\cap E(F^{(2)}\cup\{c_2\}) = \phi$, $F^{(1)}\cup F^{(2)}\cup\{c_1,c_2\} = A_1\cup\{c_1c_2\}$ is linearly independent. Let $A_2 = \bigcup_{j=1}^{m-2} A_2^{(j)}$ and $A_3 = \bigcup_{j=1}^{m-2} A_3^{(j)}$. It is easy to see that the cycles of A_i are edge pairwise disjoint for $i = 2,3$ and each cycle of A_3 contains at least one edge of the form $(\delta,j)(\delta-1,j+1)$ and $(\delta,j)(\delta-1,j-1)$ which is not in A_2. And so $A_2\cup A_3$ is linearly independent. Clearly, c_3 can not be written as a linear combination of cycles of $A_2\cup A_3$. Therefore, $A_2\cup A_3\cup\{c_3\}$ is linearly independent. Let $\mathcal{B} = A_1\cup A_2\cup A_3\cup\{c_1,c_2,c_3\}$. We now prove that $\mathcal{B}$ is a linearly independent set. Assume not i.e. there are two sets of cycles say $\{d_1,d_2,\ldots,d_{\gamma_1}\}\subset A_1\cup\{c_1,c_2\}$ and $\{f_1,f_2,\ldots,f_{\gamma_2}\}\in A_2\cup A_3\cup\{c_3\}$ such that $\sum_{i=1}^{\gamma_1} d_i = \sum_{i=1}^{\gamma_2} f_i \pmod 2$. Consequently, $E(d_1\oplus d_2\oplus\ldots\oplus d_{\gamma_1}) = E(f_1\oplus f_2\oplus\ldots\oplus f_{\gamma_2})$ where $\oplus$ is a ring sum and so $d_1\oplus d_2\oplus\ldots\oplus d_{\gamma_1}$ contains at least one edge of the form $(1,j)(\delta,j+1)$ and $(1,j+1)(\delta,j)$ for some $j\leq m-1$. Which contradicts the fact that no cycle of $A_1\cup\{c_1,c_2\}$ contains such edges. Now,

$$|\mathcal{B}| = \sum_{i=1}^{3}|A_i| + 3$$

$$
\begin{aligned}
&= n(m-2)+(m-2)+(m-2)+3 \\
&= nm-2n+2m-1 \\
&= \dim \mathcal{C}(\theta \wedge P).
\end{aligned}
$$

Hence, $\mathcal{B}$ is a basis of $\theta \wedge P$. To complete the proof of the theorem, we should show that $\mathcal{B}$ is of 3-fold. Let $e \in E(\theta \wedge P)$. Then (1) if $e=(i,j)(i+1,j+1)$, or $e=(n,j)(1,j+1)$ where $1 \leq i \leq n-1$ and $2 \leq j \leq m-2$, then $f_{A_1}(e)=2, f_{A_2 \cup A_3}(e) \leq 1$ and $f_{\cup_{i=1}^3 \{c_i\}}(e)=0$, and so $f_{\mathcal{B}}(e) \leq 3$. (2) If $e=(i,j)(i+1,j-1)$, or $e=(n,j)(1,j-1)$ where $1 \leq i \leq n-1$ and $3 \leq j \leq m-1$, then $f_{A_1}(e)=2, f_{A_2 \cup A_3}(e) \leq 1$ and $f_{\cup_{i=1}^3 \{c_i\}}(e)=0$, and so $f_{\mathcal{B}}(e) \leq 3$. (3) If $e=(i,1)(i+1,2)$, or $e=(1,1)(n,2)$ where $1 \leq i \leq n-1$, then $f_{A_1}(e)=1, f_{A_2 \cup A_3}(e) \leq 1$ and $f_{\cup_{i=1}^3 \{c_i\}}(e)=1$, and so $f_{\mathcal{B}}(e) \leq 3$. (4) If $e=(i,2)(i+1,1)$, or $e=(1,2)(n,1)$ where $1 \leq i \leq n-1$, then $f_{A_1}(e)=1, f_{A_2 \cup A_3}(e) \leq 1$ and $f_{\cup_{i=1}^3 \{c_i\}}(e)=1$, and so $f_{\mathcal{B}}(e) \leq 3$. (5) If $e=(1,j)(\delta,j+1)$, where $1 \leq j \leq m-2$, then $f_{A_1}(e)=0, f_{A_2 \cup A_3}(e) \leq 2$ and $f_{\cup_{i=1}^3 \{c_i\}}(e)=0$, and so $f_{\mathcal{B}}(e) \leq 2$. (6) If $e=(1,j)(\delta,j-1)$, where $2 \leq j \leq m-1$, then $f_{A_1}(e)=0, f_{A_2 \cup A_3}(e) \leq 2$ and $f_{\cup_{i=1}^3 \{c_i\}}(e)=0$, and so $f_{\mathcal{B}}(e) \leq 2$. (7) If $e=(i,m-1)(i+1,m)$, or $e=(i,m)(i+1,m-1)$, or $e=(1,m)(n,m-1)$ where $1 \leq i \leq n-1$, then $f_{A_1}(e)=1, f_{A_2 \cup A_3}(e) \leq 1$ and $f_{\cup_{i=1}^3 \{c_i\}}(e) \leq 1$, and so $f_{\mathcal{B}}(e) \leq 3$. (8) If $e=(1,m)(\delta,\, m-1)$, or $e=(1,m-1)(\delta,m)$, then $f_{A_1}(e)=0, f_{A_1 \cup A_2}(e) \leq 1$ and $f_{\cup_{i=1}^3 \{c_i\}}(e) \leq 1$, and so $f_{\mathcal{B}}(e) \leq 2$. Therefore $\mathcal{B}$ is a 3-fold basis.

Case 1b. n is odd. Then we may assume δ is even. Now, consider the following sets of cycles: A_1, A_2, and A_3 are as in Case 1a and

$$
\begin{aligned}
c_1 &= (1,m)(2,m-1)(3,m)\ldots(\delta,m-1)(1,m), \\
c_2 &= (1,m-1)(2,m)(3,m-1)\ldots(\delta,m)(1,m-1), \\
c_3 &= (1,1)(2,2)(3,1)\ldots(n,1)(1,2)(2,1)\ldots(n,2)(1,1).
\end{aligned}
$$

Let $\mathcal{B}=(\bigcup_{i=1}^3 A_i)\bigcup(\bigcup_{i=1}^3 \{c_i\})$. Since $E(c_1)\cap E(c_2)=\phi$, $\{c_1,c_2\}$ is linearly independent. Since $\delta \geq 4$ and so c_1 contains the edge of the form $(2,m-1)(3,m)$ and c_2 contains $(2,m)(3,m-1)$ and each of these two edges is not in any cycle of $A_2 \cup A_3$. Thus $A_2 \cup A_3 \cup \{c_1,c_2\}$ is linearly independent. Next, we show that $A_1 \cup \{c_3\}$ is linearly independent. Let $R_i=E(C \wedge i(i+1))$ where C is as in Cas 1a. Note that $\{R_1,R_2,\ldots,R_{m-1}\}$ is a partition of $E(C \wedge P)$. Also, $E(c_3)=R_1$ and $R_1 \cup R_2=E(A_1^{(1)})$. Thus, if $\{c_3\}$ can be written as a linear combination of some cycles of A_1, say $\{k_1,k_2,\ldots,k_r\}$, then $A_1^{(1)} \subset \{k_1,k_2,\ldots,k_r\}$. Since $R_2 \cup R_3=E(A_1^{(2)})$ and no edge of R_2 belong to $E(c_3)$, $A_1^{(2)} \subset \{k_1,k_2,\ldots,k_r\}$, and so on. This implies that $A_1^{(m-2)} \subset \{k_1,k_2,\ldots,k_r\}$. Note that $R_{m-1} \subset E(A_1^{(m-2)})$ and each edge of R_{m-1} appears only in one cycle of A_1. Thus $R_{m-1} \subset E(c_3)$, which is a

contradiction. Hence $A_1 \cup \{c_3\}$ is linearly independent. Now, by the same argument as in Case 1a we can prove that $\mathcal{B}$ is linearly independent.
Case 2. $\theta \wedge P$ is disconnected. Then by Theorem 1.3 $\theta \wedge P$ consists of two components and both of n and δ are even. Consider the following sets of cycles: A_1, A_2, A_3, c_1, and c_2 as in Case 1b, and $D_1 = c_1$ and $D_2 = c_2$ where c_1, c_2 are as in Case 1a. In a way similar to the ways in Case 1a and Case 1b we can prove that $\mathcal{B} = (\bigcup_{i=1}^{3} A_i) \bigcup \{c_1, c_2\} \bigcup \{D_1, D_2\}$ is a linearly independent set of 3-fold. Since

$$\begin{aligned} |\mathcal{B}| &= \sum_{i=1}^{3} |A_i| + 2 + 2 \\ &= nm - 2n + 2m \\ &= \dim \mathcal{C}(\theta \wedge P), \end{aligned}$$

$\mathcal{B}$ is a 3-fold basis.
ACKNOWLEDGMENT. The publication of this paper was supported by Yarmouk University Research Council.

References

[1] A.A. Ali, *The basis number of the direct products of paths and cycles*, Ars Combin. 27 (1989), 155-163.

[3] A.A. Al-Rhayyel and M.M.M. Jaradat, *On the basis number of the direct product of some graphs*, Bull. Cal. Math. Soc. 88 (1996), no. 6, 509-516.

[4] J.A. Banks and E.F. Schmeichel, *The basis number of n-cube*, J. Combin. Theory Ser. B 33 (1982), no. 2, 95-100.

[5] J.A. Bondy and U.S. Murty, *"Graph theory with applications"*, American Elsevier Publishing Co. Inc., New York, 1976.

[6] F. Harary, *"Graph theory"*, Addison-Wesley Publishing Co., Reading, Mass.- Menlo Park, Calif.-London 1969.

[7] M.M.M. Jaradat, *On the basis number of the direct product of graphs,* Australas. J. Combin. 27, 293-306 (2003)..

[8] S. MacLane, *A combinatorial condition for planar graphs,* Fundamenta Math. 28 (1937), 22-32.

[9] E.F. Schmeichel, *The basis number of a graph*, J. Combin. Theory Ser. B 30 (1981), no. 2, 123-129.