On the Hull Number of the Composition of Graphs

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ABSTRACT

The convex hull of a subset A of $V(G)$, where G is a connected graph, is defined as the smallest convex set in G containing A. The hull number of G is the cardinality of a smallest set A whose convex hull is $V(G)$. In this paper, we give the hull number of the composition of two connected graphs.

1 Introduction

Given a connected graph G, the couple $(V(G), d_G)$, where $V(G)$ is the vertex set of G and $d_G(u, v)$ is the length of a shortest path connecting vertices u and v in G, is a metric space on $V(G)$. Any $u - v$ path of length $d_G(u, v)$ is called a *$u - v$ geodesic*. A subset C of $V(G)$ is *convex* if for every two vertices $u, v \in C$, the vertex set of every $u - v$ geodesic is contained in C.

If u and v are vertices of a graph G, then the set $I_G[u, v]$ is the closed interval consisting of u, v and all vertices lying on a $u - v$ geodesic of G. If $S \subseteq V(G)$, then

$$I_G[S] = \bigcup_{u,v \in S} I_G[u, v].$$

When no confusion arises, we simply refer $I_G[S]$ as $I[S]$. A set S is *convex* in G if $I[S] = S$. The *convex hull* $[S]$ *of* S is the smallest convex set containing S.

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It can be formed from the sequence $\{I^p[S]\}$, where p is a nonnegative integer, $I^0[S] = S$, $I^1[S] = I[S]$, and $I^p[S] = I[I^{p-1}[S]]$ for $p \geq 2$. For some p, we must have $I^q[S] = I^p[S]$ for all $q \geq p$. Further, if p is the smallest nonnegative integer such that $I^q[S] = I^p[S]$ for all $q \geq p$, then $I^p[S] = [S]$.

A set S of vertices of G is called a *hull set* in G if $[S] = V(G)$, and a hull set of minimum cardinality is a *minimum hull set* in G. The cardinality of a minimum hull set in G is called the *hull number* $h(G)$ of G. These concepts were introduced by Everett and Siedman [4] and investigated further in [3]. For other graph theoretic terms which are assumed here, readers are advised to refer to [5].

2 Results

We first state a simple lemma.

Lemma 2.1 *Let G be a nontrivial connected graph. If A and B are nonempty subsets of $V(G)$ with $A \subseteq B$, then $I^p[A] \subseteq I^p[B]$ for all natural numbers p.*

The *composition* of two graphs G and H, denoted by $G[H]$, is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u,v)(u',v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

Lemma 2.2 *Let G and H be nontrivial connected graphs and $u, v \in V(H)$ with $d_H(u,v) = 2$. Let $x \in V(G)$ and $A_x = \{(x,u),(x,v)\} \subseteq V(G[H])$. If $y \in N(x)$ ($N(x)$ is the set of neighbors of x in G), then $(y,w) \in I_{G[H]}[A_x]$ for all $w \in V(H)$.*

Proof. Let $u, v \in V(H)$ with $d_H(u,v) = 2$, and $A_x = \{(x,u),(x,v)\} \subseteq V(G[H])$ for $x \in V(G)$. If $y \in N(x)$ and $w \in V(H)$, then $[(x,u),(y,w),(x,v)]$ is an $(x,u)-(x,v)$ geodesic in $G[H]$. Thus, $(y,w) \in I_{G[H]}[(x,u),(x,v)]$. Therefore, $(y,w) \in I_{G[H]}[A_x]$ for all $w \in V(H)$. ■

Lemma 2.3 *Let G and H be nontrivial connected graphs and $u, v \in V(H)$ with $d_H(u,v) = 2$. Let $x \in V(G)$ and $A_x = \{(x,u),(x,v)\} \subseteq V(G[H])$. If $N(x) \neq \emptyset$, then $(x,w) \in I^2_{G[H]}[A_x]$ for all $w \in V(H)$.*

Proof. Let $u, v \in V(H)$ with $d_H(u,v) = 2$, and $A_x = \{(x,u),(x,v)\} \subseteq V(G[H])$ for $x \in V(G)$. Obviously, $(x,u),(x,v) \in I^2_{G[H]}[A_x]$.

Let $y \in N(x)$. By Lemma 2.2, $(y,w) \in I_{G[H]}[A_x]$ for all $w \in V(H)$. Let $B = \{(y,u),(y,v)\}$. Then $B \subseteq I_{G[H]}[A_x]$. Thus, by Lemma 2.1, $I_{G[H]}[B] \subseteq I_{G[H]}[I_{G[H]}[A_x]] = I^2_{G[H]}[A_x]$. Since $x \in N(y)$, $(x,w) \in I_{G[H]}[B]$ for all $w \in V(H)$ by Lemma 2.2. Therefore, $(x,w) \in I^2_{G[H]}[A_x]$ for every $w \in V(H)$. ■

Lemma 2.4 *Let G and H be nontrivial connected graphs, $u, v \in V(H)$ with $d_H(u, v) = 2$ and $x \in V(G)$. Let $A_x = \{(x, u), (x, v)\}$. For each $r \geq 1$, if $y \in V(G)$ and $d_G(x, y) = r$, then $(y, w) \in I^r_{G[H]}[A_x]$ for all $w \in V(H)$.*

Proof. Let $u, v \in V(H)$ with $d_H(u, v) = 2$, and $A_x = \{(x, u), (x, v)\} \subseteq V(G[H])$ for $x \in V(G)$. By Lemma 2.2, the assertion holds for $r = 1$.

Suppose the assertion holds for $r = k$ $(k \geq 2)$, that is, if $y \in V(G)$ and $d_G(x, y) = k$, then $(y, w) \in I^k_{G[H]}[A_x]$ for all $w \in V(H)$. Suppose $y \in V(G)$ and $d_G(x, y) = k + 1$. Consider an $x - y$ geodesic $P_{k+1} = [x, x_1, x_2, \ldots, x_k, y]$. Since $d_G(x, x_k) = k$, $(x_k, w) \in I^k_{G[H]}[A_x]$ for all $w \in V(H)$ by the inductive hypothesis. Let $B_k = \{(x_k, u), (x_k, v)\}$. Then $B_k \subseteq I^k_{G[H]}[A_x]$ and so $I_{G[H]}[B_k] \subseteq I^{k+1}_{G[H]}[A_x]$ by Lemma 2.1. Since $y \in N(x_k)$, we have $(y, w) \in I_{G[H]}[B_k]$ for all $w \in V(H)$ by Lemma 2.2. Thus, $(y, w) \in I^{k+1}_{G[H]}[A_x]$ for all $w \in V(H)$. This completes the proof of the lemma. ■

Theorem 2.5 *Let G and H be connected graphs. If H is non-complete, then*

$$h(G[H]) = \begin{cases} h(H) & \text{if } G = K_1, \\ 2 & \text{if } G \neq K_1. \end{cases}$$

Proof. Let G and H be connected graphs, where H is non-complete. If $G = K_1$, then $G[H] \cong H$. This implies that, $h(G[H]) = h(H)$.

Suppose $G \neq K_1$. Since H is non-complete, there exist $u, v \in V(H)$ such that $d_H(u, v) = 2$. Choose $x \in V(G)$ and let $A_x = \{(x, u), (x, v)\}$. Since $N(x) \neq \emptyset$, $(x, w) \in I^2_{G[H]}[A_x] \subseteq [A_x]$ for all $w \in V(H)$ by Lemma 2.3. Now, let $y \in V(G)\backslash\{x\}$ and let $d_G(x, y) = r$ $(r \geq 1)$. By Lemma 2.4, $(y, w) \in I^r_{G[H]}[A_x] \subseteq [A_x]$ for all $w \in V(H)$. Since $[A_x] \subseteq V(G[H])$, it follows that $[A_x] = V(G[H])$. Therefore, $h(G[H]) = 2$. ■

The following remark is a direct consequence of Theorem 2.5.

Remark 2.6 *Let n and m be positive integers. Then*

1. $h(P_n[P_m]) = 2$ *for* $n, m \geq 3$;
2. $h(P_n[C_m]) = h(C_m[P_n]) = 2$ *for* $n \geq 3$ *and* $m \geq 4$;
3. $h(C_n[C_m]) = 2$ *for* $n, m \geq 4$;
4. $h(K_n[P_m]) = 2$ *for* $n \geq 2$ *and* $m \geq 3$;
5. $h(W_n[P_m]) = h(P_m[W_n]) = 2$ *for* $n \geq 4$ *and* $m \geq 3$;
6. $h(K_n[C_m]) = 2$ *for* $n \geq 2$ *and* $m \geq 4$;
7. $h(C_n[W_m]) = h(W_m[C_n]) = 2$ *for* $n, m \geq 4$;

8. $h(K_n[W_m]) = 2$ *for* $n \geq 2$ *and* $m \geq 4$ *and*

9. $h(W_n[W_m]) = 2$ *for* $n, m \geq 4$.

A vertex v in a connected graph G is an *extreme vertex* if the subgraph induced by $N(v)$ (the set of neighbors of v) is complete.

Theorem 2.7 [3] *If v is a vertex of a graph G such that $\langle N(v) \rangle$ is complete (that is, v is an extreme vertex in G), then v belongs to every hull set and every geodesic set of G.*

Lemma 2.8 [2] *Let G be a connected graph. A vertex x is an extreme vertex in G if and only if (x, a) is an extreme vertex in $G[K_m]$ for all $a \in V(K_m)$.*

Lemma 2.9 *Let G be a connected graph, $T \subseteq V(G[K_m])$ and $T_f = \{x \in V(G) : (x, v) \in T$ for some $v \in V(K_m)\}$. For every n, the following holds: If $u \in I_G^n[T_f] \backslash T_f$, then $(u, v) \in I_{G[K_m]}^n[T]$ for all $v \in V(K_m)$.*

Proof. Let $T \subseteq V(G[K_m])$ and $T_f = \{x \in V(G) : (x, v) \in T$ for some $v \in V(K_m)\}$. Suppose $v \in V(K_m)$. If $u \in I_G[T_f] \backslash T_f$, then there exist $s, t \in T_f$ such that $u \in I_G[s, t]$. Let $P = [s, u_1, u_2, \ldots, u_r, t]$, where $u = u_k$ $(1 \leq k \leq r)$, be an $s - t$ geodesic in G. Let $v_o, v' \in V(K_m)$ such that $(s, v_o), (t, v') \in T$. Then $P^* = [(s, v_o), (u_1, v), (u_2, v), \ldots, (u_r, v), (t, v')]$ is an $(s, v_o) - (t, v')$ geodesic in $G[K_m]$. Since $(s, v_o), (t, v') \in T$ and $(u, v) \in I_{G[K_m]}[(s, v_o), (t, v')]$, it follows that $(u, v) \in I_{G[K_m]}[T]$.

Suppose the assertion holds for $n = k > 1$, that is, if $u \in I_G^k[T_f] \backslash T_f$, then $(u, v) \in I_{G[K_m]}^k[T]$. Let $u \in I_G^{k+1}[T_f] \backslash T_f$. Then there exist $p, q \in I_G^k[T_f]$ such that $u \in I_G[p, q]$. We consider three cases.

Case 1. Both p and q belong to T_f. As proved earlier, we have $(u, v) \in I_{G[K_m]}[T] \subseteq I_{G[K_m]}^{k+1}[T]$ for all $v \in V(K_m)$.

Case 2. Exactly one of p and q belongs to T_f, say $p \in T_f$ and $q \notin T_f$. Then $(p, v_o) \in T \subseteq I_{G[K_m]}^k[T]$ for some $v_o \in V(K_m)$ and $q \in I_G^k[T_f] \backslash T_f$. Thus, $(p, v_o) \in I_{G[K_m]}^k[T]$ and $(q, v) \in I_{G[K_m]}^k[T]$ by the inductive hypothesis. Since $(u, v) \in I_{G[K_m]}[(p, v_o), (q, v)]$, $(u, v) \in I_{G[K_m]}^{k+1}[T]$.

Case 3. Both p and q do not belong to T_f. Then $p, q \in I_G^k[T_f] \backslash T_f$. By the inductive hypothesis, $(p, v), (q, v) \in I_{G[K_m]}^k[T]$. Since $(u, v) \in I_{G[K_m]}[(p, v), (q, v)]$, $(u, v) \in I_{G[K_m]}^{k+1}[T]$. Therefore, the assertion of the lemma holds. ■

In what follows, A_e and D_e denote the sets of extreme vertices in $V(G)$ and $V(G[K_m])$, respectively.

Lemma 2.10 *Let G be a connected graph and $T \subseteq V(G[K_m])$ such that $T_f = \{x \in V(G) : (x, v) \in T$ for some $v \in V(K_m)\}$ is a hull set in G. If $x \in T_f \backslash A_e$, then $(x, v) \in [T]$ for all $v \in V(K_m)$.*

Case 2. Suppose, without loss of generality, that $(p,a) \in T$ and $(q,c) \notin T$. Then $p \in T_f \subseteq I_G^k[T_f]$ and $q \in I_G^k[T_f]$ by the inductive hypothesis. If $p = q$, then $d_{G[K_m]}((p,a),(q,c)) = 1$. Hence, $(u,b) = (q,c)$. Thus, $u = q \in I_G^k[T_f] \subseteq I_G^{k+1}[T_f]$. If $p \neq q$, then $u \in I_G[p,q]$. Therefore, $u \in I_G^{k+1}[T_f]$.

Case 3. Suppose $(p,a),(q,c) \notin T$. Then $p,q \in I_G^k[T_f]$ by the inductive hypothesis. Thus, $u \in I_G^{k+1}[T_f]$. Therefore, the assertion holds. ■

Lemma 2.13 *Let G be a connected graph, $T \subseteq V(G[K_m])$ and $T_f = \{x \in V(G) : (x,a) \in T$ for some $a \in V(K_m)\}$. If T is a hull set in $G[K_m]$, then T_f is a hull set in G.*

Proof. Let T be a hull set in $G[K_m]$, $[T] = I_{G[K_m]}^n[T] = V(G[K_m])$ and $u \in V(G)$. Choose $b \in V(K_m)$. If $(u,b) \in T$, then $u \in T_f \subseteq [T_f]$. If $(u,b) \notin T$, then $(u,b) \in I_{G[K_m]}^n[T] \backslash T$. Hence $u \in I_G^n[T_f] \subseteq [T_f]$ by Lemma 2.12. Therefore, T_f is a hull set in G. ■

Theorem 2.14 *Let G be a connected graph. If T is a hull set in $G[K_m]$, then $T = C \cup D_e$, where $C \cap D_e = \emptyset$, and $T_f = \{x \in V(G) : (x,v) \in T$ for some $v \in V(K_m)\}$ is a hull set in G.*

Proof. Suppose T is a hull set in $G[K_m]$. Then $T = C \cup D_e$ by Theorem 2.7, where $C \cap D_e = \emptyset$. Furthermore, $T_f = \{x \in V(G) : (x,v) \in T$ for some $v \in V(K_m)\}$ is a hull set in G by Lemma 2.13. ■

Corollary 2.15 *Let G be a connected graph. If A is a minimum hull set in G, then $T = [(A\backslash A_e) \times \{v_o\}] \cup D_e$ is a minimum hull set in $G[K_m]$ for every $v_o \in V(K_m)$.*

Proof. Let A be a minimum hull set in G, $v_o \in V(K_m)$ and $T = [(A\backslash A_e) \times \{v_o\}] \cup D_e$. By Theorem 2.11, T is a hull set in $G[K_m]$. Suppose T is not a minimum hull set. Let T' be a minimum hull set in $G[K_m]$. Then $T' = C \cup D_e$, where $C \cap D_e = \emptyset$, and $T'_f = \{x : (x,u) \in T'$ for some $u \in V(K_m)\}$ is a hull set in G by Theorem 2.14. Further, $|T'| < |T|$. By Theorem 2.11, $T^* = [(T'_f \backslash A_e) \times \{v_o\}] \cup D_e$ is a hull set in $G[K_m]$ and $|T^*| = |T'_f| + (m-1)|A_e| \leq |T'|$. Since T' is minimum, $|T'| = |T'_f| + (m-1)|A_e| < |T| = |A| + (m-1)|A_e|$. Thus, $|T'_f| < |A|$, a contradiction. Therefore, T is a minimum hull set in $G[K_m]$. ■

Theorem 2.16 *Let G be a connected graph and K_m the complete graph of order m. Then $h(G[K_m]) = h(G) + (m-1)|A_e|$.*

Proof. Let A be a minimum hull set in G and $v_o \in V(K_m)$. By Corollary 2.15, $T = [(A\backslash A_e) \times \{v_o\}] \cup D_e$ is a minimum hull set in $G[K_m]$. Since T is minimum and $|T| = |((A\backslash A_e) \times \{v_o\}) \cup D_e| = h(G) + (m-1)|A_e|$, we obtain the desired result. ■

The next results follow directly from Theorem 2.16.

Proof. Let $x \in T_f \backslash A_e$ and $[T_f] = I_G^p[T_f] = V(G)$ for some nonnegative integer p. Then there exist $y, z \in N(x)$ such that $d_G(y, z) = 2$. Hence, $d_{G[K_m]}((y, u), (z, w)) = 2$ and $(y, u), (z, w) \in N((x, v))$ for any $u, w, v \in V(K_m)$. Thus, $(x, v) \in I_{G[K_m]}[(y, u), (z, w)]$ for all $u, w, v \in V(K_m)$. Consider the following cases:

Case 1. Suppose $y, z \in T_f$. Then for some $u_o, w_o \in V(K_m)$, we have $(y, u_o), (z, w_o) \in T$. Since $(x, v) \in I_{G[K_m]}[(y, u_o), (z, w_o)]$ for all $v \in V(K_m)$, it follows that $(x, v) \in I_{G[K_m]}[T] \subseteq [T]$ for all $v \in V(K_m)$.

Case 2. Suppose, without loss of generality, that $y \in T_f$ and $z \notin T_f$. Then $(y, v_o) \in T \subseteq I_{G[K_m]}^p[T]$ for some $v_o \in V(K_m)$ and $z \in I_G^p[T_f] \backslash T_f$. Thus, $(y, v_o) \in I_{G[K_m]}^p[T]$ and $(z, v_o) \in I_{G[K_m]}^p[T]$ by Lemma 2.9. Hence, $(x, v) \in I_{G[K_m]}^{p+1}[T] = [T]$ for all $v \in V(K_m)$.

Case 3. Suppose $y, z \notin T_f$. Since $y, z \in I_G^p[T_f] \backslash T_f$, $(y, v), (z, v) \in I_{G[K_m]}^p[T]$ for all $v \in V(K_m)$ by Lemma 2.9. Since $(x, v) \in I_{G[K_m]}[(y, v), (z, v)]$, it follows that $(x, v) \in I_{G[K_m]}^{p+1}[T] = [T]$ for all $v \in V(K_m)$.

Therefore, in any case, $(x, v) \in [T]$ for all $v \in V(K_m)$. ■

Theorem 2.11 *Let G be a connected graph. If A is a hull set in G, then $T = C \cup D_e$, where $C \subseteq V(G[K_m])$ and $A \subseteq T_f = \{x \in V(G) : (x, v) \in T$ for some $v \in V(K_m)\}$, is a hull set in $G[K_m]$.*

Proof. Suppose A is a hull set in G and $T = C \cup D_e$, where $A \subseteq T_f = \{x \in V(G) : (x, v) \in T$ for some $v \in V(K_m)\}$. Then T_f is a hull set in G and $A_e \subseteq T_f$ by Theorem 2.7. Let $(x, v) \in V(G[K_m])$. If $x \in A_e$, then $(x, v) \in D_e \subseteq T \subseteq [T]$ by Lemma 2.8. If $x \in T_f \backslash A_e$, then $(x, v) \in [T]$ by Lemma 2.10. Finally, if $x \notin T_f$, then $x \in [T_f] \backslash T_f = I_G^q[T_f] \backslash T_f$ for some positive integer q. This implies that $(x, v) \in I_{G[K_m]}^q[T] \subseteq [T]$ by Lemma 2.9. Therefore, $[T] = V(G[K_m])$. ■

Lemma 2.12 *Let G be a connected graph, $T \subseteq V(G[K_m])$ and $T_f = \{x \in V(G) : (x, a) \in T$ for some $a \in V(K_m)\}$. For every n, the following holds: If $(u, b) \in I_{G[K_m]}^n[T] \backslash T$, then $u \in I_G^n[T_f]$.*

Proof. If $(u, b) \in I_{G[K_m]}[T] \backslash T$, then there exist $(v, a), (w, c) \in T$ such that $(u, b) \in I_{G[K_m]}[(v, a), (w, c)]$. Since $(u, b) \notin T$, $(u, b) \neq (v, a)$ and $(u, b) \neq (w, c)$. Hence, $d_{G[K_m]}((v, a), (w, c)) \neq 1$. This implies that v and w are distinct elements of T_f and $u \in I_G[v, w]$. Thus, $u \in I_G[T_f]$ and the lemma holds for $n = 1$.

Suppose the assertion holds for $n = k > 1$, that is, if $(u, b) \in I_{G[K_m]}^k[T] \backslash T$, then $u \in I_G^k[T_f]$. Let $(u, b) \in I_{G[K_m]}^{k+1}[T] \backslash T$. Then there exist $(p, a), (q, c) \in I_{G[K_m]}^k[T]$ such that $(u, b) \in I_{G[K_m]}[(p, a), (q, c)]$. Consider the following cases:

Case 1. Suppose $(p, a), (q, c) \in T$. Then, as in the above argument, $u \in I_G[p, q]$. Hence, $u \in I_G^{k+1}[T_f]$.

Corollary 2.17 *Let G be a connected graph of order $n \geq 4$. If G has no extreme vertex, then $h(G[K_m]) = h(G)$.*

The following examples follow from Corollary 2.17.

Example 2.18 *Let n and m be positive integers. Then*

1. *$h(C_n[K_m]) = h(C_n)$ for $n \geq 4$ and $m \geq 2$ and*

2. *$h(W_n[K_m]) = h(W_n)$ for $n \geq 4$ and $m \geq 2$.*

Corollary 2.19 *Let G be a connected graph of order n. If A_e is a hull set in G, then $h(G[K_m]) = m|A_e|$.*

The following examples follow from Corollary 2.19.

Example 2.20 *Let n and m be positive integers. Then*

1. *$h(K_n[K_m]) = mn$ for $n, m \geq 2$ and*

2. *$h(P_n[K_m]) = mh(P_n) = 2m$ for $n \geq 3$ and $m \geq 2$.*

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