The non planar vertex deletion of $C_n \times C_m$

C. F. X. de Mendonça[1] E. F. Xavier[2] J. Stolfi[2]
L. Faria[3] C. M. H. de Figueiredo[4]

Abstract

The non planar vertex deletion or vertex deletion $vd(G)$ of a graph $G = (V, E)$ is the smallest non negative integer k, such that the removal of k vertices from G produces a planar graph. Hence, the maximum planar induced subgraph of G has precisely $|V| - vd(G)$ vertices. The problem of computing vertex deletion is in general very hard, it is NP-complete. In this paper we compute the non planar vertex deletion for the family of toroidal graphs $C_n \times C_m$.

Keywords: non planar vertex deletion, maximum planar induced subgraph, non planar edge deletion, 4-regular graphs, planarity invariants, NP-complete, product of cycles.

AMS Subject Classification: 05C10, 57M15, 05C62

1 Introduction

Measures for non planarity have an important place in the study of planar graphs due to many industrial and combinatorial applications which involve planarity concepts. There are several important measures for the non planarity of a graph, for instance, the minimum number of crossings in an embedding in the plane, the genus, the minimum number of edges whose removal defines a planar graph, the minimum number of planar subgraphs disjoint in edges whose sets of edges partition the set of edges of the graph. These measures have applications to VLSI circuits with respect to minimum area, computation speed and number of layers [16, 19].

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[1]Departamento de Informática, UEM, Maringá, PR, Brazil. xavier@din.uem.br
[2]Instituto de Computação, Unicamp, Campinas, SP, Brazil. stolfi@ic.unicamp.br
[3]Departamento de Matemática, FFP-UERJ, São Gonçalo, RJ, Brazil. luerbio@cos.ufrj.br
[4]Instituto de Matemática and COPPE Sistemas e Computação, UFRJ, Rio de Janeiro, RJ, Brazil. celina@cos.ufrj.br

The corresponding decision problems for most of these interesting invariants are known to be NP-complete [8, 12, 13, 26, 27]. Approximation methods like Polynomial Time Approximation Schemes in some cases are not likely to exist [5, 7, 9, 20].

Graphs $C_n \times C_m$ have been much studied. These graphs can be drawn as regular latitude-longitude grids on the torus, and thus are also known as "toroidal rectangular grids" or similar names. They occur often as interconnection diagrams of multiprocessor computers and cellular automata [14, 16, 19], and so results about the planarity properties of the $C_n \times C_m$ graphs are relevant to the physical design of such machines.

This established difficult for computing planar invariants and this interest in the study of $C_n \times C_m$ graphs have justified many publications where a non planar invariant is computed for just one graph [1, 2, 6, 10, 23] or for a subclass [3, 17, 21, 22, 24, 25] of $C_n \times C_m$.

The *non planar vertex deletion* or *vertex deletion* $vd(G)$ of a graph G is the smallest non negative integer k such that the removal of k vertices from G produces a planar graph.

With respect to special classes of graphs, vertex deletion is known for complete graphs and complete bipartite graphs: $vd(K_n) = n - 4$ if $n > 4$ and 0 otherwise; and $vd(K_{n,m}) = \min\{n, m\} - 2$ if $\min\{n, m\} > 2$ and 0 otherwise.

The VERTEX DELETION decision problem (VD) consists in given a graph G and a positive integer k decide whether $vd(G) \leq k$. In 1978, Yannakakis [27] proved that this problem is NP-complete. More recently, Lund and Yannakakis [20] gave a proof that the optimization version of VERTEX DELETION for graphs in general is a Max SNP-hard problem.

In this paper, we determine the exact values of the vertex deletion for all $C_n \times C_m$ graphs. We give a proof that $vd(C_3 \times C_3) = 1$; $vd(C_3 \times C_4) = vd(C_3 \times C_5) = vd(C_3 \times C_6) = 2$; if $n \geq 7$, then $vd(C_3 \times C_n) = 3$; $vd(C_4 \times C_4) = 2$; $vd(C_4 \times C_5) = 3$; if $n \geq 6$, then $vd(C_4 \times C_n) = 4$; $vd(C_5 \times C_5) = 4$; if $n \geq 6$, then $vd(C_5 \times C_n) = 5$; and if $\min\{n, m\} \geq 6$, then $vd(C_n \times C_m) = min\{n, m\}$.

For the computation of $vd(C_n \times C_m)$ in Section 4, we start by evaluating upper bounds for the vertex deletion of $C_n \times C_m$. We proceed by defining for fixed n, m a subset $\Sigma \subset V(C_n \times C_m)$ of vertices, whose removal from $C_n \times C_m$ produces a planar graph, which establishes the upper bound $|\Sigma|$ for $vd(G)$. We find next lower bounds with the same values of the upper bounds exhibited. Our main tool for establishing the lower bounds is that if a graph G has a graph H as a minor, then $vd(H) \leq vd(G)$.

This article is organized as follows. In Sections 2 and 3 we establish notation, definition and main properties. In Section 4 we compute the exact values of $vd(C_n \times C_m)$.

2 Notation and Definitions

For basic concepts—*graph*, *path*, *cycle*, *complete graph*, etc.—we borrow the definitions and nomenclature from Bondy and Murty [4].

Let $G = (V, E)$ be a graph, $v \in V$ and $S \subset V$. The subgraph of G *induced* by S is the maximal subgraph of G with vertex set S. The graph $G - v$ is the subgraph of G induced by $V \setminus \{v\}$. The graph $G - S$ is the subgraph of G induced by $V \setminus S$.

A *subdivision* of an edge $e = uv$ replaces e by a path of length 2 connecting u and v, where the internal vertex of the path is a new vertex. A graph H is a *subdivision* for a graph G, if H is obtained from G by a sequence of edge subdivisions.

A *contraction* of an edge $e = uv$ replaces its endvertices u, v by a new vertex w whose neighborhood $N(w) = N(u) \cup N(v) \setminus \{u, v\}$, (i.e., w is adjacent to every other vertex that was adjacent to u or v, except to u and v). We say that a graph G is *contractible* to a graph H if H is obtained from G by a sequence of edge contractions. We say that a graph G has a graph H as a *minor* if G has as a subgraph a graph contractible to H.

We define an *open arc* as a bounded subset of the plane $\mathbb{R}^2$ homeomorphic to the real line $\mathbb{R}$ in the standard topology. A *drawing* of a graph G is a mapping η_G of the vertices of G to points of the plane, and of the edges of G to open arcs—the *vertices* and *edges* of the drawing, respectively—such that (1) the vertices of the drawing are pairwise distinct, and disjoint from all its edges; (2) any two edges of the drawing are either disjoint, or cross at a single point; (3) for every edge $e = uv$ of G, the external frontier of $\eta_G(e)$ is $\{\eta_G(u), \eta_G(v)\}$; and (4) no three edges of the drawing go through the same point.

We say that a graph is *planar* if it has a drawing $D(G)$ without crossing edges, and in this case we say that $D(G)$ is a *plane* drawing.

We denote by K_n the complete graph on n vertices, and by $K_{m,n}$ the complete bipartite graph with vertex parts of size m and n. In our proofs, we rely heavily on Kuratowski's theorem [18] which says that a graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor. In particular, every planar graph has only planar graphs as minors. Every time we argue that a graph is not planar, we exhibit as certificate a $K_{3,3}$-minor or a K_5-minor (see Figure 1). In particular, we use the fact that the class of planar graphs is closed under edge contraction.

For $n \geq 3$, we denote by C_n the chordless *cycle* with n vertices and n edges. The $n \times m$ *toroidal grid* $C_n \times C_m$ is the graph-theoretic product of C_n and C_m; that is, the graph with nm vertices $\{v_{ij} : 0 \leq i < n,\ 0 \leq j < m\}$, and $2nm$ edges $\{v_{ij}v_{(i+1) \bmod n, j}, v_{ij}v_{i(j+1) \bmod m} : 0 \leq i < n,\ 0 \leq j < m\}$. The vertex $v_{ij} \in V(C_n \times C_m)$ is represented by a point on the plane with coordinates (i, j). Note that $C_n \times C_m$ has a planar embedding on the torus.

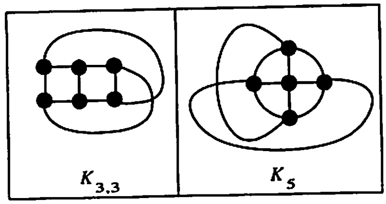

Figure 1: $K_{3,3}$ and K_5.

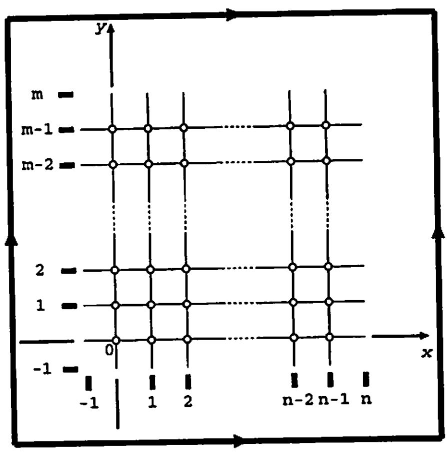

Figure 2: Toroidal drawings for $C_n \times C_m$.

Call *toroidal drawing* for $C_n \times C_m$ the usual drawing for $C_n \times C_m$ on the torus as a latitude-longitude grid depicted in Figure 2. We represent $C_n \times C_m$ inside a rectangle with sides aligned with the Cartesian axis such that non consecutive sides of the rectangle are identifyed by the orientations. In this drawing, the edges $v_{ij}v_{(i+1)j}$, $0 \leq i < n-1$, $0 \leq j < m$ and $v_{ij}v_{i(j+1)}$, $0 \leq i < n$, $0 \leq j < m-1$, are respectively, represented by horizontal straight segment connecting the points (i,j) and $(i+1,j)$ and by the vertical segment connecting the points (i,j) and $(i,j+1)$; and the edges $v_{(n-1)j}v_{0j}, 0 \leq j < m$; and $v_{i(m-1)}v_{i0}, 0 \leq i < n$ respectively, represented by the 2 horizontal segments connecting the points $(-1,j)$ and $(0,j)$ and the points $(n-1,j)$ and (n,j); and by the 2 vertical segments connecting the points $(i,-1)$ and $(i,0)$ and the points $(i,m-1)$ and (i,m). Based on this convention, we call the two families of edges above *horizontal* and *vertical*, respectively. In a toroidal drawing for $C_n \times C_m$ we omit the axis and the sides of the rectangle.

A cycle of $C_n \times C_m$ is called a *meridian* if it uses only vertical edges, and a *parallel* if it uses only horizontal edges. Thus the $n \times m$ toroidal grid has n meridians isomorphic to C_m, and m parallels isomorphic to C_n. The meridian $(v_{i0}, v_{i1}, v_{i2}, \ldots, v_{i(m-1)}, v_{i0})$ containing all vertices with the first coordinate $i, 0 \leq j < m$ is called *meridian* i; the parallel $(v_{0j}, v_{1j}, v_{2j}, \ldots, v_{(n-1)j}), v_{0j})$ containing all vertices with the second coordinate $j, 0 \leq i < n$ is called *parallel* j.

Two graphs G and H are said to be *isomorphic* if there is a pair $\alpha = (\alpha_V, \alpha_E)$ of bijections, where $\alpha_V: V(G) \to V(H)$ and $\alpha_E: E(G) \to E(H)$, such that $uv \in E(G)$ if and only if $\alpha_E(uv) = \alpha_V(u)\alpha_V(v) \in E(H)$. The pair α is called an *isomorphism* from G to H. An *automorphism* of a graph is an isomorphism from the graph to itself.

We say a graph H is a *spanning subgraph* of a graph G if H is a subgraph of G and $V(H) = V(G)$.

In this work we define several automorphisms $\alpha = (\alpha_V, \alpha_E)$ of a graph G, where G is a spanning subgraph of $C_n \times C_m$. In our paper we define these pairs of functions by defining, for each pair i, j with $0 \leq i < n$ and $0 \leq j < m$, which vertex is the image $\alpha_V(v_{ij}) \in V(C_n \times C_m)$ of $v_{ij} \in V(G)$, i.e., we define just the restriction $\alpha_V : V(G) \to V(C_n \times C_m)$ of α. The definition of the restriction $\alpha_E : E(G) \to E(C_n \times C_m)$ of α is indirectly done by considering that if $uv \in E(G)$, then $\alpha_E(uv) = \alpha_V(u)\alpha_V(v) \in E(\alpha(G))$.

Let G be a spanning subgraph of $C_n \times C_m$. Say that G has *horizontal symmetry* if the function $\alpha : G \to G$ is an automorphism, where $\alpha(v_{ij}) = v_{n-1-i,j}$. Analogously, we say that G has *vertical symmetry* if the function $\alpha : G \to G$ is an automorphism, where $\alpha(v_{ij}) = v_{i,m-1-j}$. Denote $G_{ij} = G - v_{ij}$. If G has horizontal symmetry, then $vd(G_{ij}) = vd(G_{n-1-i,j})$, because by definition G_{ij} is isomorphic to $G_{n-1-i,j}$; if G has vertical symmetry, then $vd(G_{ij}) = vd(G_{i,n-1-j})$; and if G has horizontal and vertical symmetry, then $vd(G_{ij}) = vd(G_{n-1-i,m-1-j})$.

We observe also that if G has horizontal symmetry, then the collection of graphs $G_{00}, G_{01}, G_{02}, \ldots, G_{0(m-1)}, G_{10}, G_{11}, G_{12}, \ldots, G_{1(m-1)}, \ldots, G_{\lceil \frac{n}{2} \rceil 0}, G_{\lceil \frac{n}{2} \rceil 1}, G_{\lceil \frac{n}{2} \rceil 2}, \ldots, G_{\lceil \frac{n}{2} \rceil (m-1)}$ contains all graphs obtained from G with one vertex removed. Analogously, if G has vertical symmetry, then the collection of graphs $G_{00}, G_{10}, G_{20}, \ldots, G_{(n-1)0}, G_{01}, G_{11}, G_{21}, \ldots, G_{(n-1)1}, \ldots, G_{0(m-1)}, G_{1(m-1)}, G_{2(m-1)}, \ldots, G_{(n-1)\lceil \frac{m}{2} \rceil}$ contains all graphs obtained from G by removing a vertex. In addition, if G has horizontal and vertical symmetry, then the collection of graphs $G_{00}, G_{10}, G_{20}, \ldots, G_{\lceil \frac{n}{2} \rceil 0}, G_{01}, G_{11}, G_{21}, \ldots, G_{\lceil \frac{n}{2} \rceil 1}, \ldots, G_{0\lceil \frac{m}{2} \rceil}, G_{1\lceil \frac{m}{2} \rceil}, G_{2\lceil \frac{m}{2} \rceil}, \ldots, G_{\lceil \frac{n}{2} \rceil \lceil \frac{m}{2} \rceil}$ contains all graphs obtained from G by removing a vertex.

When considering the computation of $vd(C_n \times C_m)$, we shall use that several subgraphs of $C_n \times C_m$ have horizontal and vertical symmetry, as proved in Section 3. We observe that these symmetries allow us to reduce the number of cases we have to consider.

3 Some properties of vertex deletion

In this section we state some general properties of the vertex deletion parameter used in our proofs.

Lemma 1 *If H is a subgraph of G, then $vd(H) \leq vd(G)$.*

Lemma 2 *If a vertex v of a graph G has at most one neighbor, then $vd(G) = vd(G - v)$.*

Lemma 3 *If G is contractible to H, then $vd(H) \leq vd(G)$.*

Proof: It is enough to prove that given a graph G, an edge $e = uv$ of G, and H the graph obtained from G by contracting e into a vertex w, we have $vd(H) \leq vd(G)$. Let $S \subset V(G)$ be a set of vertices whose removal defines a planar graph $G' = G - S$ from G. If $\{u, v\} \cap S = \emptyset$, then we set $S' = S$, else we set $S' = \{S \cup \{w\}\} \setminus \{u, v\}$. Obviously, $|S'| \leq |S|$. We consider the graph $H' = H - S'$. If $\{u, v\} \cap S = \emptyset$, i.e., $S' = S$, then a plane drawing $D(H')$ for H' is obtained from a plane drawing $D(G')$ for G' by contracting e in $D(G')$, which by Kuratowski Theorem defines H' as a planar graph; if $\{u, v\} \cap S \neq \emptyset$, i.e., $|S'| < |S|$, then we have that G' has H' as a subgraph, which also defines H' as a planar graph. □

Corollary 1 *If G has H as a minor, then $vd(H) \leq vd(G)$.*

Corollary 2 *If G is a subdivision of H, then $vd(H) = vd(G)$.*

Fact 1 *Let G be a graph. A set $\Sigma = \{v_1, v_2, v_3, \ldots, v_{|\Sigma|}\} \subset V(G)$ satisfies that $G - \Sigma$ is a planar graph and $|\Sigma| = vd(G)$ if and only if the sequence $(G_k)_{k \in \mathbb{N}}$ of graphs satisfies $vd(G_k) = |\Sigma| - k$, where $G_0 = G$ and $G_k = G - \{v_1, v_2, v_3, \ldots, v_k\}$, $k \in \{1, 2, 3, \ldots, |\Sigma|\}$.*

Proof: Suppose $\Sigma = \{v_1, v_2, v_3, \ldots, v_{|\Sigma|}\} \subset V(G)$ satisfies that $G - \Sigma$ is a planar graph and that $|\Sigma| = vd(G)$. We argue by induction on the number of vertices removed. Let $G_0 = G$ and $G_k = G - \{v_1, v_2, v_3, \ldots, v_k\}$, $k \in \{1, 2, 3, \ldots, |\Sigma|\}$. It remains to prove that $vd(G_k) = |\Sigma| - k$ for each $k \in \{0, 1, 2, 3, \ldots, |\Sigma|\}$. The result is true if $k = 0$, i.e., $vd(G_0) = vd(G) = |\Sigma| - 0$. Suppose $vd(G_k) = |\Sigma| - k$ with $0 \leq k < |\Sigma|$. It follows from the definition of Σ that $G_{k+1} - \{v_{k+2}, v_{k+3}, v_{k+4}, \ldots, v_{|\Sigma|}\}$ is a planar graph, hence $vd(G_{k+1}) \leq |\Sigma| - (k+1)$. On the other hand, since $G_{k+1} = G_k - v_{k+1}$, we have that $vd(G_k) \leq vd(G_{k+1}) + 1$, hence $vd(G_{k+1}) \geq vd(G_k) - 1 = |\Sigma| - (k+1)$. The converse of the Fact 1 follows from the definition of the planar graph $G_{|\Sigma|}$. □

Let $\mathcal{F}$ be a family of isomorphic subgraphs of a graph G. We say that G is *$\mathcal{F}$-transitive* if for any two elements F and H of $\mathcal{F}$ there is an automorphism of G that takes F to H.

Fact 2 *If G is a non planar vertex transitive graph, then $vd(G) = vd(G - v) + 1$.*

Facts 3 to 7 together with Corollaries 3 and 4 below relate the values for vertex deletion in the family $C_n \times C_m$ as follows.

Fact 3 *$C_n \times C_m$ is vertex transitive.*

Fact 4 *$C_n \times C_m$ is parallel transitive.*

Fact 5 *$C_n \times C_m$ is meridian transitive.*

Fact 6 *If $p \leq n$ and $q \leq m$, then $C_n \times C_m$ has $C_p \times C_q$ as a minor.*

Corollary 3 *If $n, m \geq 4$, then $C_n \times C_m - v$ has $C_{n-1} \times C_{m-1}$ as a minor.*

Corollary 4 *If $p \leq n$ and $q \leq m$, then $vd(C_p \times C_q) \leq vd(C_n \times C_m)$.*

Fact 7 *If $n, m \geq 4$ and v is a vertex of $C_n \times C_m$, then $vd(C_n \times C_m) = vd(C_n \times C_m - v) + 1 \geq vd(C_{n-1} \times C_{m-1}) + 1$.*

4 The vertex deletion of $C_n \times C_m$

In this section we establish the vertex deletion for all graphs in the family $C_n \times C_m$. In order to establish the values presented in Table 1, we first establish each claimed value as an upper bound by exhibiting in each case a plane drawing of the graph obtained after removing $vd(C_n \times C_m)$ vertices from $C_n \times C_m$.

Second we establish each claimed value as a lower bound by using Corollary 1, by proving that removing any set with fewer than $vd(C_n \times C_m)$ vertices leaves a graph with either $K_{3,3}$ or K_5 as a minor, and by using auxiliary graphs with vertex deletion 2, 3 and 4.

4.1 Upper bounds for $vd(C_n \times C_m)$

Lemma 4 *$vd(C_3 \times C_3) \leq 1$; $vd(C_3 \times C_4)$, $vd(C_3 \times C_5)$, $vd(C_3 \times C_6)$, $vd(C_4 \times C_4) \leq 2$; $vd(C_4 \times C_5) \leq 3$; $vd(C_5 \times C_5) \leq 4$; and if $n, m \geq 3$, then $vd(C_n \times C_m) \leq \min\{n, m\}$.*

Proof: Figure 3 is used in order to prove Lemma 4. In Figure 3, we show 6 pictures. Each picture contains a toroidal drawing for a graph $C_n \times C_m$ on the left, and a plane drawing of an induced subgraph of $C_n \times C_m$ on the right. Figures 3(a), 3(b), 3(c), 3(d), 3(e) and 3(f) correspond, respectively, to $C_3 \times C_3$, $C_3 \times C_6$, $C_4 \times C_4$, $C_4 \times C_5$, $C_5 \times C_5$ and $C_n \times C_m$, with

m / n	3	4	5	6	7	8	$\dots$	i	$\dots$
3	1	2	2	2	3	3	$\dots$	3	$\dots$
4	2	2	3	4	4	4	$\dots$	4	$\dots$
5	2	3	4	5	5	5	$\dots$	5	$\dots$
6	2	4	5	6	6	6	$\dots$	6	$\dots$
7	3	4	5	6	7	7	$\dots$	7	$\dots$
8	3	4	5	6	7	8	$\dots$	8	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
i	3	4	5	6	7	8	$\dots$	i	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1: $vd(C_n \times C_m)$.

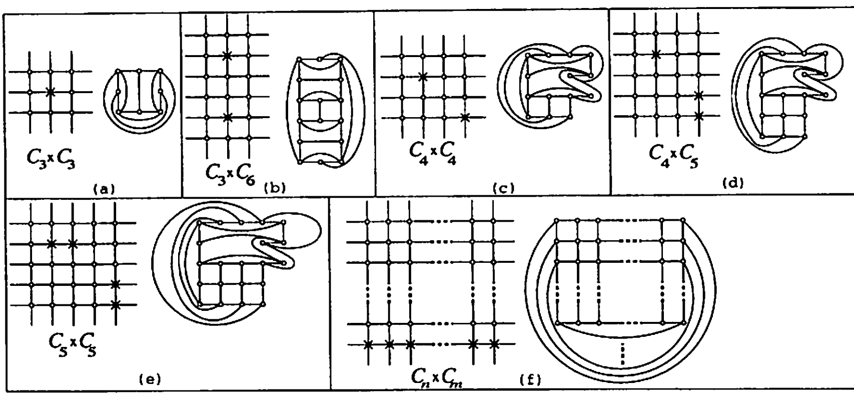

Figure 3: Upper bounds for $vd(C_n \times C_m)$.

$n, m \geq 3$. Each picture is used to exhibit an upper bound for the vertex deletion of the corresponding graph $C_n \times C_m$. This upper bound is obtained, in each picture, by counting the number of vertices of the subset of vertices of $V(C_n \times C_m)$, whose removal from $C_n \times C_m$ defines the corresponding induced planar subgraph. Each removed vertex is indicated at the drawing for $C_n \times C_m$ by symbol ✱. The corresponding plane drawing consists in a drawing for the subgraph induced by vertices of $C_n \times C_m$ without the removed subset of vertices. The upper bound for $vd(C_n \times C_m)$ is then defined by the size of the removed subset of vertices. For the convenience of the reader, in each picture we have the coordinates of the vertices of the induced subgraph on the right side agreeing with their original coordinates in the drawing for $C_n \times C_m$ on the left side.

Consider first Figure 3(a). The toroidal drawing for $C_3 \times C_3$ on the left side contains one vertex indicated by ⁎. On the right side we depict a plane drawing for the subgraph induced by the vertices of $C_3 \times C_3$ without the vertex indicated by ⁎. This means that $vd(C_3 \times C_3) \leq 1$.

On the left side of Figures 3(b) and 3(c) are displayed, respectively, the toroidal drawings for graphs $C_3 \times C_6$ and $C_4 \times C_4$ each with two vertices indicated by ⁎. The corresponding plane drawings on the right prove that, $vd(C_3 \times C_6) \leq 2$ and $vd(C_4 \times C_4) \leq 2$. As by Fact 6, the graph $C_3 \times C_6$ has $C_3 \times C_4$ and $C_3 \times C_5$ as a minor, it follows from Corollary 4 that $vd(C_3 \times C_4) \leq 2$ and that $vd(C_3 \times C_5) \leq 2$.

In a similar way Figures 3(d) and 3(e) allow us to define the upper bounds: $vd(C_4 \times C_5) \leq 3$ and $vd(C_5 \times C_5) \leq 4$. Finally, Figure 3(f) displays a toroidal drawing for the graph $C_n \times C_m$ and a plane drawing for the planar subgraph induced by $V(C_n \times C_m)$ without a subset with $\min\{n, m\}$ vertices. Thus, $vd(C_n \times C_m) \leq \min\{n, m\}$. □

4.2 Lower bounds for vertex deletion of $C_n \times C_m$

Lemma 5 $vd(C_3 \times C_3) \geq 1$.

Proof: It is enough to prove that $C_3 \times C_3$ is non planar. We remark that although Harary et al. [15] proved that $C_3 \times C_3$ is a non planar graph, we give here another proof in order to define the strategy of proof used for higher values of n and m.

In Figure 4 we show that $C_3 \times C_3$ is non planar by defining a subdivision for $K_{3,3}$ as a subgraph of $C_3 \times C_3$, and hence proving that $1 \leq vd(C_3 \times C_3)$.

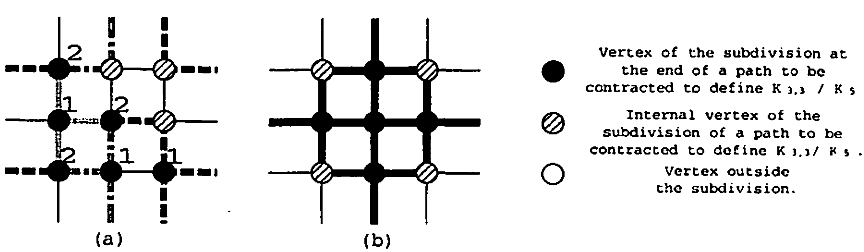

Figure 4: Lower bound $vd(C_3 \times C_3) \geq 1$.

The definition of the subdivision for $K_{3,3}$ is done by a vertex colouring, where the vertices are depicted by white, black and stripped colours. Black and stripped vertices belong to the subdivision for $K_{3,3}$. White vertices do not belong to the subdivision. We depict in dashed edges the set of edges of the subdivision for $K_{3,3}$ that is a subgraph of $C_3 \times C_3$. This convention is adopted from now on for the remaining results of the paper. Based on this convention, in Figure 4 we exhibit two subdivisions: one for $K_{3,3}$ (4(a)) and one for K_5 (4(b)). In these drawings we have no vertex in white color, we observe that this will not be the case in general, white vertcies will appear in Figure 6.

For the convenience of the reader the paths of the subdivision to be contracted in order to define the graph $K_{3,3}$ are defined by the paths in dashed edges containing only stripped vertices. We also label the two colour classes of the subdivision for $K_{3,3}$, respectively with 1 and 2, and we use a different pattern of dashed in the paths going from a different vertex labeled with 1.

Therefore, the graph $C_3 \times C_3$ is not planar, which implies $1 \leq vd(C_3 \times C_3)$. □

Lemma 6 $vd(C_3 \times C_4) \geq 2$.

Proof: Let $v \in V(C_3 \times C_4)$ and $G = C_3 \times C_4 - v$. By Fact 3 we can assume that v is the vertex v_{00}. Graph G contains the subdivision of $K_{3,3}$ shown in Figure 5. Hence, $vd(C_3 \times C_4 - v) \geq 1$. By Fact 7 we have $vd(C_3 \times C_4) \geq 2$. □

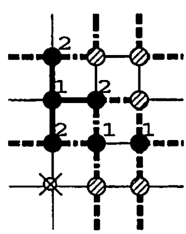

Figure 5: Lower bound $vd(C_3 \times C_4) \geq 2$.

Corollary 5 *The vertex deletion of each of the graphs:* $C_3 \times C_5$, $C_3 \times C_6$ *and* $C_4 \times C_4$ *is at least* 2.

Proof: It follows from Corollary 4 and Lemma 6. □

Lemma 7 $vd(C_3 \times C_7) \geq 3$.

Proof: Figure 6 is used in order to prove Lemma 7. Figure 6 shows three copies of the toroidal drawing for $C_3 \times C_7$. Each drawing defines one subdivision for K_5 as a subgraph of $C_3 \times C_7$.

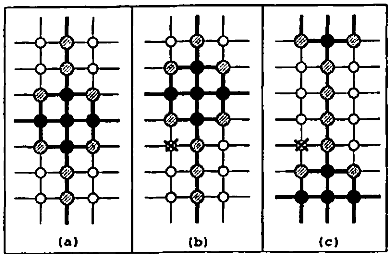

Figure 6: Lower bound $vd(C_3 \times C_7) \geq 3$.

Suppose, by contradiction, that $vd(C_3 \times C_7) \leq 2$. Let Σ be a subset $\Sigma \subset V(C_3 \times C_7)$, with $|\Sigma| = 2$, whose removal from $C_3 \times C_7$ defines the planar graph $G = C_3 \times C_7 - \Sigma$. As $|\Sigma| = 2$, there is a parallel and a meridian with no vertex in Σ. By Facts 4 and 5, we may assume that meridian 1 and parallel 3 have no vertex in Σ. Note that at least one of the vertices: $v_{02}, v_{04}, v_{22}, v_{24}$ must belong to Σ, otherwise we have that the graph induced by the set of vertices in meridian 1 and in parallel 3 plus vertices $v_{02}, v_{04}, v_{22}, v_{24}$ contains a subdivision for K_5 shown in Figure 6(a) as a subgraph of G. We consider a suitable automorphism and assume v_{02} in Σ as depicted in Figure 6(b). Again by the same argument, one of the vertices: $v_{04}, v_{05}, v_{24}, v_{25}$ must belong to Σ, otherwise, we have that the graph induced by the set of vertices in meridian 1, plus vertices in parallel 3, plus vertices $v_{04}, v_{05}, v_{24}, v_{25}$ contain the subdivision for K_5 shown in Figure 6(b) as a subgraph of the graph G. Because $|\Sigma| = 2$, this means that parallels 6, 0 and 1 have no vertex in Σ. Now, the graph induced by the set of vertices in meridian 1, plus vertices in parallel 6, 0 and 1 contain the subdivision for K_5 shown in Figure 6(c) as a subgraph of the graph G, a contradiction. □

Corollary 6 *If $m \geq 7$, then $vd(C_3 \times C_m) \geq 3$.*

Proof: It follows from Corollary 4 and Lemma 7. □

Lemma 8 $vd(C_4 \times C_5) \geq 3$.

Proof: It follows from Fact 7 and Lemma 6. □

Now, we discuss a lower bound for $vd(C_4 \times C_6)$. From Lemma 8, Corollary 4 and Fact 6 we have that $vd(C_4 \times C_6) \geq 3$. We prove in Lemma 11 that $vd(C_4 \times C_6) \geq 4$ by proving that the removal of 3 vertices from $C_4 \times C_6$ does not produce a planar graph. For, we use two auxiliary lemmas: Lemma 9 and Lemma 10. In these lemmas we prove that if a set Σ of vertices of

$C_4 \times C_6$, with $|\Sigma| = 3$ is such that $C_4 \times C_6 - \Sigma$ is planar, then no meridian (Lemma 9) nor parallel (Lemma 10) has 2 vertices in Σ. In Lemma 11 we prove that: if Σ is a set with $|\Sigma| = 3$, $\Sigma \subset V(C_4 \times C_6)$, such that Σ has no pair of vertices in a same meridian or parallel of $C_4 \times C_6$, then the graph $C_4 \times C_6 - \Sigma$ is non planar, which implies $vd(C_4 \times C_6) \geq 4$.

Lemma 9 *If there is a subset Σ of vertices of $C_4 \times C_6$ with $|\Sigma| = 3$, whose removal from $C_4 \times C_6$ defines a planar graph G, then Σ has no pair of vertices in a same meridian of $C_4 \times C_6$.*

Proof: Let Σ be a subset of vertices of $V(C_4 \times C_6)$, with $|\Sigma| = 3$, whose removal from $C_4 \times C_6$ defines a planar graph $G = C_4 \times C_6 - \Sigma$. We argue by contradiction. Suppose there is a meridian of $C_4 \times C_6$ with two vertices u and v in Σ. Since the graph $C_4 \times C_6 - \{u, v\}$ defined from $C_4 \times C_6$ by removing u and v has a subdivision of $C_3 \times C_4$ as a subgraph, by Lemmas 1 and 6 there are 2 additional vertices in Σ, contradicting the size $|\Sigma| = 3$. Thus, there is at most one vertex of Σ for each meridian of $C_4 \times C_6$. □

Lemma 10 *If there is a subset Σ of vertices of $C_4 \times C_6$ with $|\Sigma| = 3$, whose removal from $C_4 \times C_6$ defines a planar graph G, then Σ has no pair of vertices in a same parallel of $C_4 \times C_6$.*

Proof: Let Σ be a subset of vertices of $V(C_4 \times C_6)$, with $|\Sigma| = 3$, whose removal from $C_4 \times C_6$ defines a planar graph $G = C_4 \times C_6 - \Sigma$. We argue by contradiction. Suppose there is a parallel of $C_4 \times C_6$ with two vertices in Σ. We prove that G is non planar contradicting the hypothesis. As $|\Sigma| = 3$, there is one meridian with no vertices in Σ. We consider a suitable automorphism and assume that meridian 2 has no vertex in Σ and that parallel 0 has 2 vertices in Σ. Let u and v be the two vertices of Σ in parallel 0. There are 2 possibilities for u and v according to $uv \notin E(C_4 \times C_6)$ or $uv \in E(C_4 \times C_6)$. Figure 7 is used in order to prove that if there are 2 vertices of Σ at the same parallel, then G is non planar. Figure 7 shows four

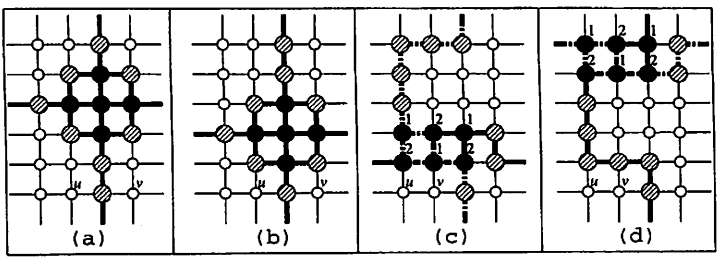

Figure 7: Two vertices of $C_4 \times C_6$ at the same parallel.

copies of the toroidal drawing for $C_4 \times C_6$. Figures 7(a) and 7(b) are used to prove that if $uv \notin E(C_4 \times C_6)$, then G is non planar. In this case, G has as a subgraph at least one of the subdivisions for K_5 defined in Figures 7(a) or 7(b) . Figures 7(c) and 7(d) are used to prove that if $uv \in E(C_4 \times C_6)$, then G is non planar. In this case, G has as a subgraph at least one of the subdivisions for $K_{3,3}$ defined in Figures 7(c) or 7(d).

We analize first the case when $uv \notin E(C_4 \times C_6)$. In this case, $u = v_{10}$ and $v = v_{30}$. As $|\Sigma| = 3$, by Lemma 9 the vertex of Σ in meridian 1 is v_{10} and the vertex of Σ in meridian 3 is v_{30}. Hence, as by supposition, meridian 2 has no vertex in Σ, the third vertex of Σ is one of the vertices in meridian 0, i.e., vertices: $v_{00}, v_{01}, v_{02}, v_{03}, v_{04}$ and v_{05}. The two subdivisions for K_5 in Figures 7(a) and 7(b) show that for each choice in meridian 0 for the third vertex of Σ there is a suitable subdivision for K_5 as a subgraph of the graph G.

The second case to analize is when $uv \in E(C_4 \times C_6)$. In this case, we consider a suitable automorphism and assume that $uv = v_{00}v_{10}$. As $|\Sigma| = 3$, by Lemma 9 the vertex of Σ in meridian 0 is v_{00} and the vertex of Σ in meridian 1 is v_{10}. Hence, the third vertex of Σ is one of the vertices $v_{30}, v_{31}, v_{32}, v_{33}, v_{34}, v_{35}$. The two subdivisions for $K_{3,3}$ in Figures 7(c) and 7(d) show that for each choice of the vertex of Σ in meridian 3 there is a subdivision for $K_{3,3}$ as a subgraph of the graph G. □

Lemma 11 $vd(C_4 \times C_6) \geq 4$.

Proof: We argue by contradiction. Let Σ be a subset of vertices of $V(C_4 \times C_6)$, with $|\Sigma| = 3$ vertices, whose removal from $C_4 \times C_6$ defines a planar subgraph $G = C_4 \times C_6 - \Sigma$.

As Σ has size 3, by Lemmas 9 and 10 there is at most one vertex of Σ in each one of the meridians and at most one vertex of Σ in each one of the parallels. As there are four meridians and six parallels in $C_4 \times C_6$, there is exactly one meridian with no vertex in Σ and there are exactly three parallels with no vertex in Σ. By Facts 4 and 5 we assume that meridian 2 and parallel 5 have no vertex in Σ.

We show in Figure 8(a) a drawing of a subdivision for K_5. Note that, at least one of the vertices: $v_{10}, v_{14}, v_{30}, v_{34}$ must be in Σ, otherwise, vertices $v_{10}, v_{14}, v_{30}, v_{34}$ plus vertices in meridian 2 and parallel 5 define the subdivision in Figure 8(a) as a subgraph of the planar graph G. We consider a suitable automorphism and assume that v_{34} is in Σ.

Now, we analize the vertex of Σ at meridian 1. As $v_{34} \in \Sigma$, by Lemmas 9 and 10 it follows that v_{34} is the vertex of meridian 3 in Σ and it is the vertex of parallel 4 in Σ. We consider in Figures 8(b) and (c) two subdivisions of $K_{3,3}$ as subgraphs of $C_4 \times C_6$. As, by supposition, parallel 5 has no vertex in Σ and $v_{34} \in \Sigma$, the vertex of meridian 1 in Σ must be one of the

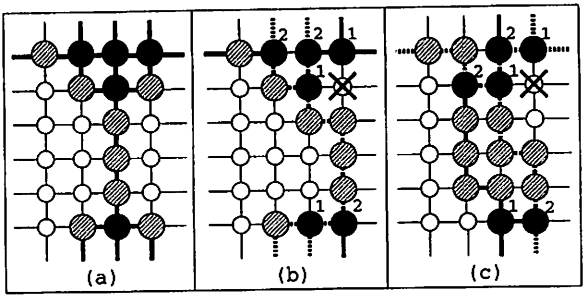

Figure 8: Lower bound $vd(C_4 \times C_6) \geq 4$.

vertices v_{10}, v_{11}, v_{12} or v_{13}. If one of the vertices: v_{11}, v_{12}, v_{13} is in Σ, then there is the subdivision of $K_{3,3}$ in Figure 8(b) as a subgraph of the planar graph G, a contradiction. If $v_{10} \in \Sigma$, then there is the subdivision of $K_{3,3}$ in Figure 8(c) as a subgraph of the planar graph G, a contradiction. □

Corollary 7 *If $m \geq 6$, then $vd(C_4 \times C_m) \geq 4$.*

Proof: It follows from Lemma 11 and Corollary 4. □

We remark that, from Fact 1, in order to prove that a graph G has vertex deletion at least 2, it is enough to prove that for each vertex v of G the graph $H = G - v$ is non planar. Analogously, to prove that a graph G has vertex deletion at least 3, it is enough to prove that for each vertex v of G the graph $H = G - v$ has vertex deletion at least 2. Lemmas 12 and 13 prove that two auxiliary graphs C and D have vertex deletion at least 2. We use these two graphs and an argument by graph minors to prove in Lemmas 14 and 15 that other two graphs: A and B have vertex deletion at least 3.

Graphs C and D are subgraphs of $C_4 \times C_4$. Hence their vertex deletions have value at most $vd(C_4 \times C_4) = 2$. Hence, when we prove that C and D have vertex deletion at least 2, we in fact prove that $vd(C) = vd(D) = 2$.

Lemma 12 *If C is the graph obtained from $C_4 \times C_4$ by removing the edges $v_{01}v_{31}$, $v_{02}v_{32}$, $v_{01}v_{02}$ and $v_{31}v_{32}$, i.e., $C = (V(C_4 \times C_4), E(C_4 \times C_4) \setminus \{v_{01}v_{31}, v_{02}v_{32}, v_{01}v_{02}, v_{31}v_{32}\})$, then $vd(C) = 2$.*

Proof: In Figure 9(a) we show a toroidal drawing for C. We note that the graph $C' = (V(C_4 \times C_4), E(C_4 \times C_4) \setminus \{v_{00}v_{30}, v_{03}v_{33}, v_{00}v_{03}, v_{30}v_{33}\})$ in Figure 9(b) is isomorphic to C with the isomorphism $\alpha(v_{ij}) = v_{i,(j+2) \bmod 4}$. We make this observation because we do not distinguish C from C' when a graph has C as a minor.

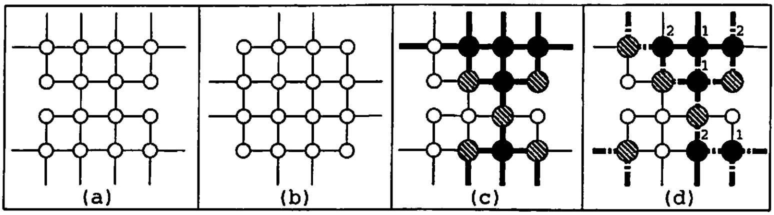

Figure 9: Lower bound $vd(C) \geq 2$.

Figures 9(c) and 9(d) show two copies of a toroidal drawing for C. Let $C_{i,j} = C - v_{ij}, 0 \leq i < 4$ and $0 \leq j < 4$. We note that C has vertical and horizontal symmetry. Hence, graphs $C_{0,0}, C_{0,1}$, $C_{1,0}$ and $C_{1,1}$ represent, up to some isomorphism, the class of subgraphs of C with one vertex removed. Then, in order to prove that $vd(C) > 1$ it is enough to prove that $C_{0,0}, C_{0,1}, C_{1,0}$ and $C_{1,1}$ are all non planar graphs. In Figure 9(c) we exhibit a subdivision for K_5 as a subgraph of $C_{0,0}, C_{0,1}$ and $C_{1,1}$ and in Figure 9(d) is exhibited a subdivision for $K_{3,3}$ as a subgraph of $C_{0,1}, C_{1,0}$ and $C_{1,1}$. □

Lemma 13 *If D is the graph obtained from $C_4 \times C_4$ by removing the edges $v_{00}v_{30}$, $v_{02}v_{32}$, $v_{00}v_{03}$ and $v_{20}v_{23}$, i.e., $D = (V(C_4 \times C_4), E(C_4 \times C_4) \setminus \{v_{00}v_{30}, v_{02}v_{32}, v_{00}v_{03}v_{20}v_{23}\})$, then $vd(D) = 2$.*

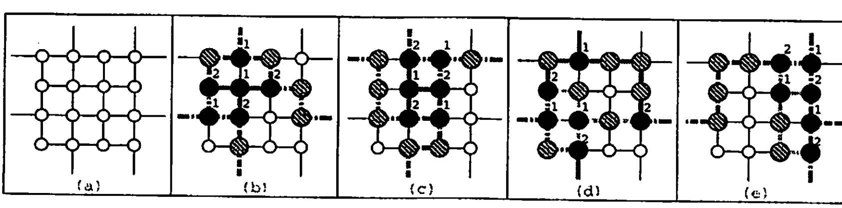

Figure 10: $vd(D) = 2$.

Proof: In Figure 10(a) we show a toroidal drawing for D. In order to prove Lemma 13 it is enough to prove that $vd(D) > 1$.

Let $D_{i,j} = D - v_{ij}, 0 \leq i < 4$ and $0 \leq j < 4$. Consider the automorphism of $C_4 \times C_4$ given by $\alpha(v_{ij}) = v_{ji}$. Since, $D = (V(C_4 \times C_4), E(C_4 \times C_4) \setminus \{v_{00}v_{30}, v_{02}v_{32}, v_{00}v_{03}v_{20}v_{23}\})$ and $\alpha(v_{ij}) = v_{ji}$, we have that the non existence of the edges: $(v_{00}v_{30})$, $(v_{02}v_{32})$, $(v_{00}v_{03})$ and $(v_{20}v_{23})$ in D implies, respectively, the non existence for the edges $(v_{00}v_{03})$, $(v_{20}v_{23})$, $(v_{00}v_{30})$ and $(v_{02}v_{32})$ in $\alpha(D)$. Hence, the image $\alpha(D)$ is a subgraph of $C_4 \times C_4$ without the edges $v_{00}v_{30}, v_{02}v_{32}, v_{00}v_{03}, v_{20}v_{23}$, i.e., α is an automorphism of D.

Hence, graphs $D_{0,0}$, $D_{1,0}$, $D_{1,1}, D_{2,0}$, $D_{2,1}, D_{2,2}$, $D_{3,0}, D_{3,1}$, $D_{3,2}$ and $D_{3,3}$ represent, up to some isomorphism, all subgraphs that can be obtained from D by removing a vertex because $D_{i,j}$ is isomorphic to $D_{j,i}$. Then, in order to prove that $vd(D) > 1$ it is enough to prove each $D_{0,0}, D_{1,0}, D_{1,1}, D_{2,0}$, $D_{2,1}, D_{2,2}$, $D_{3,0}, D_{3,1}$, $D_{3,2}$ and $D_{3,3}$ is a non planar graph. In Figure 10(b) we exhibit a subdivision for $K_{3,3}$ as a subgraph of $D_{0,0}, D_{2,0}, D_{2,1}, D_{3,0}$ and $D_{3,3}$; in Figure 10(c) we exhibit a subdivision for $K_{3,3}$ as a subgraph of $D_{0,0}, D_{3,0}, D_{3,1}$ and $D_{3,2}$; in Figure 10(d) we exhibit a subdivision for $K_{3,3}$ as a subgraph of $D_{2,0}, D_{2,2}$ and $D_{3,0}$; and in Figure 9(e) we exhibit a subdivision for $K_{3,3}$ as a subgraph of $D_{0,0}, D_{1,0}, D_{1,1}$ and $D_{1,2}$. □

Lemma 14 *If A is the subgraph of $C_4 \times C_5$ defined by removing the edges $v_{01}v_{31}$ and $v_{03}v_{33}$ from $C_4 \times C_5$, i.e., $A = (V(C_4 \times C_5), E(C_4 \times C_5) \setminus \{v_{01}v_{31}, v_{03}v_{33}\})$, then $vd(A) = 3$.*

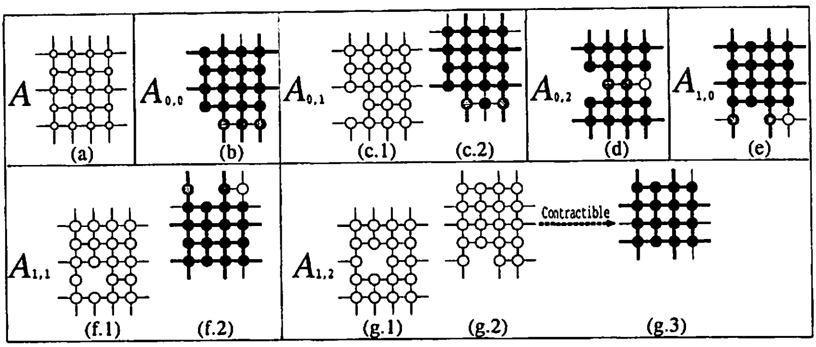

Figure 11: $vd(A) = 3$.

Proof: First of all, we observe that graph A has horizontal and vertical symmetry. Hence, in order to prove that $vd(A) \geq 3$ it is enough to prove that for each pair i, j, with $i \in \{0,1\}, j \in \{0,1,2\}$, $vd(A_{i,j}) \geq 2$. For, we show that the removal of an arbitrary vertex of $A_{i,j}$ yields a graph containing a minor for the graph C or for the graph D. For the convenience of the reader in the process of identifying these minors, we use some isomorphisms on $A_{i,j}$ which produce other graphs where it is easier to identify the minors of C or of D as a subgraph of $A_{i,j}$.

Figure 11(a) shows a toroidal drawing for A.

Figure 11(b) shows a toroidal drawing for $A_{0,0}$, where it is depicted a subdivision for D as a subgraph of $A_{0,0}$.

Figure 11(c.1) shows a toroidal drawing for $A_{0,1}$. For the convenience of the reader we show in Figure 11(c.2) a toroidal drawing for the graph

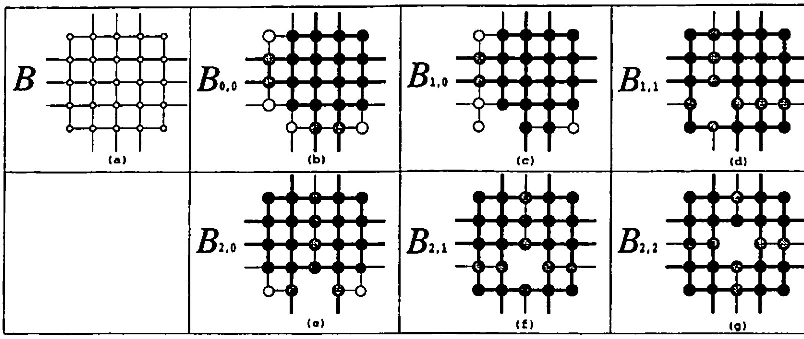

Figure 12: Lower bound $vd(B) \geq 3$.

obtained from $A_{0,1}$ by the isomorphism $\alpha(v_{ij}) = v_{i,(j-1) \bmod 4}$ and a subdivision for D as a subgraph.

Figure 11(d) shows a toroidal drawing for $A_{0,2}$, where it is depicted a subdivision for C as a subgraph of $A_{0,2}$.

Figure 11(e) shows a toroidal drawing for $A_{1,0}$, where it is depicted a subdivision for D as a subgraph of $A_{1,0}$.

Figure 11(f.1) shows a toroidal drawing for $A_{1,1}$. For the convenience of the reader we show in Figure 11(f.2) a toroidal drawing for the graph obtained from $A_{1,1}$ by the isomorphism $\alpha(v_{ij}) = v_{i,(j-2) \bmod 4}$ and a subdivision for D as a subgraph.

Figure 11(g.1) shows a toroidal drawing for $A_{1,2}$. For the convenience of the reader we show in Figure 11(g.2) a toroidal drawing for the graph obtained from $A_{1,2}$ by the isomorphism $\alpha(v_{ij}) = v_{i,(j-2) \bmod 4}$ and indicate the contraction of the edges $v_{02}v_{03}$, $v_{22}v_{23}$ and $v_{32}v_{33}$, whose define D as a minor of $A_{1,2}$. □

Lemma 15 *If B is the subgraph of $C_5 \times C_5$ defined by removing the edges $v_{00}v_{04}$, $v_{00}v_{40}$, $v_{04}v_{44}$ and $v_{40}v_{44}$, then $vd(B) \geq 3$.*

Proof: We prove that $vd(B) \geq 3$ by an argument similar to that of Lemma 14. In Figure 12(a) we show a toroidal drawing for B. First of all, we note that B has horizontal and vertical symmetry and that the function $\alpha(v_{ij}) = v_{ji}$ is an automorphism on B. Hence, in order to show that $vd(B) \geq 3$ it is enough to prove that for each pair i, j, with $0 \leq j \leq i \leq 2$ we have that $vd(B_{ij}) \geq 2$. For, in Figures 12(b), 12(c),12(d), 12(e), 12(f) and 12(g), we show, respectively, toroidal drawings for $B_{0,0}$, $B_{1,0}$, $B_{1,1}$, $B_{2,0}$, $B_{2,1}$, $B_{2,2}$ and, in each case a subdivision for graph C. □

Lemmas 16 and 17 use Lemmas 14 and 15 in order to prove that two auxiliary graphs have vertex deletion at least 4.

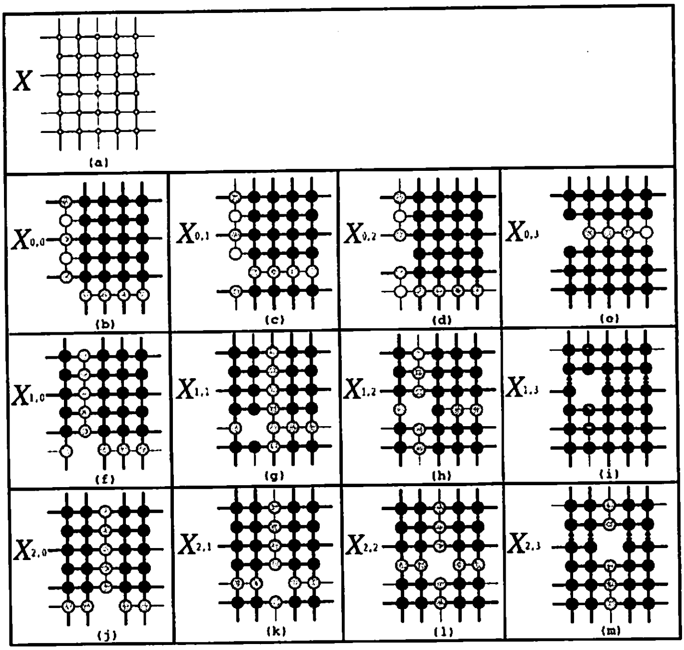

Figure 13: Lower bound $vd(X) \geq 4$.

Lemma 16 *If X is the graph obtained from $C_5 \times C_6$ by removing the edges $v_{02}v_{42}$, $v_{04}v_{44}$, i.e., $X = (V(C_5 \times C_6), E(C_5 \times C_6) \setminus \{v_{02}v_{42}, v_{04}v_{44}\})$, then $vd(X) \geq 4$.*

Proof: In Figure 13(a) we exhibit a toroidal drawing for X. Let $X_{i,j}$ be the graph obtained from X by removing vertex v_{ij}, with $i \in \{0,1,2,3,4\}$ and $j \in \{0,1,2,3,4,5\}$. We note that graph X has horizontal symmetry. Hence, to prove that $vd(X) \geq 4$ it is enough to prove that the removal of each vertex in meridian 0, 1 and 2 from X produces a graph with vertex deletion at least 3.

Note that, graphs $X_{0,1}$, $X_{0,2}$, $X_{1,1}$, $X_{1,2}$, $X_{2,1}$, $X_{2,2}$, are respectively isomorphic to $X_{0,5}$, $X_{0,4}$, $X_{1,5}$, $X_{1,4}$, $X_{2,5}$, $X_{2,4}$ by the isomorphism $\alpha(v_{ij}) = v_{i,(6-j) \bmod 6}$.

Hence, to prove that $vd(X) \geq 4$, it is enough to prove that if $i \in \{0,1,2\}$ and $j \in \{0,1,2,3\}$, then $X_{i,j}$ has a graph with vertex deletion at least 3 as a minor, which implies that $vd(X_{i,j}) \geq 3$. In Figures 13(b), 13(c), 13(d), 13(e), 13(f), 13(g), 13(h), 13(i), 13(j), 13(k), 13(l) and 13(m) we have, respectively, a toroidal drawing of graphs $X_{0,0}$, $X_{0,1}$, $X_{0,2}$, $X_{0,3}$, $X_{1,0}$, $X_{1,1}$, $X_{1,2}$, $X_{1,3}$, $X_{2,0}$, $X_{2,1}$, $X_{2,2}$ and $X_{2,3}$. We note that graphs $X_{0,0}$,

$X_{0,1}, X_{0,2}, X_{1,0}, X_{1,1}, X_{1,2}, X_{2,0}, X_{2,1}$ and $X_{2,2}$ have a subdivision of A as a subgraph; and note that $X_{0,3}$ has a subdivision of B as a subgraph. We observe that graphs $X_{1,3}$ and $X_{2,3}$ have a graph contractible to A as a subgraph, where we contract in $X_{1,3}$ edges $v_{03}v_{04}$, $v_{23}v_{24}$, $v_{33}v_{34}$, and $v_{43}v_{44}$ and we contract in $X_{2,3}$ edges $v_{03}v_{04}$, $v_{13}v_{14}$, $v_{33}v_{34}$, and $v_{43}v_{44}$. Hence, by Lemmas 1, 1, 3, 14 and 15, we have that $vd(X) \geq 4$. □

Lemma 17 *If Y is the graph obtained from $C_5 \times C_6$, by removing edges $v_{01}v_{41}$ and $v_{04}v_{44}$ i.e., $Y = (V(C_5 \times C_6), E(C_5 \times C_6) \setminus \{v_{01}v_{41}, v_{04}v_{44}\})$, then $vd(Y) \geq 4$.*

Proof: In Figure 14(a) we exhibit a toroidal drawing for Y. Let $Y_{i,j} = Y - v_{ij}$ be the graph obtained by removing vertex v_{ij} from Y. We note that Y has horizontal and vertical symmetry. We also note that the graphs $Y_{0,0}, Y_{1,0}, Y_{2,0}$ are, respectively, isomorphic to graphs $Y_{0,2}, Y_{1,2}, Y_{2,2}$, by the function $\alpha(v_{ij}) = v_{i,(j-3) \bmod 6}$. Hence, to prove that $vd(Y) \geq 4$ it is enough to prove that each one of the graphs $Y_{0,0}, Y_{0,1}, Y_{1,0}, Y_{1,1}$ $Y_{2,0}$ and $Y_{2,1}$ has as a subgraph a graph with vertex deletion at least 3. In Figures 14(b), 14(c), 14(d), 14(e), 14(f), 14(g), we exhibit, respectively, graphs: $Y_{0,0}, Y_{0,1}, Y_{1,0}, Y_{1,1}$ $Y_{2,0}$ and $Y_{2,1}$. We depict in each one of these graphs a subdivision of graph A as a subgraph. □

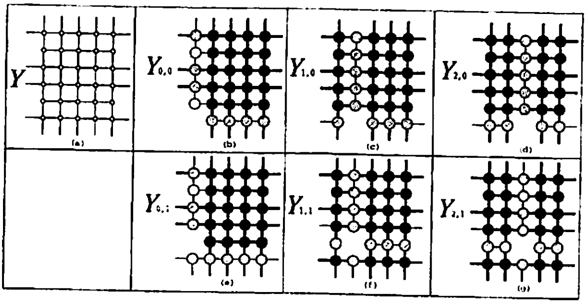

Figure 14: Lower bound $vd(Y) \geq 4$.

Lemma 18 $vd(C_5 \times C_5) \geq 4$.

Proof: It follows from Fact 7, Lemma 14 and that if $v \in C_5 \times C_5$, then the graph $C_5 \times C_5 - v$ has A as a subgraph. □

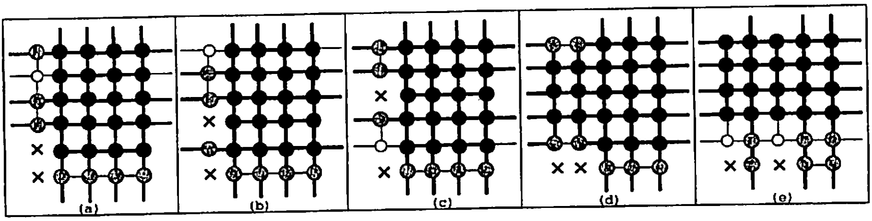

Figure 15: An obstruction for Lemma 19.

Lemma 19 *If there is a subset* Σ *of vertices of* $C_5 \times C_6$ *with* $|\Sigma| < 5$, *whose removal from* $C_5 \times C_6$ *define a planar graph, then*

1. Σ *has no pair of vertices in a same meridian of* $C_5 \times C_6$;
2. Σ *has no pair of vertices in a same parallel of* $C_5 \times C_6$.

Proof: Note that, if Σ has two vertices u and v in a same meridian of $C_5 \times C_6$, then the distance from u to v can be 1, 2 or 3. Hence, Figures 15(a), 15(b) and 15(c) show all the cases, up to isomorphism, of the graph obtained from $C_5 \times C_6$ by removing two vertices in a same meridian. Analogously, Figures 15(d) and 15(e) show all the cases, up to isomorphism, of the graph obtained from $C_5 \times C_6$ by removing two vertices u and v in a same parallel, according to the distance from u to v is 1 or 2. The proof of the lemma 19 follows from the fact that each graph in Figures 15(a), 15(b), 15(c) and 15(e), has as subgraph a subdivision of the graph A; and that the graph in Figure 15(d) has as subgraph a subdivision of the graph B depicted in bold edges. By Lemmas 14 and 15, at least 3 additional vertices are required in Σ, which implies $|\Sigma| \geq 5$, a contradiction. □

Lemma 20 $vd(C_5 \times C_6) \geq 5$.

Proof: Suppose, by contradiction, that $vd(C_5 \times C_6) < 5$. Let Σ be a subset of $V(C_5 \times C_6)$ with size $|\Sigma| = 4$ whose removal from $C_5 \times C_6$ defines a planar graph $G = C_5 \times C_6 - \Sigma$. By Lemma 19, Σ has no pair of vertices in a same meridian or parallel. As $C_5 \times C_6$ has five meridians, there are four meridians each one of them with just one vertex in Σ and just one meridian with no vertex in Σ.

Assume, meridian 2 and parallel 0 to be, respectively, one parallel and one meridian with no vertices in Σ as shown in Figure 16(a).

We claim that at least one of the vertices: $v_{11}, v_{31}, v_{15}, v_{35}$ is a vertex of Σ. Otherwise, these four vertices, plus vertices in meridian 2 and parallel 0 induce a subdivision for K_5 as a subgraph of the planar graph G.

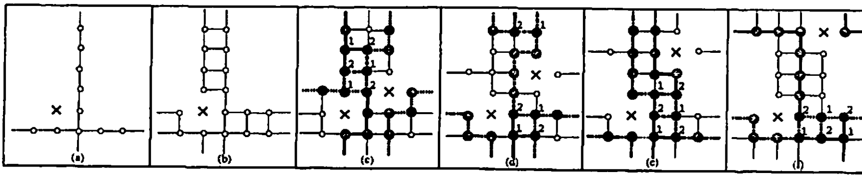

Figure 16: An obstruction for Lemma 20.

We consider a suitable automorphism and assume v_{11} in Σ as shown in Figure 16(a).

As v_{11} is in Σ and as $vd(C_5 \times C_6) = 4 < 5$, by Lemma 19 the vertex of Σ in meridian 1 is v_{11} and the vertex of Σ in parallel 1 is v_{11}. The remaining vertices in meridian 1 and in parallel 1, plus the vertices in meridian 2 and parallel 0 induce, so far, the graph in Figure 16(b) as a subgraph of the planar graph $G = C_5 \times C_6 - \Sigma$.

Now we analize the vertex of Σ in meridian 3. In Figures 16(c), 16(d), 16(e) and 16(f) we show the 4 possibilities for the vertex $v_{3i}, i \in \{2,3,4,5\}$ of Σ in meridian 3. We observe that for each case, where v_{3i} is in Σ, $i \in \{2,3,4,5\}$, Lemma 19 and the hypothesis of $|\Sigma| < 5$ force the remaining vertices in meridian 3 and in parallel i to be not in Σ.

As for each case, there is a subdivision for $K_{3,3}$ as a subgraph of the planar graph $G = C_5 \times C_6 - \Sigma$, we have the contradiction. Thus $vd(C_5 \times C_6) \geq 5$. □

Corollary 8 *If $m \geq 6$, then $vd(C_5 \times C_m) \geq 5$.*

Proof: It follows from Corollary 4 and Lemma 20. □

Lemma 21 *If there is a subset Σ of vertices of $C_6 \times C_6$ with $|\Sigma| < 6$, whose removal from $C_6 \times C_6$ defines a planar graph $G = C_6 \times C_6 - \Sigma$, then*

1. *Σ has no pair of vertices in a same meridian of $C_6 \times C_6$.*
2. *Σ has no pair of vertices in a same parallel of $C_6 \times C_6$.*

Proof: Suppose, by contradiction, that Σ has two vertices in a same parallel or in a same meridian.

Note that, if Σ has two vertices u and v in a same meridian or parallel of $C_6 \times C_6$, then the distance from u to v must be 1, 2 or 3. Hence, Figures 17(a), 17(b) and 17(c) show all the cases, up to isomorphism, of the graph obtained from $C_6 \times C_6$ by removing two vertices in a same meridian or parallel.

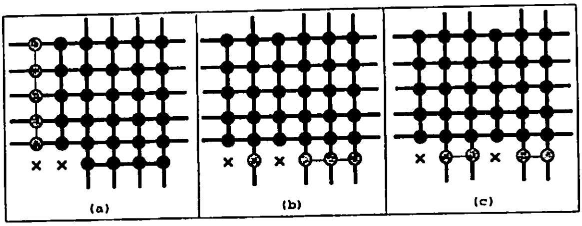

Figure 17: Non isomorphic graphs with 2 vertices removed in a same meridian or parallel of $C_6 \times C_6$ for Lemma 21.

As the graph in Figure 17(a) has a subdivision of $C_5 \times C_6 - v$, the graph in Figure 17(b) has a subdivision for graph X, and the graph in Figure 17(c) has a subdivision for graph Y; and by Lemmas 16, 17 and 20, each one of these graphs: $C_5 \times C_6 - v$, X and Y has vertex deletion at least 4, then by Corollary 1 and Fact 7 there are at least four additional vertices in Σ, contradicting the fact that $|\Sigma| < 6$. □

Lemma 22 $vd(C_6 \times C_6) \geq 6$.

Proof: Suppose, by contradiction, that $vd(C_6 \times C_6) < 6$. Let Σ be a set of vertices with size $|\Sigma| = 5$ defining a planar graph $G = C_6 \times C_6 - \Sigma$. By Lemma 21, Σ has no pair of vertices in a same meridian or parallel. Then, as $C_6 \times C_6$ has six parallels and six meridians, we assume parallel 4 and meridian 4 to be, respectively, the parallel and the meridian of $C_6 \times C_6$ with no vertex in Σ as depicted in bold edges in Figure 18(a).

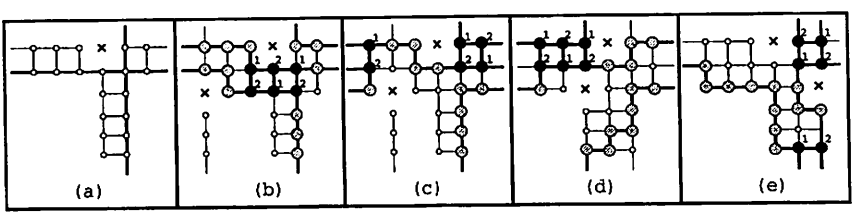

Figure 18: An obstruction for Lemma 22.

We claim that at least one of the vertices: v_{33}, v_{35}, v_{53}, v_{55} is a vertex of Σ. Otherwise, these four vertices plus vertices in meridian 4 and parallel 4 induce a subdivision for K_5 in the planar graph $G = C_6 \times C_6 - \Sigma$. We consider a suitable automorphism and assume that v_{35} is in Σ as shown in Figure 18(a).

As v_{35} is in Σ and as $vd(C_6 \times C_6) = 5 < 6$, by Lemma 21 the vertex of Σ in meridian 3 is v_{35} and the vertex of Σ in parallel 5 is v_{35}. The remaining vertices of meridian 3 and parallel 5, plus vertices in meridian 4 and parallel 4 induce in the planar graph G, the subgraph of the Figure 18(a).

Now we consider the vertex of Σ in parallel 3. In Figures 18(b), 18(c), 18(d) and 18(e) we examine each possibility for vertex $v_{i3}, i \in \{0,1,2,5\}$ of the parallel 3 in Σ. Note that, for each case there is a subdivision for $K_{3,3}$ as subgraph of the planar graph $G = C_6 \times C_6 - \Sigma$, a contradiction. Thus, $vd(C_6 \times C_6) \geq 6$. □

Lemma 23 *If $k \geq 6$, then $vd(C_k \times C_k) \geq k$.*

Proof: We prove this assertion by induction in k. The induction basis is the graph $C_6 \times C_6$. The induction hypothesis is $vd(C_{k-1} \times C_{k-1}) \geq k-1$, where $6 \leq k-1$. Since $k \geq 4$, it follows from Fact 7 that $vd(C_k \times C_k) \geq k$. □

Corollary 9 *If $n, m \geq 6$, then $vd(C_n \times C_m) \geq \min\{n, m\}$.*

Proof: It follows from Corollary 4 and Lemma 23. □

Theorem 1 *The vertex deletion of $C_n \times C_m$ is given by $vd(C_3 \times C_3) = 1$; $vd(C_3 \times C_4) = vd(C_3 \times C_5) = vd(C_3 \times C_6) = 2$; if $m \geq 7$, then $vd(C_3 \times C_m) = 3$; $vd(C_4 \times C_4) = 2$; $vd(C_4 \times C_5) = 3$; if $m \geq 6$, then $vd(C_4 \times C_m) = 4$; $vd(C_5 \times C_5) = 4$; if $m \geq 6$, then $vd(C_5 \times C_m) = 5$; and if $n, m \geq 6$, then ; $vd(C_n \times C_m) = \min\{n, m\}$.*

Proof: The assertion follows from Lemmas 4, 5, 6, Corollary 5, Corollary 6, Lemma 8, Corollary 7, Lemma 18, Corollary 8 and Corollary 9. □

5 Conclusion

In this work, we have determined the exact values of the vertex deletion for all $C_n \times C_m$ graphs. We observe that the results of Yannakakis [27] and Lund and Yannakakis [20] which proved, respectively, that VERTEX DELETION decision and optimization versions for graphs in general are NP-complete and Max SNP-hard problems, left open the smallest maximum allowed degree for an instance which insures NP-completeness and Max SNP-hardness. In [11] we consider the complexity of computing this parameter and we answer these questions by proving that VERTEX DELETION is NP-complete and Max SNP-hard even for cubic graphs.

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