New Classes of Combinatorial Identities

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Abstract

Using generating functions of the author ([1], [2]), we obtain three infinite classes of combinatorial identities involving partitions with "$n+t$ copies of n" introduced by the author and G.E. Andrews [3], and lattice paths studied by the author and D.M. Bressoud [4].

Key words and phrases: Partitions with "$n+t$ copies of n", weighted differences, lattice paths, generating functions, combinatorial identities.

1 Introduction, definitions and the main results

We shall prove three generalized theorems. Each of these theorems gives us infinitely many combinatorial identities. First, we recall the following definitions of partitions with "$n+t$ copies of n" and weighted differences from [3]:

Definition 1.1. A partition with "$n+t$ copies of n", $t \geq 0$, is a partition in which a part of size n, $n \geq 0$, can come in $n+t$ different colors denoted by subscripts: $n_1, n_2, \ldots, n_{n+t}$. Thus, for example, the partitions of 2 with "$n+1$ copies of n" are

$$\begin{array}{llll} 2_1 & 2_1+0_1, & 1_1+1_1, & 1_1+1_1+0_1, \\ 2_2, & 2_2+0_1, & 1_2+1_1, & 1_2+1_1+0_1, \\ 2_3, & 2_3+0_1, & 1_2+1_2, & 1_2+1_2+0_1. \end{array}$$

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Note that zeros are permitted if and only if t is greater than or equal to one. Also, in no partition are zeros permitted to repeat.

Definition 1.2. The weighted difference of two elements m_i, n_j, $m \geq n$ is defined by $m - n - i - j$ and is denoted by $((m_i - n_j))$.

Next, we recall the following description of lattice paths from [4] which we shall be considering in this paper:

All paths will be of infinite length lying in the first quadrant. Only three moves are allowed at each step:

northeast: from (i, j) to $(i+1, j+1)$
southeast: from (i, j) to $(i+1, j-1)$, only allowed if $j > 0$
horizontal: from $(i, 0)$ to $(i+1, 0)$, only allowed along x-axis

The following terminology will be used in describing lattice paths.

Peak: Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.

Valley: A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.

Mountain: A section of the path which starts on either the x- or y-axis, which ends on the x-axis and which does not touch the x-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.

Plain: A section of path consisting of only horizontal steps which starts either on the y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.

The **Height** of a vertex is its y-coordinate. The **Weight** of a vertex is its x-coordinate. The **Weight of a Path** is the sum of the weights of its peaks.

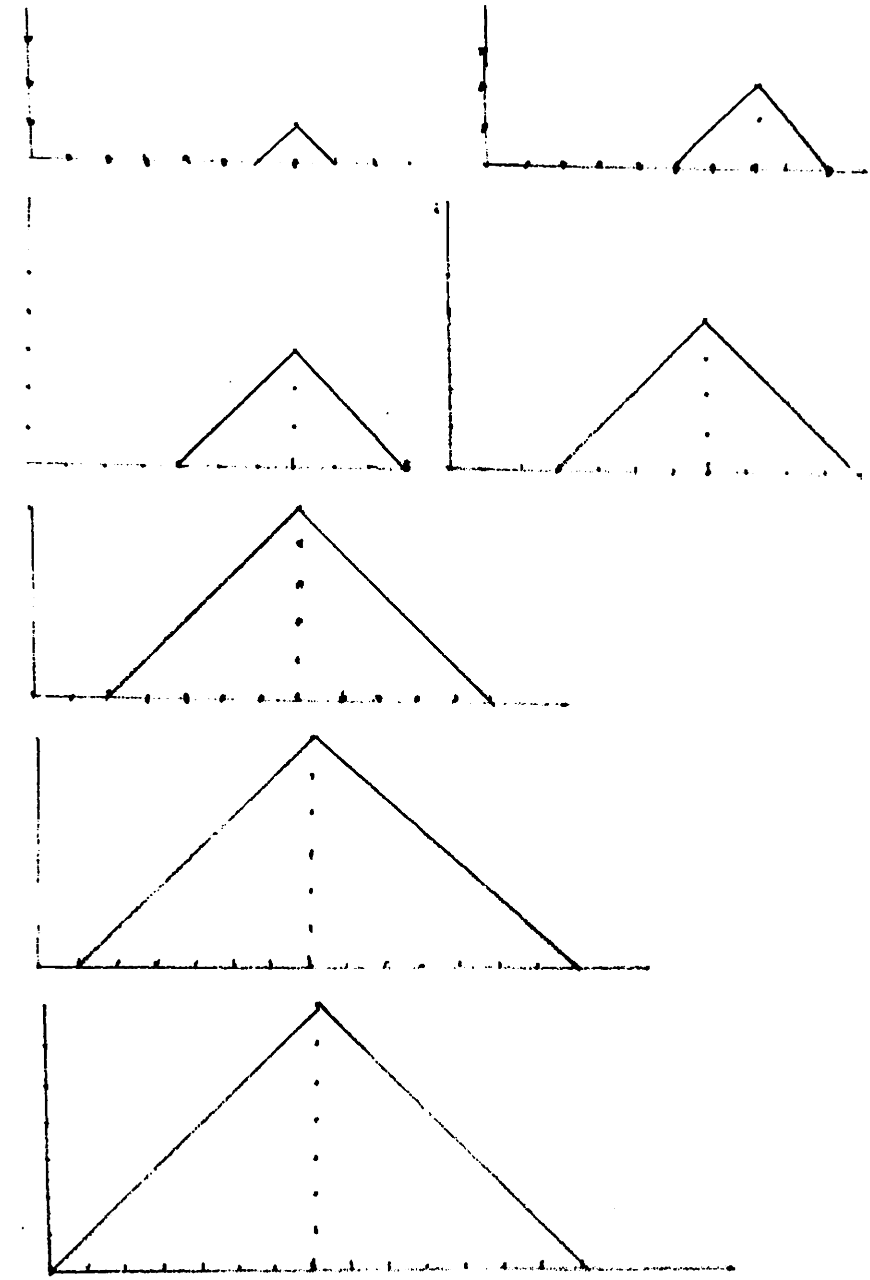
We shall prove the following theorems:

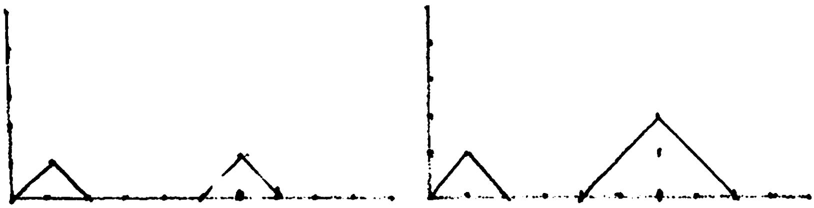
Theorem 1. *For $k \geq -1$, let $A_1^k(\nu)$ denote the number of lattice paths of weight ν which start at a point on the x-axis have no valley above height 0 if $k = -1$ and no valleys at all if $k \geq 0$, and there is a plain of minimal length $k+1$ $(k \geq 0)$ between any two mountains. Let $B_1^k(\nu)$ denote the number of partitions of ν with "n copies of n" such that the weighted difference of each pair of parts is greater than k. Then $A_1^k(\nu) = B_1^k(\nu)$, for all ν.*

Example. Consider the case when $\nu = 7$, $k = 1$.

$B_1^1(7) = 9$, since the relevant partitions of 7 with "n copies of n" are

$7_1, 7_2, 7_3, 7_4, 7_5, 7_6, 7_7, 6_1 + 1_1, 6_2 + 1_1$. Also, $A_1^1(7) = 9$, since the relevant lattice paths of weight 7 are:





Theorem 2. *For $k \geq -1$, let $A_2^k(\nu)$ denote the number of lattice paths of weight ν which start from $(0,1)$ have no valley above height 0 if $k = -1$, no valleys at all if $k > -1$ and there is a plain with minimum length $k+1$ $(k \geq 0)$ between any two mountains. Let $B_2^k(\nu)$ denote the number of partitions of ν with "$n+1$ copies of n" such that the weighted difference of each pair of parts is greater than k, and for some i, i_{i+1} is a part. Then $A_2^k(\nu) = B_2^k(\nu)$, for all ν.*

Theorem 3. *For $k \geq -1$, let $A_3^k(\nu)$ denote the number of lattice paths of ν which start from $(0,2)$, have no valley above height 0 if $k = -1$ and no valleys at all if $k > -1$ and there is a plain with minimal length $k+1$ between any two mountains. Let $B_3^k(\nu)$ denote the number of partitions of ν with "$n+2$ copies of n" such that the weighted difference of each pair of parts is greater than k, and for some i, i_{i+2} is a part. Then $A_3^k(\nu) = B_3^k(\nu)$, for all ν.*

In the proofs of Theorems 1-3, we shall make use of the following generating functions from [1,2]:

$$\sum_{\nu=0}^{\infty} B_1^k(\nu) q^\nu = \sum_{m=0}^{\infty} \frac{q^{m[1+\frac{(k+3)(m-1)}{2}]}}{(q;q)_m (q;q^2)_m}, \tag{1}$$

$$\sum_{\nu=0}^{\infty} B_2^k(\nu) q^\nu = \sum_{m=0}^{\infty} \frac{q^{\frac{m(m+1)(k+3)}{2}}}{(q;q)_m (q;q^2)_{m+1}}, \tag{2}$$

$$\sum_{\nu=0}^{\infty} B_3^k(\nu) q^\nu = \sum_{\nu=0}^{\infty} \frac{q^{m[1+(m+1)(k+3)/2]}}{(q;q)_m (q;q^2)_{m+1}}, \tag{3}$$

where $|q| < 1$ and for any complex constant a,

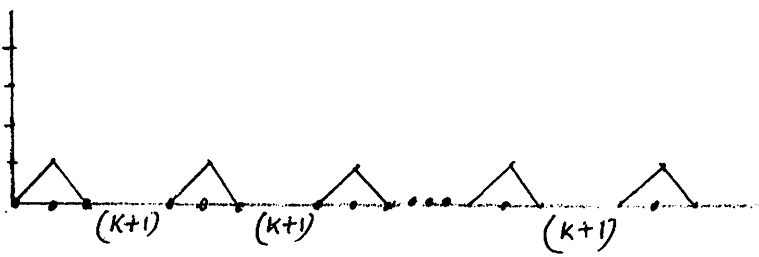
$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+1})}.$$

We shall provide two different proofs of Theorem 1-3. In the first proof we shall show that for $1 \leq i \leq 3$ the right-hand side of Equation (1) also generates $A_i^k(\nu)$, while the second proof is bijective.

In Section 2 we illustrate the method of proof by proving Theorem 1 completely. In Section 3 we sketch the proofs of the other two theorems. We remark that Equations (1)-(3) are also valid for $k = -2$ and -3. But $A_i^{-2}(\nu)$ and $A_i^{-3}(\nu)$ $(1 \leq i \leq 3)$ are not defined.

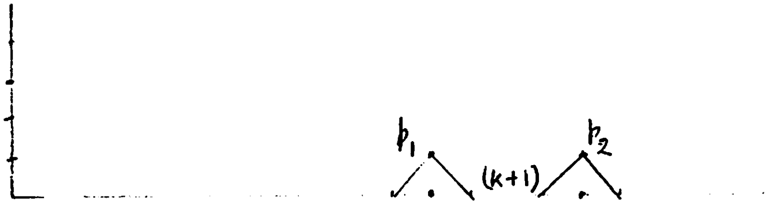
2 Proof of Theorem 1

In $\frac{q^{m[1+\frac{(k+3)(m-1)}{2}]}}{(q;q)_m(q;q^2)_m}$ the factor $q^{m[1+\frac{(k+3)(m-1)}{2}]}$ generates a lattice path from $(0,0)$ to $(2m+(m-1)(k+1),0)$ having m peaks each of height 1 and a plain of length $k+1$ between any two successive peaks. Thus the path begins as



Graph A

In the above graph we consider two successive peaks, say, ith and $(i+1)st$ and denote them by p_1 and p_2, respectively.



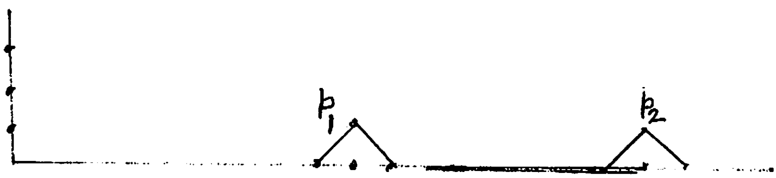
Graph B

Clearly in Graph B

$$\begin{aligned} p_1 &= ((2i-1)+(i-1)(k+1),1), \text{ and} \\ p_2 &= ((2i+1)+i(k+1),1). \end{aligned}$$

The factor $\frac{1}{(q;q)_m}$ generates m non-negative integers, say $a_1 \geq a_2 \geq \ldots \geq a_m \geq 0$, which are encoded by inserting a_m horizontal steps in front of the first mountain and $a_i - a_{i+1}$ horizontal steps in front of the $(m-i+1)st$ mountain, $1 \leq i \leq m-1$.

Thus, the x-coordinate of the ith peak is increased by $a_m + (a_{m-1} - a_m) + (a_{m-2} - a_{m-1}) + \ldots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$, and the x-coordinate of the $(i+1)st$ peak is increased by a_{m-i}. Graph B now becomes Graph C.

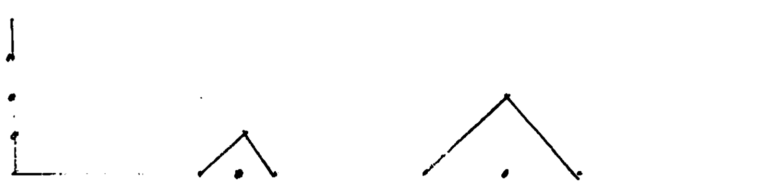


Graph C

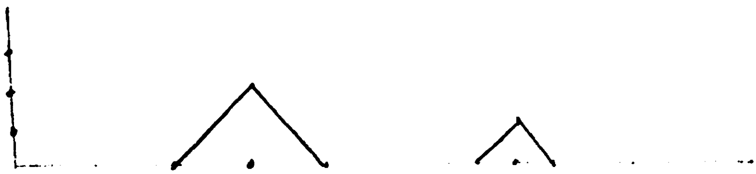
In Graph C,

$$
\begin{aligned}
p_1 &= ((2i-1) + (i-1)(k+1) + a_{m-i+1}, 1), \text{ and} \\
p_2 &= ((2i+1) + i(k+1) + a_{m-i}, 1).
\end{aligned}
$$

The factor $\frac{1}{(q;q^2)_m}$ generates non-negative multiples of $(2i-1)$, $1 \leq i \leq m$, say $b_1 \times 1, b_2 \times 3, \ldots, b_m \times (2m-1)$. This is encoded by having the ith peak grow to height $b_{m-i+1} + 1$. Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two. Graph C now changes to Graph D or to Graph E depending on whether $b_{m-i} > b_{m-i+1}$ or $b_{m-i} < b_{m-i+1}$. Note that if $b_{m-i} = b_{m-i+1}$ then the new graph will look like Graph C.



Graph D



Graph E

In Graph D (or Graph E),

$$\begin{aligned} p_1 &= ((2i-1)+(i-1)(k+1)+a_{m-i+1}+2(b_m+\ldots+b_{m-i+2}) \\ &\quad +b_{m-i+1}, b_{m-i+1}+1), \\ p_2 &= ((2i+1)+i(k+1)+a_{m-i}+2(b_m+\ldots+b_{m-i+1})+b_{m-i}, b_{m-i}+1). \end{aligned}$$

We see that every lattice path enumerated by $A_1^k(\nu)$ is uniquely generated in this manner. This proves that the right-hand side of Equation (1) also generates $A_1^k(\nu)$.

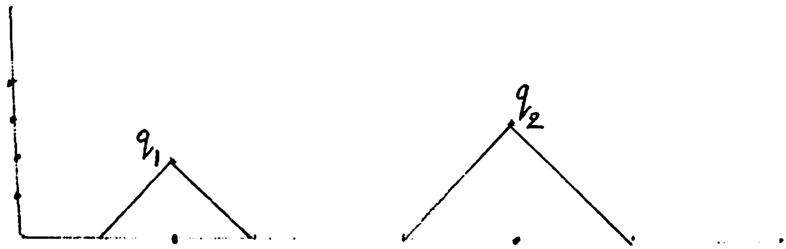
We now establish a $1-1$ correspondence between the lattice path enumerated by $A_1^k(\nu)$ and the partitions with "n copies of n" enumerated by $B_1^k(\nu)$. We do this by encoding each path as the sequence of the weights of the peaks with each weight subscripted by the height of the respected peak. Thus, if we denote the two peaks in Graph D (or Graph E) by A_x and B_y, respectively, then

$$\begin{aligned} A &= (2i-1)+(i-1)(k+1)+a_{m-i+1}+2(b_m+\ldots+b_{m-i+2})+b_{m-i+1} \\ x &= b_{m-i+1}+1 \\ B &= 2i+i(k+1)+a_{m-i}+2(b_m+\ldots+b_{m-i+1})+b_{m-i} \\ y &= b_{m-i}+1 \end{aligned}$$

The weighted difference of these two parts is

$$((B_y - A_x)) = B - A - x - y = k+1+(a_{m-i}-a_{m-i+1}) > k.$$

To see the reverse implication we consider two parts of a partition enumerated by $B_1^k(\nu)$, say C_u and D_v. Let $q_1 \equiv (C,u)$ and $q_2 \equiv (D,v)$ be the corresponding peaks in the associated lattice path.

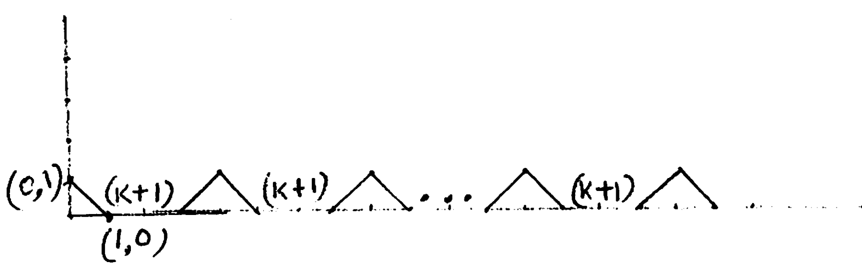


Graph F

The length of the plain between the two peaks is $D - C - u - v$ which is the weighted difference between the two parts C_u and D_v and is hence greater than k. This shows that if $k = -1$ then there are no valleys above height 0 and if $k \geq 0$ then there is a plain of minimal length $k+1$ between any two mountains.

3 Sketch of the proofs of Theorems 2-3

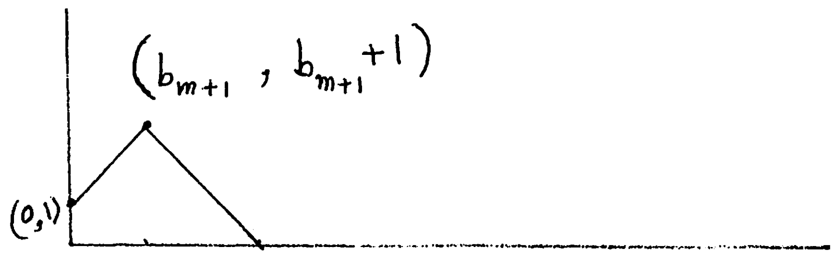
Proof of Theorem 2. The extra factor of $(k+2)m$ puts a southeast step from $(0,1)$ to $(1,0)$ followed by a plain of length $k+1$ at the front of lattice path. So in this case the path begins with $(m+1)$ peaks starting from $(0,1)$ and ending at $(1+(k+3)m, 0)$ and with a plain of length $k+1$ between two successive peaks.



Graph G

The m non-negative integers $a_1 \geq a_2 \geq \ldots \geq a_m \geq 0$ generated by the factor $\frac{1}{(q;q)_m}$ are encoded by inserting a_m horizontal steps in front of the second mountain and $a_i - a_{i+1}$ horizontal steps in front of the $(m-i+2)nd$

mountain $1 \le i \le m-1$. The m non-negative multiples of $(2i-1)$, $1 \le i \le m$, say $b_1 \times 1, b_2 \times 3, \ldots, b_m \times (2m-1)$ generated by $\frac{1}{(q;q^2)_m}$ are encoded by having the *ith* peak grow to height $b_{m-i+2}+1$, $2 \le i \le m+1$. Also, the extra factor of $(1-q^{2m+1})$ introduces a non-negative multiple of $(2m+1)$, say $b_{m+1} * (2m+1)$. This is encoded by having the first peak grow to height $b_{m+1}+1$ in the northeast direction. Considering these changes in the Graph D (or Graph E) we see that the *ith* and $(i+1)st$ peaks now become $(i+1)st$ and $(i+2)nd$ peaks, respectively, and the first peak looks like



Graph H

So if we denote the $(i+1)st$ and $(i+2)nd$ peaks by (A, x) and (B, y), respectively, then

$$\begin{aligned}
A &= 2b_{m+1} + 2(i-1) + (i-1)(k+1) + a_{m-i+1} + 2(b_m + b_{m-1} + \ldots \\
&\quad + b_{m-i+2}) + b_{m-i+1} \\
x &= b_{m-i+1} + 1 \\
B &= 2b_{m+1} + 2i + i(k+1) + a_{m-i} + 2(b_m + \ldots + b_{m-i+1}) + b_{m-i} \\
y &= b_{m-i} + 1
\end{aligned}$$

The weighted difference of the corresponding colored parts is $((B_y - A_x)) = (k+1) + (a_{m-i} - a_{m-i+1}) > k$. The first part is $(b_{m+1})_{b_{m+1}} + 1$ which is of the form i_{i+1} and shows that we are using $n+1$ copies of n.

Proof of Theorem 3. In this case we have an extra factor of q^m than in the previous case. This is encoded by putting two southeast steps $(0,2)$ to $(1,1)$ and $(1,1)$ to $(2,0)$. So in this case the path begins with $(m+1)$ peaks starting from $(0,2)$ and ending at $(2+(k+3)m, 0)$ and with a plain of length $(k+1)$ between every two successive peaks. In this case the non-negative multiple of $(2m+1)$, say, b_{m+1} is encoded by having the first peak grow to height $b_{m+1}+2$ in the northeast direction. So if we denote the $(i+1)st$

and $(i+2)nd$ peaks by (A,x) and (B,y) respectively, then

$$\begin{aligned} A &= 2b_{m+1}+2(i-1)+1+(i-1)(k+1)+a_{m-i+1}+2(b_m+b_{m-1} \\ &\quad +\ldots+b_{m-i+2})+b_{m-i+1} \\ x &= b_{m-i+1}+1 \\ B &= 2b_{m+1}+2i+1+i(k+1)+a_{m-i}+2(b_m+\ldots+b_{m-i+2})+b_{m-i} \\ y &= b_{m-i}+1 \end{aligned}$$

The weighted difference of the colored parts A_x and B_y is $((B_y-A_x)) = B-A-x-y = k+1+a_{m-i}-a_{m-i+1} > k$. The first part is $(b_{m+1})_{b_{m+1}+2}$, which is of the form i_{i+2} and shows that we are using $n+2$ copies of n.

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