Every abelian group of odd order has a narcissistic terrace

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Abstract

A recent series of papers by Anderson and Preece has looked at half-and-half terraces for cyclic groups of odd order, particularly focusing on those terraces which are narcissistic. We give a new direct product construction for half-and-half terraces which allows us to construct a narcissistic terrace for every abelian group of odd order. We also show that infinitely many non-abelian groups have narcissistic terraces.

1 Introduction

Let G be a multiplicatively written group of order n with identity element e. Let **a** be the arrangement $(a_1, a_2, \dots, a_n)$ of the elements of G and let $\mathbf{b} = (b_1, b_2, \dots, b_{n-1})$, where $b_i = a_i^{-1}a_{i+1}$ for $1 \leq i \leq n-1$. If **b** contains each involution of G exactly once, and for each $g \in G$ with $g^2 \neq e$ the sequence **b** contains:

- two occurrences of g and no occurrences of g^{-1},
- one occurrence of g and one occurrence of g^{-1} or
- no occurrences of g and two occurrences of g^{-1},

then **a** is a *terrace* for G and **b** is a *2-sequencing* of G. A group that has a terrace is called *terraced.*

Terraces were defined by Bailey [10] for use in the construction of quasi-complete latin squares. Prior to this, Williams [17] had implicitly used them

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for cyclic groups for a similar purpose. Bailey [10] showed that elementary abelian 2-groups of order greater than 2 are not terraced and conjectured that these were the only groups that do not have terraces. In [10] it was also shown that all abelian groups of odd order are terraced. We give an alternative proof of this in Section 2.

Let a be the terrace $(a_1, a_2, \ldots, a_n)$ for a group G, with 2-sequencing $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$. If $a_1 = e$ then the terrace is *basic* [10]. Multiplying each element of a on the left by a_1^{-1} gives a basic terrace.

Suppose that n is odd, say $n = 2m+1$. If the sequences $(b_1, b_2, \ldots, b_m)$ and $(b_{m+1}, b_{m+2}, \ldots, b_{2m})$ both contain exactly one occurrence from the set $\{g, g^{-1}\}$ for each $g \in G$ then the terrace a is called *half-and-half* [6]. If b is the same as its reverse, in which case a is necessarily half-and-half, then b is *reflective* [2] and a is *narcissistic* [7]. Anderson and Preece introduced half-and-half terraces for use in the construction of a particular type of carry-over design [6]. In later papers [7, 8, 9] they construct elegant half-and-half terraces for cyclic groups, many of these being narcissistic.

We give two examples of half-and-half terraces for cyclic groups from the literature. We denote the cyclic group of order n by $\mathbb{Z}_n$ and write it additively. The sequence

$$(0, 1, n-1, 2, n-2, \ldots)$$

is the *Lucas-Walecki-Williams* terrace, or *LWW* terrace, for $\mathbb{Z}_n$, so named [11] as it was implicitly used for even n by Lucas, who gave credit to Walecki, in [13] and for both even and odd n by Williams [17]. For odd n, say $n = 2m+1$, the LWW terrace is narcissistic, with 2-sequencing

$$(1, 2m-1, 3, 2m-3, \ldots, 2m-1, 1).$$

For our second example—the *triangular numbers* terrace—we require that n is an odd prime. Define $\tau_i = \frac{i(i+1)}{2}$, so τ_i is the ith triangular number. Take r to be a non-square element of $\mathbb{Z}_n$. Set

$$\Delta_r(n) = (0, \tau_1, \tau_2, \ldots, \tau_m, \tau_m - rm, \tau_m - rm - r(m-1), \ldots, \tau_m - \frac{rm(m+1)}{2}).$$

Then $\Delta_r(n)$ is a terrace with 2-sequencing

$$(1, 2, 3, \ldots, m, -mr, -(m-1)r, \ldots, -r).$$

The terrace was first given in its full generality by B. A. Anderson [3, 4], though the special case of $n \equiv 3 \pmod 4$ and $r = -1$ was used by Williams [17]. The terrace is always half-and-half and when $n \equiv 3 \pmod 4$ the terrace $\Delta_{-1}(n)$ is narcissistic.

In the next section we show that all abelian groups of odd order have narcissistic terraces (and hence have half-and-half terraces). In Section 3 we show that infinitely many non-abelian groups have narcissistic terraces.

2 The construction

The construction in Theorem 1 takes terraces for two groups of odd order and gives a terrace for their direct product. It is this construction which allows us to build the terraces we want.

Theorem 1 *Let G and H be groups of odd order. Let G have terrace* **a** *and H have half-and-half terrace* **c**. *Then $G \times H$ is terraced.*

Proof: Let G have order $2l+1$ and let **a** have 2-sequencing **b**, given by $\mathbf{a} = (a_1, a_2, \ldots, a_{2l+1})$ and $\mathbf{b} = (b_1, b_2, \ldots, b_{2l})$. Let H have order $2m+1$ and let **c** have 2-sequencing **d**, given by $\mathbf{c} = (c_1, c_2, \ldots, c_{2m+1})$ and $\mathbf{d} = (d_1, d_2, \ldots, d_{2m})$.

We represent the elements of $G \times H$ by a $(2l+1) \times (2m+1)$ array of points, where the point in the ith row and jth column represents the element (a_i, c_j). We then consider a particular directed Hamiltonian path through these points and show that the sequence of vertices (in order, starting at (a_1, c_1)) of this path is a terrace for $G \times H$.

Figure 1 gives the directed Hamiltonian path for $l = 2$ and $m = 3$; it generalises in the natural way for other values of l and m.

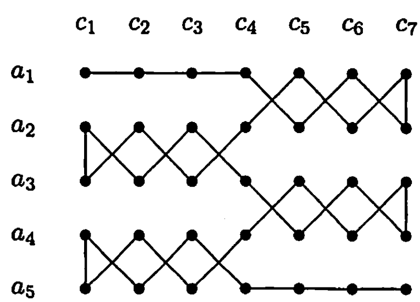

Figure 1: The Hamiltonian path through $G \times H$

We need to check that the "2-sequencing" associated with this path is indeed a 2-sequencing.

Let e and f be the identity elements of G and H respectively. Elements of the "2-sequencing" that are of the form (e, h) are given by the horizontal edges. In fact, the $2m$ horizontal edges yield the elements (e, d_j) for $1 \leq j \leq 2m$; these elements have the correct properties as **c** is a terrace.

Similarly, elements of the form (g, f) are given by the vertical edges. In fact, the $2l$ vertical edges give the elements $(b_i^{\pm 1}, f)$ for $1 \leq i \leq 2l$; these elements have the correct properties as **a** is a terrace.

Now, fix $g \in G \setminus \{e\}$ and $h \in H \setminus \{f\}$ and consider elements of the form $(g^{\pm 1}, h^{\pm 1})$. There are exactly two pairs (i, j) such that there is a edge between (a_i, c_j) and (a_{i+1}, c_{j+1}), where $b_i = g^{\pm 1}$ and $d_i = h^{\pm 1}$. The pair of edges from (a_i, c_j) to (a_{i+1}, c_{j+1}) and from (a_{i+1}, c_j) to (a_i, c_{j+1}) are traversed in one of the following ways:

- south-east and south-west,
- north-east and north-west.

So in both cases we pick up exactly one from each pair $\{(g, h), (g^{-1}, h^{-1})\}$ and $\{(g^{-1}, h), (g, h^{-1})\}$. Thus the diagonal edges give us exactly the elements we need. So our Hamiltonian path gives a terrace for $G \times H$. □

B. A. Anderson [5] proved a similar theorem, but required that one of the terraces was "starter-translate" rather than half-and-half.

We now consider what happens when the terraces for G and H in Theorem 1 are half-and-half or indeed narcissistic.

Theorem 2 *Let G and H be groups which have half-and-half terraces. Then $G \times H$ has a half-and-half terrace.*

Proof: Let G have order $2l + 1$ and half-and-half terrace **a**. Let H have order $2m + 1$ and half-and-half terrace **c**. Let $(2l + 1)(2m + 1) = 2n + 1$. Apply Theorem 1 to give a terrace **e** for $G \times H$ whose 2-sequencing **f** is $(f_1, f_2, \ldots, f_{2n})$.

Take k, with $1 \leq k \leq n$, and let $f_k = (g, h)$. Then $f_{k'} = (g, h)^{\pm 1}$ for some k' with $n+1 \leq k' \leq 2n$. This is true when $g = e$ as **c** is half-and-half. It is true for other values of g as **a** is half-and-half. □

Corollary 3 *Every abelian group of odd order has a half-and-half terrace.*

Proof: It is well-known that any abelian group may be written as a direct product of cyclic groups [16, chapter 4]. Cyclic groups of every odd order have at least one half-and-half terrace—for example, the LWW terrace—and so the result follows from repeated applications of Theorem 1. □

Theorem 4 *Let G and H be groups which have narcissistic terraces. Then $G \times H$ has a narcissistic terrace.*

Proof: Let G have order $2l + 1$ and narcissistic terrace **a**. Let H have order $2m + 1$ and narcissistic terrace **c**. Let $(2l + 1)(2m + 1) = 2n + 1$. Apply Theorem 1 to give a terrace **e** for $G \times H$ whose 2-sequencing **f** is $(f_1, f_2, \ldots, f_{2n})$.

Take k, with $1 \leq k \leq n$ and let $f_k = (g, h)$. Then $f_{2n+1-k} = (g, h)$. This follows from the narcissism of **a** and **c** and the rotational symmetry of the Hamiltonian path. □

Corollary 5 *All abelian groups of odd order have narcissistic terraces.*

Proof: As in the proof of Corollary 3, we may write an abelian group of odd order as a direct product of cyclic groups. Each cyclic group of odd order has at least one narcissistic terrace; for example the LWW terrace. Repeated applications of Theorem 4 give the result. □

Example 6 *The (narcissistic) LWW terrace for* $\mathbb{Z}_5$ *is* $(0, 1, 4, 2, 3)$. *Applying Theorem 1 gives the following narcissistic terrace for* $\mathbb{Z}_5 \times \mathbb{Z}_5$ *(where we have omitted brackets and commas from the direct product notation):*

$$(00, 01, 04, 12, 03, 13, 02, 14, 41, 10, 40, 11, 44, \\ 22, 43, 23, 42, 24, 31, 20, 30, 21, 34, 32, 33)$$

which has reflective 2*-sequencing*

$$(01, 03, 13, 41, 10, 44, 12, 32, 24, 30, 21, 33, \\ 33, 21, 30, 24, 32, 12, 44, 10, 41, 13, 03, 01).$$

If p and q are coprime then $\mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$. So Theorem 1 allows us to construct many half-and-half (and narcissistic) terraces for cyclic groups of non-prime-power order.

Example 7 *Consider* $\mathbb{Z}_7 \times \mathbb{Z}_3$. *Taking* $r = 5$ *we get the triangular numbers terrace* $(0, 1, 3, 6, 5, 2, 4)$ *for* $\mathbb{Z}_7$. *The LWW terrace for* $\mathbb{Z}_3$ *is* $(0, 1, 2)$. *Applying Theorem 1 to these terraces with* $G = \mathbb{Z}_7$ *and* $H = \mathbb{Z}_3$ *gives the following half-and-half terrace for* $\mathbb{Z}_7 \times \mathbb{Z}_3$ *(again omitting brackets and commas):*

$$(00, 01, 12, 02, 11, 30, 10, 31, 62, 32, 61, 50, 60, 51, 22, 52, 21, 40, 20, 41, 42).$$

The associated 2*-sequencing is*

$$(01, 11, 60, 12, 22, 50, 21, 31, 40, 32, 62, 10, 61, 41, 30, 42, 22, 50, 21, 01).$$

3 Non-abelian groups

Let G_{21} be the non-abelian group of order 21:

$$G_{21} := \langle u, v : u^7 = e = v^3, vu = u^4 v \rangle.$$

The following terrace is a half-and-half terrace for G_{21}:

$$(e, u, u^4, v^2, u^5, uv, u^3v^2, u^3, uv^2, u^4v^2, u^5v,$$
$$u^6v, u^5v^2, u^2, u^2v^2, u^4v, v, u^2v, u^6v^2, u^3v, u^6).$$

The associated 2-sequencing is

$$(u, u^3, u^3v^2, u^6v, u^3v, u^4v, v, u^5v^2, u^5, u^4v^2,$$
$$u^2, u^5v, u^2v, v^2, uv^2, u^6, u^4, uv, u^2v^2, u^6v^2).$$

Note that every non-identity element of G_{21} occurs exactly once in the 2-sequencing. A terrace whose 2-sequencing has this property is called *directed*, see [14] for a survey of results concerning directed terraces.

The following terrace is a narcissistic terrace for G_{21}:

$$(e, v, u^5v, u, u^2, u^4v, u^2v^2, u^6v, u^3v^2, v^2, u^3,$$
$$uv, u^2v, u^6v^2, u^3v, uv^2, u^5, u^6, u^5v^2, u^4v^2, u^4).$$

The associated 2-sequencing is

$$(v, u^3, u^6v^2, u, u^2v, u^3v, u^2v^2, uv, u^2, u^5v,$$
$$u^5v, u^2, uv, u^2v^2, u^3v, u^2v, u, u^6v^2, u^3, v).$$

These terraces were found using a heuristic algorithm implemented in GAP [12]. The algorithm was closely based on the ideas of [1] and [15, chapter 8].

Proposition 8 *Infinitely many non-abelian groups have half-and-half terraces. Moreover, infinitely many non-abelian groups have narcissistic terraces.*

Proof: We can find a half-and-half terrace for G_{21}^k, for any positive integer k, by applying Theorem 2 $(k-1)$ times, using one of the above half-and-half terraces for G_{21}. If we use the second of the two terraces then our half-and-half terrace for G_{21} is narcissistic (by Theorem 4). □

We conclude with three questions:

Question 1 *Does every non-abelian group of odd order have a half-and-half terrace?*

Question 2 *Does every non-abelian group of odd order have a directed half-and-half terrace?*

Question 3 *Does every non-abelian group of odd order have a narcissistic terrace?*

With regard to Question 2 we observe that it is not even known whether every non-abelian group of odd order has a directed terrace.

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