Trees with Equal Domination and Paired-domination Numbers

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Abstract

A paired-dominating set of a graph G is a dominating set of vertices whose induced subgraph has a perfect matching. The paired-domination number of G is the minimum cardinality of a paired-dominating set of G, and is obviously bounded below by the domination number of G. We give a constructive characterization of the trees with equal domination and paired-domination numbers.

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1 Introduction

For a graph $G = (V, E)$, a set S is a *dominating set* if every vertex in $V - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. We call a dominating set of cardinality $\gamma(G)$ a $\gamma(G)$*-set* and use similar notation for other parameters. Domination and its many variations have been surveyed in [7, 8].

We are interested in a variation of domination called *paired-domination* where the dominating set has the additional property that all the vertices in the set can be matched (paired). Paired-domination was introduced by Haynes and Slater in [9, 10] as a model for assigning backups to guards for security purposes. (See also [1, 3, 4, 6, 12]). Formally, a set S is a *paired-dominating set* if it dominates V and the induced subgraph $\langle S \rangle$ contains at least one perfect matching. A *paired-dominating set S with matching M* is a dominating set $S = \{v_1, v_2, \ldots, v_{2t-1}, v_{2t}\}$ with independent edge set $M = \{e_1, e_2, \ldots, e_t\}$, where each edge e_i joins two elements of S, that is, M is a perfect matching (not necessarily induced) in the induced subgraph $\langle S \rangle$. If $v_j v_k = e_i \in M$, we say that v_j and v_k are *paired* in S. The *paired-domination number* $\gamma_{\rm pr}(G)$ is the minimum cardinality of a paired-dominating set of G.

Since every paired-dominating set is a dominating set, $\gamma(G) \leq \gamma_{\rm pr}(G)$ for all graphs G without isolated vertices. We call the graphs obtaining the lower bound $(\gamma, \gamma_{\rm pr})$-graphs. It is our goal in this paper to characterize $(\gamma, \gamma_{\rm pr})$-trees. Recently, Qiao, Kang, Cardei and Ding-Zhu [13] gave a characterization of $(\gamma, \gamma_{\rm pr})$-trees. The characterization we present is a constructive characterization using labelings that is simpler than that presented in [13].

In general we follow the notation and graph theory terminology in [2, 7]. Specifically, a vertex of degree one is called a *leaf* and its neighbor is called a *support* vertex. A subset $S \subseteq V$ is a *packing* in G if the vertices of S are pairwise at distance at least three apart in G. The *packing number* $\rho(G)$ is the maximum cardinality of a packing in G. We will need the following fact from [11].

Theorem 1 (Moon and Meir [11]) *For a tree T, $\gamma(T) = \rho(T)$.*

2 $(\gamma, \gamma_{\rm pr})$-Trees

Our aim in this section is to give a constructive characterization of the trees T having $\gamma(T) = \gamma_{\rm pr}(T)$. The key to our constructive characterization is to find a labeling of the vertices that indicates the roles each vertex plays in the sets associated with both parameters. This idea of labeling the vertices is introduced in [5], where trees with equal domination and independent domination numbers as well as trees with equal domination and total domination numbers are characterized.

We define a **ρ-$\gamma_{\rm pr}$-labeling** of a tree $T = (V, E)$ as a weak partition $S = (S_A, S_B, S_C, S_D)$ of V such that (i) $S_A \cup S_D$ is a $\gamma_{\rm pr}(T)$-set, (ii) $S_C \cup S_D$ is a $\rho(T)$-set, and (iii) $|S_A| = |S_C|$. (The order and labeling are for compatibility with previous work.) We will refer to the pair (T, S) as a **ρ-$\gamma_{\rm pr}$-tree**. The *label* or *status* of a vertex v, denoted sta(v), is the letter $x \in \{A, B, C, D\}$ such that $v \in S_x$.

Lemma 2 *A tree T is a (γ, γ_{pr})-tree if and only if T has a ρ-$\gamma_{\rm pr}$-labeling.*

Proof. Suppose (T, S) is a ρ-$\gamma_{\rm pr}$-tree. Then Theorem 1 implies that $\rho(T) = \gamma(T) \leq \gamma_{pr}(T) = |S_A \cup S_D| = |S_C \cup S_D| = \rho(T)$, and so $\gamma(T) = \gamma_{pr}(T)$, i.e., T is (γ, γ_{pr})-tree. Suppose that T is a (γ, γ_{pr})-tree. Let P be a $\gamma_{pr}(T)$-set and L be a $\rho(T)$-set. Then a ρ-$\gamma_{\rm pr}$-labeling is given by $S_A = P \setminus L$, $S_B = V \setminus (P \cup L)$, $S_C = L \setminus P$, and $S_D = L \cap P$. Since $\rho(T) = \gamma(T) = \gamma_{pr}(T)$, it follows that $|S_A| = |S_C|$. □

We will need the following lemma:

Lemma 3 *Consider a ρ-$\gamma_{\rm pr}$-labeling of a tree T. If $v \in S_A$ (resp. S_C), then v is adjacent to exactly one vertex of S_C (resp. S_A). Moreover, S_D is empty.*

Proof. Since $S_C \cup S_D$ is a packing, a vertex in S_A is adjacent to at most one vertex in $S_C \cup S_D$. Every vertex in S_C is adjacent to at least one vertex in S_A, since it is dominated by $S_A \cup S_D$ and has no neighbor in S_D. Since a vertex in S_A is adjacent to at most one vertex in $S_C \cup S_D$ and $|S_A| = |S_C|$, a vertex in S_C cannot have two neighbors in S_A (otherwise some other vertex in S_C has no neighbor in S_A), and every vertex in S_A is adjacent to a vertex in S_C. In particular, every vertex of S_D has neighbors only in S_B. Thus, $S_D = \emptyset$. □

We now describe a procedure to build ρ-γ_{pr}-trees. By a *labeled* P_4, we shall mean a P_4 with leaves of status C and support vertices of status A.

Let $\mathcal{F}$ be the minimum family of labeled trees that:

(i) contains a labeled P_4; and

(ii) is closed under the four operations $\mathcal{F}_j$ $(j = 1, \ldots, 4)$ listed below, which extend the tree T by attaching a tree to the vertex $v \in V(T)$.

- **Operation $\mathcal{F}_1$.** Attach to a vertex v of status A a vertex of status B.
- **Operation $\mathcal{F}_2$.** Add a labeled P_4 and join a vertex v of status A or B to a support vertex of the P_4.
- **Operation $\mathcal{F}_3$.** Add a labeled P_4 and join a vertex v of status B that has no neighbor of status C to a leaf of the P_4.
- **Operation $\mathcal{F}_4$.** Attach to a vertex v of status B or C a vertex of status B and join that vertex to a support vertex of a labeled P_4.

These operations are illustrated in Figure 1 where by status B^* we mean a status B vertex that has no neighbor of status C.

We are now is a position to characterize ρ-γ_{pr}-trees. The proof of the following result is similar to that presented in [5] to characterize trees with equal domination and total domination numbers.

Theorem 4 *A labeled tree is a ρ-γ_{pr}-tree if and only if it is in $\mathcal{F}$.*

Proof. It is clear that the four operations $\mathcal{F}_i$, $1 \leq i \leq 4$, preserve a ρ-γ_{pr}-labeling, whence every element of $\mathcal{F}$ is a ρ-γ_{pr}-tree.

The proof that every ρ-γ_{pr}-tree (T, S) is in $\mathcal{F}$ is by induction on the order of T. The smallest ρ-γ_{pr}-tree is a labeled P_4 which is indeed in $\mathcal{F}$. So fix a ρ-γ_{pr}-tree (T, S) of order at least 5, and assume that any smaller ρ-γ_{pr}-tree is in $\mathcal{F}$. By Lemma 3, there is no vertex in the set S_D, and so the set S_A is a $\gamma_{pr}(T)$-set while the set S_C is a $\rho(T)$-set. In particular, each vertex of S_B is adjacent to at least one vertex of S_A and at most one vertex of S_C, and each vertex of S_A has degree at least 2 in T. Further, by Lemma 3, the set of edges between the vertices in S_A and S_C induce a perfect matching in $\langle S_A \cup S_C \rangle$.

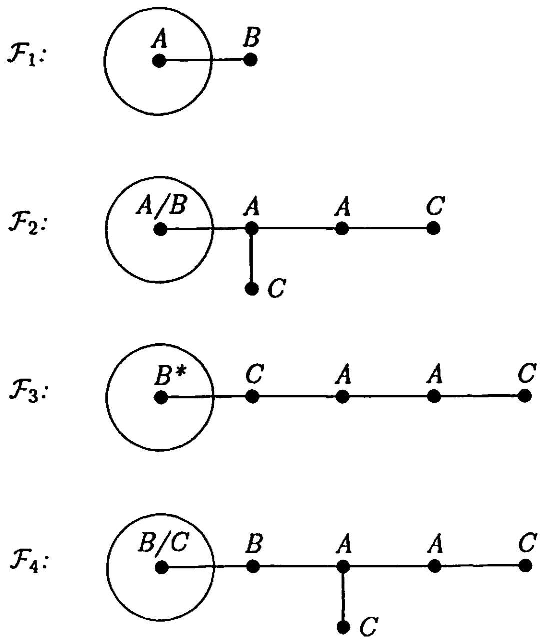

Figure 1: The four $\mathcal{F}_i$ operations.

To complete the proof, we need to identify a set P of vertices that can be pruned to leave a ρ-$\gamma_{\rm pr}$-tree, and an operation $\mathcal{R}$ that restores the pruned vertices.

Since S_A is a paired-dominating set, each vertex is adjacent to a vertex of S_A. In particular, every leaf has status B or C. If there is a leaf u in S_B, then we can take $P = \{u\}$ and $\mathcal{R} = \mathcal{F}_1$. So we may assume that all leaves have status C. Hence all support vertices have status A and only one leaf as a neighbor.

Consider a diametrical path $v_0, v_1, v_2, \ldots, v_{\mathrm{diam}(T)}$. It follows from the above discussion that $\mathrm{diam}(T) \geq 4$ and that $v_0 \in S_C$, $v_1, v_2 \in S_A$ and v_1 has degree 2. We consider three possibilities.

Suppose $v_3 \in S_A$. Then, v_2 has a neighbor v_2' in S_C that is a leaf and v_2 has degree 3. Since S_A is a paired-dominating set, the vertices v_1 and v_2 are paired in $\langle S_A \rangle$, and so the vertex v_3 must have a neighbor, different from v_2, of status A. Thus we can take $P = \{v_0, v_1, v_2, v_2'\}$ and $\mathcal{R} = \mathcal{F}_2$.

Suppose $v_3 \in S_B$. Then, v_2 has a neighbor v_2' in S_C that is a leaf and v_2 has degree 3. If v_3 has a neighbor in S_A different from v_2, then take $P = \{v_0, v_1, v_2, v_2'\}$ and $\mathcal{R} = \mathcal{F}_2$. On the other hand, suppose that v_2 is the only neighbor of v_3 in S_A. Then, v_4 has status B or C. By the diameter constraint, v_3 cannot have another neighbor in S_B, and so v_3 has degree 2. Thus we can take $P = \{v_0, v_1, v_2, v_2', v_3\}$ and $\mathcal{R} = \mathcal{F}_4$.

Suppose $v_3 \in S_C$. Then, $v_4 \in S_B$ and v_2 has degree 2. By the diameter constraint, v_3 cannot have another neighbor in S_B, and so v_3 has degree 2. Since each vertex of S_B is adjacent to at most one vertex of S_C, all neighbors of v_4 other than v_3 have status A or B. Thus we can take $P = \{v_0, v_1, v_2, v_3\}$ and $\mathcal{R} = \mathcal{F}_3$. □

By Theorem 1 and Lemma 2, it follows that:

Corollary 5 *The $(\gamma, \gamma_{\rm pr})$-trees are precisely those trees T such that $(T, S) \in \mathcal{F}$ for some labeling S.*

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