Note on transformation of posets with the same semi bound graphs

Hiroshi ERA
Faculty of Information and Communication
Bunkyo University
Chgasaki 253-0007, Japan
Kenjiro OGAWA
Department of Mathematical Sciences, Tokai University
Hiratsuka 259-1292, Japan
Morimasa TSUCHIYA*
Department of Mathematical Sciences, Tokai University
Hiratsuka 259-1292, Japan
and
Department of Mathematics, MIT
Cambridge MA 02139-4307, USA
e-mail: tsuchiya@ss.u-tokai.ac.jp

Abstract

In this paper, we consider transformations between posets P and Q, whose semi bound graphs are the same. Those posets with the same double canonical posets can be transformed into each other by a finite sequence of two kinds of transformations, called d_additions and d_deletions.

1 Introduction

In this paper, we consider finite undirected simple graphs. For a poset $P = (X, \leq)$, the *semi bound graph* (SB-graph) of $P = (X, \leq)$ is the graph $SB(P) = (X, E_{SB(P)})$, where $uv \in E_{SB(P)}$ if and only if $u \neq v$ and there exists a common lower bound of u and v in P or a common upper bound u and v in P. We introduced this concept and gave characterizations of semi

*Corresponding author

bound graphs in [3]. We already know other kinds of graphs on posets, that is, upper bound graphs and double bound graphs. In [5] and [6] we deal with properties of transformations on upper bound graphs. And we also obtained properties of transformations on double bound graphs in [2] and [7].

Figure 1 shows two different posets which have the common semi bound graphs. This example induces an interest in properties of posets with the same SB-graph: how to obtain any corresponding poset of an SB-graph from any other one in finitely many steps? In this paper, we shall answer this question, introducing two kinds of transformations of such posets.

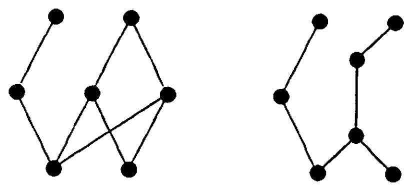

Figure 1: Posets P and Q.

2 Transformations of posets

For a poset $P = (X, \leq)$ and $x \in X$, $L_P(x) = \{y \in X; y < x\}$ and $U_P(x) = \{y \in X; y > x\}$. Furthermore $Max(P)$ is the set of all maximal elements of P, $Min(P)$ is the set of all minimal elements of P.

For a vertex v in G, the *neighborhood* of v is the set of vertices which are adjacent to v in G, and denoted by $N_G(v)$. A *clique* in the graph G is the vertex set of maximal complete subgraph. In some cases, we consider that a clique is a maximal complete subgraph. A family $\mathcal{C}$ of complete subgraphs *edge covers* G if and only if for each edge $uv \in E(G)$, there exists $C \in \mathcal{C}$ such that $u, v \in C$.

For a graph G with fixed two disjoint independent subsets M and N of $V(G)$ and $u \in V(G) - (M \cup N)$, define the sets $U_G(v) = \{u \in M; uv \in E(G)\}$, $L_G(v) = \{u \in N; uv \in E(G)\}$.

Theorem 1. ([6]) *Let G be a graph with n vertices. G is an SB-graph if and only if G has spanning subgraphs H and K satisfying the following conditions:*

(0) $E(G) = E(H) \cup E(K)$,

(1) *there exists a family $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ of cliques of H and disjoint independent subsets M and N such that*

  (a) *$\mathcal{C}$ edge covers H,*

(b) *for each* C_i, *there exist* $m_i \in M$, $n_i \in N$ *such that* $\{m_i, n_i\} \subseteq C_i$ *and* $\{m_i, n_i\} \not\subseteq C_j$ *for all* $i \neq j$, *and*

(c) *for each* $v \in V(H) - (M \cup N)$, $|U_H(v)| \times |L_H(v)|$ *equals the number of elements of* C *containing* v.

(2) $uv \in E(K)$ *if and only if* $M \cap N_H(u) \cap N_H(v) \neq \emptyset$ *and* $N \cap N_H(u) \cap N_H(v) = \emptyset$, *or* $M \cap N_H(u) \cap N_H(v) = \emptyset$ *and* $N \cap N_H(u) \cap N_H(v) \neq \emptyset$. □

For an SB-graph G and an edge clique cover $C = \{C_1, C_2, \ldots, C_n\}$ satisfying the conditions of Theorem 1, M is called an *upper kernel* $UK_{SB}(G)$ of G and N is called a *lower kernel* $LK_{SB}(G)$ of G. In the following sections, we consider a fixed labeled connected SB-graph G with a fixed upper kernel $UK_{SB}(G)$ and a fixed lower kernel $LK_{SB}(G)$. We know that for a corresponding posets P of an SB-graph G, $UK_{SB}(G)$ corresponds to the set $Max(P)$ and $LK_{SB}(G)$ corresponds to the set $Min(P)$.

For an SB-graph G, $\mathcal{P}_{SB}(G) = \{P; SB(P) = G, Max(P) = UK_{SB}(G), Min(P) = LK_{SB}(G)\}$. Each poset P in $\mathcal{P}_{SB}(G)$ is identified with the set of comparability's in P. Thus $\mathcal{P}_{SB}(G)$ is a poset by set inclusions. For a poset P, the *double canonical poset* of P is the poset $d\_can(P) = (V(P), \leq_{d\_can(P)})$, where $x \leq_{d\_can(P)} y$ if and only if (1) $y \in Max(P)$ and $x \leq_P y$, or (2) $x \in Min(P)$ and $x \leq_P y$, or (3) $x = y$.

We classify posets in $\mathcal{P}_{SB}(G)$ according to double canonical posets. That is, we classify $\mathcal{P}_{SB}(G)$ having each class a type $\mathcal{P}_{SB}(G; D) = \{P; SB(P) = G, Max(P) = UK_{SB}(G), Min(P) = LK_{SB}(G), d\_can(P) = D\}$. Each $\mathcal{P}_{SB}(G; D)$ is a subposet of $\mathcal{P}_{SB}(G)$. Note that D is the minimum element of $\mathcal{P}_{SB}(G; D)$, $Max(D) = UK_{SB}(G)$ and $Min(D) = LK_{SB}(G)$.

To consider some relations among elements of $\mathcal{P}_{SB}(G; D)$, we need some concepts as follows: For elements $x \notin Min(P)$ and $y \notin Max(P)$ in a poset P such that x is covered by y, the poset $P^{--}_{x<y}$ is obtained from P by subtracting the relation $x \leq y$ from P, and we call this transformation the *$x < y$-d_deletion*. For an incomparable pair $x \notin Min(P)$ and $y \notin Max(P)$ in a poset P such that $U_P(y) \subseteq U_P(x)$ and $L_P(y) \supseteq L_P(x)$, the poset $P^{++}_{x<y}$ is obtained from P by adding the relation $x \leq y$ to P, and we call this transformation the *$x < y$-d_addition*. We obtain the following facts on these transformations.

Fact 2. *For a poset* P,

(1) P *and* $P^{--}_{x<y}$ *have the same SB-graph,*

(2) P *and* $P^{++}_{x<y}$ *also have the same SB-graph, and*

(3) *$x < y$-d_addition and $x < y$-d_deletion are inverse transformations to each other.* □

By these facts, we obtain the following result.

Theorem 3. *Let G be an SB-graph and P, Q be posets in $\mathcal{P}_{SB}(G;D)$.*

(1) *P can be transformed into Q by a sequence of $x < y$-d_deletions and $x < y$-d_additions.*

(2) *Every poset in $\mathcal{P}_{SB}(G;D)$ is obtained from D by $x < y$-d_additions only.*

Proof. For a poset P, the number of comparable pairs in $P^{--}_{x<y}$ is less than the number of comparable pairs in P. Since P and Q are finite, we obtain the sequences of d_deletions such that

$$P \overset{d\_deletion}{\longrightarrow} \cdots \overset{d\_deletion}{\longrightarrow} d\_can(P),$$

and

$$Q \overset{d\_deletion}{\longrightarrow} \cdots \overset{d\_deletion}{\longrightarrow} d\_can(Q),$$

Since the double canonical posets of P and Q are the same, we obtain the transformations from P to Q by combining the following procedures of d_deletions and d_additions:

$$P \overset{d\_deletion}{\longrightarrow} \cdots \overset{d\_deletion}{\longrightarrow} d\_can(P) = D$$

$$Q \overset{d\_deletion}{\longleftarrow} \cdots \overset{d\_deletion}{\longleftarrow} d\_can(Q) = D.$$

□

For an SB-graph G with an upper kernel $UK_{SB}(G)$ and a lower kernel $LK_{SB}(G)$, G is a *unique SB-graph on the double canonical poset D* if it has only one realizing poset whose double canonical poset is D. By Theorem 3 we get the following result.

Proposition 4. *Let G be an SB-graph with an upper kernel $UK_{SB}(G)$ and a lower kernel $LK_{SB}(G)$. Then the following are equivalent:*

(1) *G is a unique SB-graph on the double canonical poset D,*

(2) $\mathcal{P}_{SB}(G;D) = \{D\}$,

(3) *For all $u, v \in V(G) - (UK_{SB}(G) \cup LK_{SB}(G))$, if $N_{UK}(u) \supseteq N_{UK}(v)$, then $N_{LK}(u) \not\subseteq N_{LK}(v)$, and $N_{LK}(u) \subseteq N_{LK}(v)$, then $N_{UK}(u) \not\supseteq N_{UK}(v)$, where $N_{UK}(u) = \{m_i \in UK_{SB}(G); um_i \in E(G)\}$ and $N_{LK}(u) = \{n_i \in LK_{SB}(G); un_i \in E(G)\}$.* □

3 Distances of posets on semi bound graphs

For a poset $P = (X, \le)$, the *double bound graph* (DB-graph) of $P = (X, \le)$ is the graph $DB(P) = (X, E_{DB(P)})$, where $uv \in E_{DB(P)}$ if and only if $u \ne v$ and there exist $m, n \in X$ such that $n \le u$, $v \le m$. Diny [1] gives a characterization of double bound graphs.

Theorem 5. **(Diny [1])** *A graph G is a DB-graph if and only if there exists a family $C = \{C_1, \ldots, C_n\}$ of complete subgraphs of G and disjoint independent subsets M_G and N_G such that*

(1) *C edge covers G,*

(2) *for each C_i, there exist $m_i \in M_G$ and $n_i \in N_G$ such that $\{m_i, n_i\} \subseteq C_i$ and $\{m_i, n_i\} \not\subseteq C_j$ for all $i \ne j$, and*

(3) *for each $v \in V(G) - (M_G \cup N_G)$, $|U_{M_G}(v)| \times |L_{N_G}(v)|$ equals the number of cliques of C containing v.*

Furthermore, a family C is the unique, minimal edge covering family of cliques in G. □

We already know that for a corresponding poset P of a DB-graph G, M_G corresponds to $Max(P)$ and N_G corresponds to $Min(P)$. For a fixed labeled DB-graph G with M_G and N_G, $\mathcal{P}_{DB}(G) = \{P; DB(P) = G, Max(P) = M_G, Min(P) = N_G\}$ is a poset. We define the double canonical poset on a graph as follows: the *double canonical poset* of a graph G is the poset $(V(G) \le_{d\_can(G)})$, where $x \le_{d\_can(G)} y$ if and only if (1) $y \in M_G$ and $xy \in E(G)$, or (2) $x \in N_G$ and $xy \in E(G)$, or (3) $x = y$.

We know that an SB-graph G, the graph H of Theorem 1 is a DB-graph, $M_H = UK_{SB}(G)$ and $N_H = LK_{SB}(G)$. Then the double canonical poset $d\_can(G)$ with M_H and N_H is a poset whose SB-graph is G. So we have the next result.

Proposition 6. *Every poset in $\mathcal{P}_{SB}(G; d\_can(G))$ has the same double bound graph and $\mathcal{P}_{SB}(G; D) = \mathcal{P}_{DB}(G)$ if $D = d\_can(G)$.* □

We know the following results on DB-graphs in [2] and [7]. In the following we deal with a fixed labeled DB-graph G with a fixed upper kernel M_G and a fixed lower kernel N_G.

Theorem 7. **([2])** *Let G be a DB-graph with an upper kernel M_G and a lower kernel N_G. Let P be a maximal poset on $\mathcal{P}_{DB}(D)$, $Ma(x) = \{m_i \in M_G = Max(P); x \le_P m_i\}$ and $Mi(x) = \{n_i \in N_G = Min(P); n_i \le_P x\}$ for $x \in V(G) - (M_G \cup N_G)$.*

(1) *For all $x, y \in V(G) - (M_G \cup N_G)$ such that $Ma(x) \ne Ma(y)$ or $Mi(x) \ne Mi(y)$, $x \le_P y$ if and only if $Ma(x) \supseteq Ma(y)$ and $Mi(x) \subseteq Mi(y)$.*

(2) *For all* $x, y \in V(G) - (M_G \cup N_G)$, *if* $Ma(x) = Ma(y)$ *and* $Mi(x) = Mi(y)$, *then* $x \leq_P y$ *or* $y \leq_P x$. □

For a DB-graph G with an upper kernel M_G and a lower kernel N_G, $S \subseteq M_G$ and $T \subseteq N_G$, we denote

$$I(S,T) = \cap_{\substack{m \in S,\\ n \in T}} N_G(n,m) - \cup_{\substack{m \in M_G - S,\\ n \in N_G - T}} N_G(n,m),$$

where $N_G(n,m) = \{v \in V(G) : vn, vm \in E(G)\}$.

Theorem 8. ([2]) *Let* G *be a DB-graph with an upper kernel* M_G *and a lower kernel* N_G. *Then two maximal posets on* $\mathcal{P}_{DB}(G)$ *are isomorphic. Furthermore, differences of two maximal posets on* $\mathcal{P}_{DB}(G)$ *are only total orderings of the elements in* $I(S,T)$ *for each* $\emptyset \neq S \subseteq M_G$ *and each* $\emptyset \neq T \subseteq N_G$. □

In a DB-graph G with an upper kernel M_G and alower kernel N_G, the *distance* between posets P and Q in $\mathcal{P}_{DB}(G)$, denoted by $d_{DB}(P,Q)$, is the minimum number of transformations from P to Q by d_deletions and d_additions. The *diameter* $d(\mathcal{P}_{DB}(G))$ is $max\{d_{DB}(P,Q); P, Q \in \mathcal{P}_{DB}(G)\}$.

Theorem 9. ([7]) *For a DB-graph* G *with an upper kernel* M_G *and a lower kernel* N_G,

$$\begin{aligned} d(\mathcal{P}_{DB}(G)) = 2 \quad \times \quad & \Sigma_{\substack{\emptyset \neq S \subseteq M_G,\\ \emptyset \neq T \subseteq N_G}} \binom{|I(S,T)|}{2} \\ + \quad & \Sigma_{\substack{\emptyset \neq S_1 \subset S_2 \subseteq M_G,\\ \emptyset \neq T_2 \subset T_1 \subseteq N_G}} (|I(S_1,T_1)| \times |I(S_2,T_2)|). \end{aligned}$$

□

By Proposition 6, similar results hold in SB-graphs with the double canonical poset D, an upper kernel $UK_{SB}(G)$ and a lower kernel $LK_{SB}(G)$ as follows.

Theorem 10. *Let* G *be an SB-graph with the double canonical poset* D, *an upper kernel* $UK_{SB}(G)$ *and a lower kernel* $LK_{SB}(G)$. *Let* P *be a maximal poset on* $\mathcal{P}_{SB}(G; D)$, $Ma(x) = \{m_i \in UK_{SB}(G) = Max(P); x \leq_P m_i\}$ *and* $M_i(x) = \{n_i \in LK_{SB}(G) = Min(P); n_i \leq_P x\}$ *for* $x \in V(G) - (UK_{SB}(G) \cup LK_{SB}(G))$.

(1) *For all* $x, y \in V(G) - (UK_{SB}(G) \cup LK_{SB}(G))$ *such that* $Ma(x) \neq Ma(y)$ *or* $Mi(x) \neq Mi(y)$, $x \leq_P y$ *if and only if* $Ma(x) \supseteq Ma(y)$ *and* $Mi(x) \subseteq Mi(y)$.

(2) *For all* $x, y \in V(G) - (UK_{SB}(G) \cup LK_{SB}(G))$, *if* $Ma(x) = Ma(y)$ *and* $Mi(x) = Mi(y)$, *then* $x \leq_P y$ *or* $y \leq_P x$. □

For an SB-graph G with the double canonical poset D, an upper kernel $UK_{SB}(G)$ and a lower kernel $LK_{SB}(G)$, we use

$$I(S,T) = \cap_{\substack{m \in S,\\ n \in T}} N_G(n,m) - \cup_{\substack{m \in UK_{SB}(G) - S,\\ n \in LK_{SB}(G) - T}} N_G(n,m),$$

where $S \subseteq UK_{SB}(G), T \subseteq LK_{SB}(G)$ and $N_G(n,m) = \{v \in V(G); vn, vm \in E(G)\}$. We also have the following result.

Theorem 11. *Let G be an SB-graph with the double canonical poset D, an upper kernel $UK_{SB}(G)$ and a lower kernel $LK_{SB}(G)$. Then the two maximal posets on $\mathcal{P}_{SB}(G;D)$ are isomorphic. Furthermore, differences of two maximal posets on $\mathcal{P}_{SB}(G;D)$ are only total orderings of the elements in $I(S,T)$ for each $\emptyset \neq S \subseteq UK_{SB}(G)$ and each $\emptyset \neq T \subseteq LK_{SB}(G)$.* □

Theorem 10 means that relations on a maximal poset in $\mathcal{P}_{SB}(G;D)$ are determined by the set inclusions on $Ma(x)$ and $Mi(x)$. In Theorem 11 $V(G)$ is decomposed as follows:

$$V(G) = (\cup_{\substack{\emptyset \neq S \subseteq UK_{SB}(G), \\ \emptyset \neq T \subseteq LK_{SB}(G)}} I(S,T) \cup UK_{SB}(G) \cup LK_{SB}(G).$$

Then for a maximal poset P in $\mathcal{P}_{SB}(G;D)$, $x \in I(S_1,T_1)$ and $y \in I(S_2,T_2)$,

$$x \leq_P y \text{ if and only if } S_1 \supset S_2 \text{ and } T_1 \subset T_2.$$

The elements in $I(S,T)$ also form a total order in P.

For an SB-graph G with the double canonical poset D, an upper kernel $UK_{SB}(G)$ and a lower kernel $LK_{SB}(G)$, we introduce some definitions. The *distance* between posets P and Q in $\mathcal{P}_{SB}(G;D)$, denoted by $d_{SB}(P,Q)$, is the minimum number of transformations from P to Q by d_deletions and d_additions. The *diameter* $d(\mathcal{P}_{SB}(G;D)$ is $max\{d_{SB}(P,Q) : P,Q \in \mathcal{P}_{SB}(G;D)\}$. Then we have the following result.

Theorem 12. *For an SB-graph G with the double canonical poset D, and upper kernel $UK_{SB}(G)$ and a lower kernel $LK_{SB}(G)$,*

$$d(\mathcal{P}_{SB}(G;D)) = 2 \times \Sigma_{\substack{\emptyset \neq S \subseteq UK_{SB}(G), \\ \emptyset \neq T \subseteq LK_{SB}(G)}} \binom{|I(S,T)|}{2} + \Sigma_{\substack{\emptyset \neq S_1 \subset S_2 \subseteq UK_{SB}(G), \\ \emptyset \neq T_2 \subset T_1 \subseteq LK_{SB}(G)}} (|I(S_1,T_1)| \times |I(S_2,T_2)|).$$

□

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