Enumeration of order ideals of a garland

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Summary. *We enumerate all order ideals of a garland, a partially ordered set which generalizes crowns and fences. Moreover we give some bijection between the set of such ideals and the set of certain kinds of lattice paths.*

AMS Classification: 05A15, 05A05, 05A16.

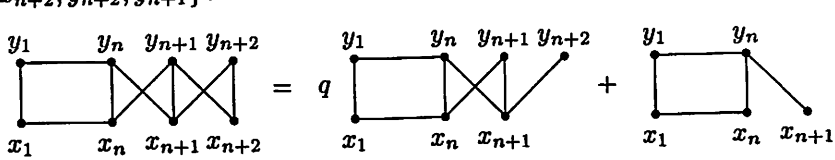
Keywords: Garland, order ideal, distributive lattice, Riordan matrix, Riordan group, self-avoiding path, non-selfintersecting path, generating function.

1 Introduction

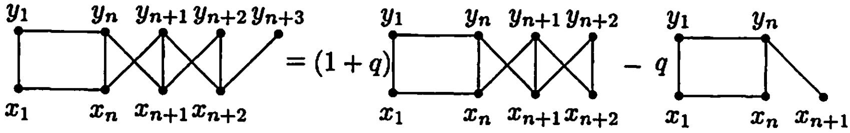
Let $\mathcal{G}_n$ be the partially ordered set (poset) with elements $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ and with cover relations $x_1 < y_1$, $x_1 < y_2$, $x_i < y_{i-1}$, $x_i < y_i$, $x_i < y_{i+1}$ for $i = 2, \ldots, n-1$ and $x_n < y_{n-1}$, $x_n < y_n$ for $n \geq 2$. Then let $\mathcal{G}_0$ be the empty set and $\mathcal{G}_1$ be a chain of length 1. The poset $\mathcal{G}_n$ will be called *garland*. Notice that $\mathcal{G}_n$ contains as subposets the *crown* C_n and the *fence* $\mathcal{Z}_{2n}$ [6, 9].

Let $g_{n,k}$ be the number of all order ideals of size k of $\mathcal{G}_n$ and similarly let g_n be the number of all order ideals of $\mathcal{G}_n$. Then let $g_n(q)$ be the rank polynomial of the lattice $J(\mathcal{G}_n)$ of all order ideals of $\mathcal{G}_n$, that is $g_n(q) = \sum_{k=0}^{2n} g_{n,k}\, q^k$.

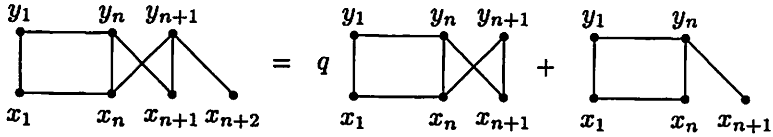
Consider the garland $\mathcal{G}_{n+2}$. Then partition all the ideals of $\mathcal{G}_{n+2}$ according to the containment of x_{n+2}. The ideals containing x_{n+2} are equivalent to the ideals of $\mathcal{G}'_{n+1} = \mathcal{G}_{n+2} \setminus \{x_{n+2}\}$, while the ideals not containing x_{n+2} are equivalent to the ideals of $\mathcal{G}''_n = \mathcal{G}_{n+2} \setminus \{x_{n+2}, y_{n+2}, y_{n+1}\}$.



Consider now $\mathcal{G}'_{n+2}$. This time the ideals can be partitioned according to the containment of y_{n+3}. The ideals not containing y_{n+3} are equivalent to the ideals of $\mathcal{G}'_{n+2} \setminus \{y_{n+3}\} = \mathcal{G}_{n+2}$, while the ideals containing y_{n+3} are equivalent to the ideals of $\mathcal{G}'_{n+2} \setminus \{y_{n+3}\} = \mathcal{G}_{n+2}$ containing x_{n+2}. These latter ideals are all the ideals of $\mathcal{G}_{n+2}$ except those not containing x_{n+2} which are equivalent to the ideals of $\mathcal{G}_{n+2} \setminus \{x_{n+2}, y_{n+2}, y_{n+1}\} = \mathcal{G}''_n$.



Finally consider $\mathcal{G}''_{n+1}$ and partition its ideals according to the containment of x_{n+2}. The ideals containing x_{n+2} are equivalent to the ideals of $\mathcal{G}''_{n+1} \setminus \{x_{n+2}\} = \mathcal{G}_n$, while the ideals not containing x_{n+2} are equivalent to the ideals of $\mathcal{G}''_{n+1} \setminus \{x_{n+2}, y_{n+1}\} = \mathcal{G}''_n$.



Let $h_n(q)$ and $i_n(q)$ be the rank polynomials of the lattices $J(\mathcal{G}'_n)$ and $J(\mathcal{G}''_n)$, respectively. Then the above analysis yields the system

$$\begin{cases} g_{n+2}(q) = q\, h_{n+1}(q) + i_n(q) \\ h_{n+2}(q) = (1+q)\, g_{n+2}(q) - q\, i_n(q) \\ i_{n+1}(q) = q\, g_{n+1}(q) + i_n(q) \end{cases} \tag{1}$$

Recall that the incremental ratio of a series $s(t) = \sum_{n \geq 0} s_n\, t^n$ is the linear operator R defined by

$$Rs(t) := \frac{s(t) - s(0)}{t} = \sum_{n \geq 0} s_{n+1}\, t^n .$$

Hence, if $g(q;t)$, $h(q;t)$ and $i(q;t)$ are respectively the ordinary generating series for the polynomials $g_n(q)$, $h_n(q)$ and $i_n(q)$, system (1) can be rewritten as

$$\begin{cases} R^2 g(q;t) = q\, Rh(q;t) + i(q;t) \\ R^2 h(q;t) = (1+q)\, R^2 g(q;t) - q\, i(q;t) \\ Ri(q;t) = q\, Rg(q;t) + i(q;t) \end{cases} \tag{2}$$

It is easy to see that the initial conditions are $g_0(q) = 1$, $g_1(q) = 1+q+q^2$, $h_0(q) = 1+q$, $h_1(q) = 1+q+2q^2+q^3$ and $i_0(q) = 1+q$. Therefore, solving system (2) with these initial conditions, we obtain the series

$$g(q;t) = \frac{1-q^2t^2}{1-(1+q+q^2)t+q^2t^2+q^3t^3}\,. \tag{3}$$

This series implies the recurrence

$$g_{n+3}(q) = (1+q+q^2)\,g_{n+2}(q) - q^2 g_{n+1}(q) - q^3 g_n(q) \tag{4}$$

which is equivalent to the recurrence

$$g_{n+3,k+3} = g_{n+2,k+3} + g_{n+2,k+2} + g_{n+2,k+1} - g_{n+1,k+1} - g_{n,k}\,. \tag{5}$$

Moreover, for $q = 1$ we obtain the generating series for the numbers g_n

$$g(t) = \sum_{n\geq 0} g_n\, t^n = g(1;t) = \frac{1-t^2}{1-3t+t^2+t^3}$$

that is

$$g(t) = \frac{1+t}{1-2t-t^2}\,. \tag{6}$$

The form of this series implies the recurrence

$$g_{n+2} = 2g_{n+1} + g_n \tag{7}$$

and the initial conditions $g_0 = 1$ and $g_1 = 3$. Then $\{g_n\}_n$ is the sequence #A001333 in [8].

2 Central ideals

Let I be an order ideal of the garland $\mathcal{G}_n$. We say that I is a *central ideal* when $|I| = n$. Let c_n be the number of all central ideals of $\mathcal{G}_n$. The first few values are 1, 1, 1, 3, 7, 15, 33, 75, 171, 391, 899, 2077, 4815, 11195, 26097, 60975, 142751. The generating function for these numbers is the diagonal of the double series $g(q,y)$. By Cauchy's integral theorem [2, 4, 10] it is given by

$$\begin{aligned} c(t) &= \sum_{n\geq 0} c_n\, t^n = \frac{1}{2\pi \mathrm{i}} \oint g\left(z;\frac{t}{z}\right)\, \frac{\mathrm{d}z}{z} \\ &= \frac{1}{2\pi \mathrm{i}} \oint \frac{1-t^2}{-tz^2+(1-t+t^2+t^3)z-t}\, \mathrm{d}z \end{aligned}$$

where the integral is taken over a simple contour containing all the singularities $s(t)$ of the series such that $s(t) \to 0$ as $t \to 0$. The polynomial $tz^2 - (1 - t + t^2 + t^3)z + t$ at the denominator has roots

$$z^{\pm} = \frac{1 - t + t^2 + t^3 \pm \sqrt{(1 - t + t^2 + t^3)^2 - 4t^2}}{2t}$$

of which only $z^- \to 0$ as $t \to 0$. Therefore, by the residue theorem, we have

$$c(t) = \lim_{z \to z^-} \frac{1 - t^2}{-t(z - z^+)} = \frac{1 - t^2}{t(z^+ - z^-)}$$

that is

$$c(t) = \frac{1 - t^2}{\sqrt{1 - 2t - t^2 - 3t^4 + 2t^5 + t^6}} = \sqrt{\frac{1 - t^2}{1 - 2t - 2t^3 - t^4}}. \qquad (8)$$

Differentiating $c(t)$ we obtain the identity

$$(1 - 2t - t^2 - t^4 + 2t^5 + t^6)c'(t) = (1 - t + 4t^2 + 2t^3 - t^4 - t^5)c(t)$$

which implies the following linear recurrence for the numbers c_n

$$\begin{aligned}(n+6)\,c_{n+6} - (2n+11)\,c_{n+5} - (n+3)\,c_{n+4} + \\ -4\,c_{n+3} - (n+4)\,c_{n+2} - (2n+3)\,c_{n+1} + (n+1)\,c_n = 0. \qquad (9)\end{aligned}$$

Finally we give a first-order asymptotic formula for c_n. Recall ([1], p. 252) that given a complex number $\xi \neq 0$ and a complex function $f(t)$ analytic at the origin, if $f(t) = (1 - t/\xi)^{-a}\psi(t)$ where $\psi(t)$ is a series with radius of convergence $R > |\xi|$ and $a \notin \{0, -1, -2, \ldots\}$, then

$$[t^n]f(t) \sim \frac{\psi(\xi)}{\xi^n} \frac{n^{\alpha-1}}{\Gamma(\alpha)}.$$

In our case we have

$$c(t) = \sqrt{\frac{1 - t^2}{(1 + t^2)(1 - 2t - t^2)}} = \sqrt{\frac{(1 - t)(1 + t)}{(1 - t/\mathrm{i})(1 + t/\mathrm{i})(1 - t/\alpha)(1 - t/\beta)}}$$

where $\alpha = -1 + \sqrt{2}$ and $\beta = -1 - \sqrt{2}$. Since α is the singularity with minimum modulus, we have that

$$c(t) = \left(1 - \frac{t}{\alpha}\right)^{-1/2} \sqrt{\frac{1 - t^2}{(1 + t^2)(1 + t/\beta)}}.$$

Hence, since $\Gamma(1/2) = \sqrt{\pi}$, we have

$$c_n \sim \frac{1}{2}\sqrt{\frac{1+\sqrt{2}}{n\pi}}\left(1+\sqrt{2}\right)^n . \tag{10}$$

In particular

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1+\sqrt{2}. \tag{11}$$

Using the generating series (6) it is easy to see that $g_n \sim \frac{1}{2}\left(1+\sqrt{2}\right)^{n+1}$. Therefore we also have

$$\lim_{n\to\infty} \frac{g_n}{c_n} = \sqrt{(1+\sqrt{2})n\pi}\,.$$

3 A Riordan Matrix

In this section we will consider the matrix

$$R = [r_{n,k}]_{n,k\geq 0} = \begin{bmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 3 & 3 & 3 & 1 \\ 7 & 6 & 6 & 4 & 1 \\ 15 & 14 & 12 & 10 & 5 & 1 \\ 33 & 32 & 27 & 22 & 15 & 6 & 1 \\ 75 & 72 & 63 & 50 & 37 & 21 & 7 & 1 \\ 171 & 164 & 146 & 118 & 88 & 58 & 28 & 8 & 1 \\ \cdots \end{bmatrix}$$

where $r_{n,k} = g_{n,n-k}$ when $k \leq n$ and $r_{n,k} = 0$ otherwise. We will prove that R is a Riordan matrix [7, 5]. First notice that recurrence (5) yields the recurrence for the coefficients $r_{n,k}$

$$r_{n+3,k+1} = r_{n+2,k} + r_{n+2,k+1} + r_{n+2,k+2} - r_{n+1,k+1} - r_{n,k+1}\,. \tag{12}$$

Then, considering the generating series $r_k(t) = \sum_{n\geq k} r_{n,k}\, t^n$ of the columns of R, recurrence (12) becomes

$$t\, r_{k+2}(t) - (1-t+t^2+t^3)\, r_{k+1}(t) + t\, r_k(t) = 0\,. \tag{13}$$

Now suppose there exist two series $c(t)$ and $f(t)$ such that $r_k(t) = c(t)f(t)^k$ for all $k \in \mathbb{N}$. Then $c(t) = r_0(t)$ is the series (8) and substituting $r_k(t) = c(t)f(t)^k$ in (13) we have that $f(t)$ is the solution of the quadratic equation

$$tf(t)^2 - (1-t+t^2+t^3)f(t) + t = 0\,,$$

that is

$$f(t) = \frac{1 - t + t^2 + t^3 - \sqrt{1 - 2t - t^2 - 3t^4 + 2t^5 + t^6}}{2t}.$$

Hence we have

$$r_k(t) = \sqrt{\frac{1 - t^2}{1 - 2t - 2t^3 - t^4}} \left(\frac{1 - t + t^2 + t^3 - \sqrt{1 - 2t - t^2 - 3t^4 + 2t^5 + t^6}}{2t} \right)^k$$

In conclusion, since $c_0 = 1$, $f_0 = 0$ and $f_1 \neq 0$, R is the Riordan matrix

$$\left(\sqrt{\frac{1 - t^2}{1 - 2t - 2t^3 - t^4}}, \frac{1 - t + t^2 + t^3 - \sqrt{1 - 2t - t^2 - 3t^4 + 2t^5 + t^6}}{2t} \right).$$

4 Self-avoiding paths

In [9] it is proved that g_n is the number of n-step self-avoiding paths starting at $(0,0)$ with steps of type $(1,0)$, $(-1,0)$ and $(0,1)$. This implies that there exists a bijection between the set $\mathcal{S}_n$ of all these paths and $J(\mathcal{G}_n)$. To obtain such a bijection it is convenient to regard the paths in $\mathcal{S}_n$ as words on the alphabet $\{x, \overline{x}, y\}$, where x, $\overline{x}$ and y stand for the steps $(1,0)$, $(-1,0)$ and $(0,1)$, subject to the restriction that $x\overline{x}$ and $\overline{x}x$ are forbidden subwords.

To each word $\alpha = a_1 \cdots a_n$ in $\mathcal{S}_n$ we can associate a 3-filtering multiset $\mu_\alpha : [n] \to \{0,1,2\}$ defined by

$$\mu_\alpha(k) = \begin{cases} 0 & \text{if } a_k = \overline{x} \\ 1 & \text{if } a_k = y \\ 2 & \text{if } a_k = x \end{cases}$$

for every $k \in [n]$. Since α is a self-avoiding path, the multiset μ_α has to satisfy the conditions: (i) if $\mu_\alpha(k) = 2$ then $\mu_\alpha(k \pm 1) \neq 0$ (ii) if $\mu_\alpha(k) = 0$ then $\mu_\alpha(k \pm 1) \neq 2$, for every $k \in [n]$. Here we consider $k+1$ and $k-1$ only when they are in $[n]$. Since the conditions (i) and (ii) are equivalent, we will consider just the first one. Let $\mathcal{M}_n$ be the set of all 3-filtering multisets on $[n]$ satisfying condition (i). It is easy to see that the map $F : \mathcal{S}_n \to \mathcal{M}_n$, defined by $F(\alpha) = \mu_\alpha$, is a bijection.

Also the ideals I of $\mathcal{G}_n$ can be described as multisets. Indeed the ideal I is equivalent to the 3-filtering multiset $\mu_I : [n] \to \{0,1,2\}$ defined by

$$\mu_I(k) = \begin{cases} 0 & \text{if } x_k, y_k \notin I \\ 1 & \text{if } x_k \in I,\ y_k \notin I \\ 2 & \text{if } x_k, y_k \in I \end{cases}$$

for every $k \in [n]$. It is easy to see that $\mu_I \in \mathcal{M}_n$ and that the map $H : J(\mathcal{G}_n) \to \mathcal{M}_n$, defined by $H(I) = \mu_I$, is a bijection. Hence we have that also the map $HF^{-1} : J(\mathcal{G}_n) \to \mathcal{S}_n$ is a bijection.

In particular, if $I \in J(\mathcal{G}_n)$, $\mu \in \mathcal{M}_n$ and $\alpha \in \mathcal{S}_n$ are equivalent in the above bijections, then

$$|I| = k \iff \mathrm{ord}(\mu) = k \iff 2\omega_\alpha(x) + \omega_\alpha(y) = k$$

where $\mathrm{ord}(\mu) = \mu(1) + \cdots + \mu(n)$ and $\omega_\alpha(x)$ is the number of occurrences of x in α, etc. When $\alpha \in \mathcal{S}_n$ then we also have $\omega_\alpha(x) + \omega_\alpha(\overline{x}) + \omega_\alpha(y) = n$. Hence it follows that $g_{n,k}$ is the number of all paths in $\mathcal{S}_n$ such that $\omega_\alpha(\overline{x}) - \omega_\alpha(x) = n - k$. In particular, $r_{n,k} = g_{n,n-k}$ is the number of all paths in $\mathcal{S}_n$ with $\omega_\alpha(\overline{x}) = \omega_\alpha(x) + k$. Moreover the central ideals of $\mathcal{G}_n$ correspond to the paths in $\mathcal{S}_n$ such that $\omega_\alpha(\overline{x}) = \omega_\alpha(x)$, i.e. to the paths with an equal number of right and left horizontal steps.

5 Central trinomial paths in the strip $[-1, 1]$

A *trinomial path* is a lattice path starting at $(0,0)$ with unitary steps of type $(1,1)$, $(1,-1)$ and $(1,0)$. In particular a *central trinomial path* is a trinomial path ending on the x-axis. Let $\mathcal{T}_n$ be the set of all central trinomial paths in the strip $[-1,1]$. In [3] it is proved that there exist a bijection between the sets $\mathcal{S}_n$ and $\mathcal{T}_{n+1}$. Here we restate such a bijection in terms of 3-filtering multisets. First of all notice that any path in $\mathcal{T}_{n+1}$ can be described as a function $f : \{0, \cdots, n+1\} \to \{-1, 0, 1\}$ subject to the restrictions (i) $f(0) = f(n+1) = 0$, (ii) if $f(k) = 1$ then $f(k \pm 1) \neq -1$ and (iii) if $f(k) = -1$ then $f(k \pm 1) \neq 1$, for every $k \in \{0, \cdots, n+1\}$. Condition (i) says that the path starts and ends on the x-axis. Conditions (ii) and (iii) are equivalent and say that the steps are unitary. Given such a function we can consider the multiset $\mu_f : [n] \to \{0,1,2\}$ defined by $\mu_f(k) = f(k) + 1$ for every $k \in [n]$. It follows that the paths in $\mathcal{T}_{n+1}$ are equivalent to the multisets in $\mathcal{M}_n$ and consequently to the paths in $\mathcal{S}_n$ and to the ideals of $\mathcal{G}_n$. In particular the set $\mathcal{T}_{n+1}$, ordered so that $f_1 \leq f_2$ if and only if $f_1(k) \leq f_2(k)$ for all $k \in [n]$, is a distributive lattice isomorphic to $J(\mathcal{G}_n)$.

Let $\max(f)$ be the number of all points k such that $f(k) = 1$ and similarly let $\min(f)$ be the number of all points k such that $f(k) = -1$. Then

$$ord(\mu_f) = n + \sum_{k=0}^{n+1} f(k) = n + \max(f) - \min(f).$$

Hence $g_{n,k}$ is also the number of all paths in $\mathcal{T}_{n+1}$ such that $\min(f) - \max(f) = n - k$. In particular the central ideals correspond to the paths in $\mathcal{T}_{n+1}$ with an equal number of maxima and minima.

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