Problems on Hall t-chromaticity

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Abstract

We find a family of graphs each of which is not Hall t-chromatic for all $t \geq 3$, and use this to prove that the same holds for the Kneser graphs $K_{a:b}$ when $a/b > 3$ and b is sufficiently large (depending on $3-(a/b)$). We also make some progress on the problem of characterizing the graphs that are Hall t-chromatic for all t.

1 Introduction

Throughout, G will be a simple graph with vertex set $V(G)$, of order $|V(G)| = n(G) = n$. The vertex independence number of G will be denoted $\alpha(G)$. (In this and other notation we will follow [7].) The *Hall ratio* of G is $\rho(G) = \max[n(H)/\alpha(H);\ H$ is a subgraph of $G]$. Clearly $\rho(G) = n(H)/\alpha(H)$ for some induced subgraph H of G.

If t is a non-negative integer and $\kappa : V(G) \to \mathbb{N} = \{0, 1, \dots\}$, a proper (t, κ)-coloring of G is a function $\varphi : V(G) \to 2^L$, for some set L with t elements, satisfying, for all $u, v \in V(G)$,

(i) $|\varphi(v)| = \kappa(v)$ and

(ii) if u and v are adjacent in G, then $\varphi(u) \cap \varphi(v) = \emptyset$. Note that (ii) is equivalent to: $(ii)'$ if $\sigma \in L$ then $\{v \in V(G); \sigma \in \varphi(v)\}$ is an independent set of vertices, in G. κ is sometimes called a *color demand* function.

We will say that G, t, and κ satisfy *Hall's condition* if and only if, for each subgraph H of G,

$$t\alpha(H) \geq \sum_{v \in V(H)} \kappa(v) \tag{$*$}$$

If G is properly (t, κ) colored, then so is each subgraph H of G. The properness of the coloring implies that each of the t colors can appear at most $\alpha(H)$ times in the subsets of L coloring the vertices of H, while the total number of appearances on those vertices is $\sum_{v \in V(H)} \kappa(v)$. Thus, satisfaction of Hall's condition is necessary for the existence of a proper (t, κ)-coloring. We will say that *G is Hall t-chromatic* if and only if Hall's condition (a condition on κ, with G and t fixed) is sufficient for the existence of a proper (t, κ)-coloring of G.

Hall's condition as given here is a special (some would say degenerate) case of a more general condition on a graph, a list assignment, and a color demand function κ; see [2]. In this special case, the list assignment is constant. The study of Hall t-chromaticity began in [1]; it turns out to have some useful connections with the study of the k-fold chromatic numbers, the fractional chromatic number, and the Hall ratio.

The k-fold chromatic number of G, $\chi^{(k)}(G)$, is the smallest integer t such that there is a proper (t, k)-coloring of G. (Here, k stands for the function on $V(G)$ with constant value k.) The fractional chromatic number of G, $\chi_f(G)$, is given by $\chi_f(G) = \min_k \frac{1}{k}\chi^{(k)}(G) = \inf_k \frac{1}{k}\chi^{(k)}(G) = \lim_{k\to\infty} \frac{1}{k}\chi^{(k)}(G)$. (See [4] and [5]; the former gives three or four other equivalent definitions of $\chi_f(G)$.)

The following easy observation and several of its corollaries appear in various forms in [1] and [2].

Lemma 1 *Suppose that k and t are positive integers. Then G, t, and $\kappa \equiv k$ satisfy Hall's condition if and only if $k\rho(G) \leq t$.*

Proof: If $k\rho(G) \leq t$, then for any subgraph H of G, $k\frac{n(H)}{\alpha(H)} \leq k\rho(G) \leq t$, so $t\alpha(H) \geq kn(H) = \sum_{v \in V(H)} \kappa(v)$ if $\kappa \equiv k$.

If $t < k\rho(G)$ then for some subgraph H of G $t\alpha(H) < kn(H)$, so G, t and $\kappa \equiv k$ do not satisfy Hall's condition. □

Corollary 1 *For all positive integers k, $\lceil k\rho(G) \rceil \leq \chi^{(k)}(G)$. If G is Hall-$\lceil k\rho(G) \rceil$-chromatic then equality holds.*

The proof is straightforward from the Lemma, the definition of $\chi^{(k)}(G)$, and the necessity of Hall's condition for a proper coloring.

Corollary 2 $\rho(G) \leq \chi_f(G)$.

This follows easily from Corollary 1 and previous remarks about $\chi_f(G)$.

Corollary 3 *If G is vertex transitive, then $\rho(G) = \chi_f(G) = n(G)/\alpha(G)$.*

Proof: Clearly $n(G)/\alpha(G) \le \rho(G)$. It is proven in [4] that if G is vertex transitive then $\chi_f(G) = n(G)/\alpha(G)$. The conclusion now follows from Corollary 2. □

Corollary 4 *If k and t are positive integers and $k\rho(G) \le t < \chi^{(k)}(G)$ then G is not Hall t-chromatic.*

Proof: By the Lemma, G, t, and $\kappa \equiv k$ satisfy Hall's condition, but $t < \chi^{(k)}(G)$ implies that there is no proper (t, k)-coloring of G. □

Let $T(G) = \{t \in \mathbb{N}; G \text{ is Hall } t\text{-chromatic}\}$.

Corollary 5 *If $\rho(G) < \chi_f(G)$ then $T(G)$ is finite.*

Corollary 5 was proved, but not stated, in [1]. Also in [1] the question was posed: is the condition in Corollary 4 the only circumstance under which a graph G is not Hall t-chromatic? That is, if G is not Hall t-chromatic, does there exist a positive integer k such that $k\rho(G) \le t < \chi^{(k)}(G)$? We shall see below that the answer is no, for easy reasons, and that even when the question is refined to rule out easy conterexamples, the answer is still no.

We might also ask if the converse of Corollary 5 is true. Again, we will see below that the answer is no, for easy reasons, but, in this case, the question has a refinement to which we do not as yet know the answer.

Compendium of other results from [1]

(1) The following are Hall t-chromatic for all $t \in \mathbb{N}$: bipartite graphs, complete multipartite graphs, and odd cycles.

(2) For any graph G, $\{0, 1, 2\} \subseteq T(G)$.

(3) If H is an induced subgraph of G, and G is Hall t-chromatic, then H is Hall t-chromatic.

(4) If G_1 and G_2 are Hall t-chromatic and $V(G_1) \cap V(G_2)$ induces a clique in both G_1 and G_2, then $G_1 \bigcup G_2$ is Hall t-chromatic.

Now we can give the easy reason why the condition of Corollary 4 is not necessary for G to be not Hall t-chromatic, and why the converse of Corollary 5 is not true. Take a graph H which is not Hall t-chromatic, or, in the second question, for which $T(H)$ is finite, and make H an induced subgraph of a graph G satisfying $k\rho(G) = \chi^{(k)}(G)$ for all k (which implies $\rho(G) = \chi_f(G)$). If this can be done, then both questions are settled, since $\mathcal{T}(G) \subseteq \mathcal{T}(H)$ by Compendium (3).

An easy way to so embed H as a subgraph of such a G is to attach a clique K_m, $m \geq \chi(H) = \chi^{(1)}(H)$, to H, either as a satellite unconnected to H, or sharing a clique K_s, $s < m$, with H. Suppose that G is formed in this way. Since $m \geq \chi(H)$, by standard arguments $\chi(G) = m$, and thus $\chi^{(k)}(G) \leq k\chi(G) = km$ for each $k = 1, 2, \ldots$, by the subadditivity of the sequence $(\chi^{(k)}(G))_k$. (See [4].) On the other hand, since G contains a K_m, $m \leq \rho(G)$. Thus $m \leq \rho(G) \leq \frac{1}{k}\chi^{(k)}(G) \leq m$ for each k, by Corollary 1, so G satisfies $\rho(G) = \frac{1}{k}\chi^{(k)}(G)$ for all k.

The refinement of the question about Corollary 5 to which we do not know the answer is the following.

Problem 1 *Suppose that $\mathcal{T}(G)$ is finite. Does G necessarily have an induced subgraph H satisfying $\rho(H) < \chi_f(H)$?*

The refinement of the other question is: suppose G is *critically* not Hall t-chromatic, meaning $t \notin \mathcal{T}(G)$ but $t \in \mathcal{T}(G - v)$ for all $v \in V(G)$. Is there necessarily some integer k such that $k\rho(G) \leq t < \chi^{(k)}(G)$? We shall see later (Corollary 10) that the answer to this question is no.

Partly because of their historic role in the study of the $\chi^{(k)}$ (see [6] and [4]), we are especially interested in the Hall t-chromaticity of the Kneser graphs. If a and b are positive integers, with $a \geq 2b$, the Kneser graph $K_{a:b}$ has for vertices the b-subsets of some a-set, with two vertices adjacent if and only if they are disjoint. A succinct account of the essential facts about and the role of these graphs in combinatorial theory may be found in [4]; and even after 27 years, there is much to be learned in [6]. Two facts we need here are that (i) $\rho(K_{a:b}) = \chi_f(K_{a:b}) = a/b$ (see [4], taking note of Corollary 3), and (ii) $\chi(K_{a:b}) = a - 2b + 2$ (This was, in modern terms, Kneser's conjecture, proven by Lovász [3]).

Corollary 6 (of Lemma 1) *If $b \geq 2$ and $a \geq 2b + 2$ then $3 \notin \mathcal{T}(K_{a:b})$.*

Proof: If $a \geq a_1 \geq 2b$ then $K_{a_1:b}$ is an induced subgraph of $K_{a:b}$. By Compendium (3), it suffices to prove the result for $a = 2b + 2$. In this case, with $G = K_{2b+2:b}$, we have $\rho(G) = 2 + 2/b \leq 3 < 2b + 2 - 2b + 2 = 4 = \chi(G)$. The conclusion follows from Corollary 4, with $k = 1$. □

We will see later that $\mathcal{T}(K_{a:b}) = \{0, 1, 2\}$ for a great many pairs a, b with $a > 3b$; we suspect that this equation holds whenever $a \geq 2b + 2$. The case $a = 2b + 1$ is special.

It is an unsolved problem to determine the numbers $\chi^{(m)}(K_{a:b})$. Stahl conjectures [6] that if $m = qb + r$, $1 \leq r \leq b$, then $\chi^{(m)}(K_{a:b}) = (q + 1)a -$

$2b+2r$, and this conjecture has withstood assault since 1976. It is known to hold whenever $a = 2b$ or $a = 2b+1$; or when $b \leq 3$; or when $r = b$; or when $q = 0$ and $1 \leq r \leq b$. If the conjecture holds whenever $m = qb+1$, then it holds for all m and $a \geq 2b$. Note that if $b \geq 2$ and $a \geq 2b+2$ and $\chi^{(qb+1)}(K_{a:b}) = (q+1)a - 2b + 2$, $q = 0, 1, 2, \ldots$, then there are infinitely many t, namely all $t \in \{\lceil (qb+1)\frac{a}{b} \rceil, \ldots, \chi^{(qb+1)}(K_{a:b}) - 1\}$, $q = 0, 1, 2, \ldots$, such that $K_{a:b}$ is not Hall t-chromatic, by Corollary 4. (We leave it as an exercise to verify that $a \geq 2b+2$, $b \geq 2$, implies that

$$\lceil (qb+1)\frac{a}{b} \rceil \leq (q+1)a - 2b + 1, q = 0, 1, 2, \ldots,$$

so that those intervals, $\{\lceil (qb+1)\rho(K_{a:b}) \rceil, \ldots, \chi^{(qb+1)}(K_{a:b}) - 1\}$ are non-empty, if Stahl's conjecture holds.) Thus, if it turns out , for some $a \geq 2b+2$, $b \geq 2$, that $\mathbb{N} \backslash \mathcal{T}(K_{a:b})$ is finite, or even that $\mathbb{N} \backslash \mathcal{T}(K_{a:b})$ fails to contain even one value that it must contain if Stahl's conjecture is true, then Stahl's conjecture is not true. However, so far all the evidence is on Stahl's side.

2 Results and Problems

Proofs are postponed until the last section. The first theorem generalizes part of Compendium (1).

Theorem 1 *Suppose that $S_1 \subseteq V(G_1)$ satisfies: for all $u, v \in S_1$, $N_{G_1}(u) \backslash S_1 = N_{G_1}(v) \backslash S_1$. Let $U = N_{G_1}(u) \backslash S_1$, for any $u \in S_1$. Let H be either a bipartite graph with bipartition X_1, X_2, or a complete multipartite graph with parts $X_1, \ldots, X_r$, $r \geq 3$. Suppose that $S \subseteq X_1$. Form G by taking disjoint copies of $G_1 - S_1$ and H, and making every vertex of S adjacent to every vertex of U. If G_1 is Hall t-chromatic, then so is G.*

Corollary 7 *Suppose that E is an independent set of edges of a clique K_m and G is formed by, for each $e \in E$, either deleting e or by replacing e by a path of even length. Then $\mathcal{T}(G) = \mathbb{N}$.*

Let $\vee$ denote the join operation; that is, $G \vee H$ is formed by taking disjoint copies of G and H and making all vertices of $V(G)$ adjacent to all vertices of $V(H)$. It is easy to see that $\chi^{(k)}(G \vee H) = \chi^{(k)}(G) + \chi^{(k)}(H)$, $k = 1, 2, \ldots$, and thus that $\chi_f(G \vee H) = \chi_f(G) + \chi_f(H)$.

For $m \geq 3$, let $W_m = K_1 \vee C_m$, the wheel with m spokes. Of course, $W_3 = K_4$.

Theorem 2 *If $p \geq 2$, $\mathcal{T}(W_{2p}) = \mathbb{N}$ and $\mathcal{T}(W_{2p+1}) = \{0, 1, 2\}$.*

Corollary 8 *If, for some integer p, $b \geq p \geq 2$ and $a \geq 3b + \lceil \frac{b}{p} \rceil$, then $\mathcal{T}(K_{a:b}) = \{0, 1, 2\}$.*

Corollary 9 *If* $0 < \epsilon \leq 1/2$, $b \geq \frac{1}{\epsilon}$ *and* $a/b \geq 3 + \epsilon$, *then* $\mathcal{T}(K_{a:b}) = \{0, 1, 2\}$.

Corollary 10 *There exist integers* $t \geq 3$ *and graphs* G *which are critically not Hall* t*-chromatic, such that for every positive integer* k, $t \notin \{\lceil k\rho(G)\rceil, \ldots, \chi^{(k)}(G) - 1\}$.

The next, and last, theorem will help out in the search for critically not Hall t-chromatic graphs, and thus may be of service in attacking problems 3 and 4, below.

Theorem 3 *If* G *is critically not Hall* t*-chromatic, then* $t \geq \rho(G)$.

Corollaries 9 and 10 give answers that were promised in section 1. A great many questions remain; the following tasks and questions seem to us especially notable, beyond Problem 1 and the problem of determining $\mathcal{T}(K_{a:b})$ for all $a > 2b \geq 4$.

Problem 2 *Is* $\mathcal{T}(G)$ *always a block of consecutive integers, either* $\mathbb{N}$ *or* $\{0, 1, 2, \ldots, \tau(G)\}$ *for some* $\tau(G)$*?*

If the answer is yes, the problem of determining the sets $\mathcal{T}(K_{a:b})$ would be greatly simplified, and the possible threat to Stahl's conjecture mentioned at the end of section 1 would evaporate. (From Corollary 6 it would follow that $\mathcal{T}(K_{a:b}) = \{0, 1, 2\}$ whenever $b \geq 2$ and $a \geq 2b + 2$.) It seems quite reasonable that the answer is yes; as t increases the restrictions on κ imposed by the inequalities $(*)$ ease; that is, a greater variety of color demand functions κ must be dealt with, and it seems plausible that the increased supply of κ's should overpower the advantage of having more colors. So it becomes harder for the graph to be Hall t-chromatic, so to speak, as t goes up. But we have no proof.

Problem 3 *Characterize the graphs* G *such that* $\mathcal{T}(G) = \{0, 1, 2\}$.

It will suffice to characterize the critical such graphs, the G such that $\mathcal{T}(G) = \{0, 1, 2\}$ but $\mathcal{T}(G - v) \neq \{0, 1, 2\}$ for each $v \in V(G)$; $\mathcal{T}(G) = \{0, 1, 2\}$ if and only if G contains one of those critical graphs as an induced subgraph.

The only such critical graphs that we know of so far are the $W_{2p+1}, p \geq 2$. In view of Theorem 2, to see this it suffices to note that $W_{2p+1} - v$ is Hall t-chromatic for all t, for each $v \in V(W_{2p+1})$, by Compendium (1) and (4); $W_{2p+1} - v$ is either an odd cycle or is built up from K_3 by attaching K_3's along edges.

Problem 4 *Characterize the graphs* G *such that* $\mathcal{T}(G) = \mathbb{N}$.

Compendium (3) implies that this family of graphs has a forbidden-induced-subgraph characterization. The forbidden induced subgraphs are those H which are vertex-critical with respect to the property $\mathcal{T}(H) \neq \mathbb{N}$, and it seems rather a long, dry task to describe all of those. Compendium (4) and Theorem 1 give us hope that there may be a constructive characterization of the graphs which are Hall t-chromatic for all t, or at least a constructive component in their characterization.

Problem 5 *Find* $\mathcal{T}(K_{2b+1:b}), b \geq 2$. We think the answer is $\mathbb{N}$.

3 Proofs

Proof of Theorem 1 If $S_1 = \emptyset$ or $S = \emptyset$ then G is the disjoint union of $G_1 - S_1$ and H and is therefore Hall t-chromatic (by Compendium (1) and the obvious fact that a graph is Hall t-chromatic if and only if each component of it is). So assume $S_1 \neq \emptyset \neq S$.

Let $\kappa : V(G) \to \mathbb{N}$ be a color demand function satisfying Hall's condition, with G and t. Let $v \in S$ be such that $\kappa(v) = \max_{w \in S} \kappa(w)$. Since $S_1 \neq \emptyset$ and v seems to $G_1 - S_1$ like any vertex of S_1, $(V(G)\backslash S_1) \cup \{v\}$ induces in G a graph G_2 isomorphic to an induced subgraph of G_1, which is Hall t-chromatic; therefore G_2 has a proper (t, κ)-coloring φ. We suppose that $L = \{1, \ldots, t\}$ and that the colors are named so that $\varphi(v) = \{1, \ldots, \kappa(v)\}$. We can immediately extend φ properly to the other vertices in S by setting $\varphi(u) = \{1, \ldots, \kappa(u)\}$, $u \in S\backslash\{v\}$, invoking the hypothesis of the theorem and the choice of v. What remains is to extend φ properly to the other vertices in H.

Suppose that H is bipartite. Applying $(*)$ in the case when the subgraph is a single edge uw, $u \in X_1$, $w \in X_2$, we have $\kappa(u) + \kappa(w) \leq t$; we extend φ to $V(H)\backslash S$ by setting $\varphi(u) = \{1, \ldots, \kappa(u)\}$ if $u \in X_1\backslash S$ and $\varphi(w) = \{t - \kappa(w) + 1, \ldots, t\}$ for $w \in X_2$.

Suppose that H is complete multipartite with parts $X_1, \ldots, X_r$, $r \geq 3$. Let $p_i = \max_{w \in X_i} \kappa(w)$. Applying $(*)$ to a clique in H with vertices $w_i \in X_i$ with $\kappa(w_i) = p_i$, $i = 1, \ldots, r$, we see that $p_1 + \cdots + p_r \leq t$. Therefore $\{1, \ldots, t\}$ can be partitioned into sets $P_1, \ldots, P_r$, with $P_1 = \{1, \ldots, p_1\}$ and $|P_i| \geq p_i, i = 2, \ldots, r$. Extend φ to all of $V(H)$ by setting $\varphi(u) = \{1, \ldots, \kappa(u)\}$ for $u \in X_1\backslash S$ and, for $w \in X_i$, $i \geq 2$, $\varphi(w)$ is any subset of P_i of cardinality $\kappa(w)$ $(\leq p_i)$. □

Proof of Corollary 7 Taking the edges of E one at a time, this is a straight-forward application of the theorem in the cases when H is bipartite.

□

Proof of Theorem 2 Suppose that $p \geq 2$. Whether $m = 2p$ or $2p+1$, let u be the vertex of W_m of degree m, and let the other vertices be $v_1, \ldots, v_m$, around the cycle C_m. In what follows, the subscript i in v_i is to be read mod m.

Suppose that $t \geq 3$ and that $\kappa : V(W_{2p}) \to \mathbb{N}$ satisfies Hall's condition with W_{2p} and t. Let $L = \{1, \ldots, t\}$ and color u with $\{t - \kappa(u) + 1 \ldots, t\}$. Applying (*) with H being the K_3 induced by u, v_i, and v_{i+1}, we have that $\kappa(v_i) + \kappa(v_{i+1}) \leq t - \kappa(u)$, $i = 1, \ldots, 2p$; color v_i with $\{1, \ldots, \kappa(v_i)\}$ if i is odd, and with $\{t - \kappa(u) - \kappa(v_i) + 1, \ldots, t - \kappa(u)\}$ if i is even, and we have a proper (t, κ)-coloring of W_{2p}.

Now suppose that $t \geq 3$; we show that W_{2p+1} is not Hall t-chromatic by setting $\kappa(u) = t - 2$ and $\kappa(v_i) = 1$, $i = 1, \ldots, 2p+1$. Clearly, W_{2p+1} is not properly (t, κ)-colorable. It remains to be seen that W_{2p+1}, t, and κ satisfy Hall's condition. Check (*) for all proper induced subgraphs H of W_{2p+1} by verifying that $W_{2p+1} - w$ is properly (t, κ)-colorable for all $w \in V(W_{2p+1})$ [keep in mind that $t \geq 3$.] As for $H = W_{2p+1}$, $\sum_{w \in V(H)} \kappa(w) = 2p + t - 1 \leq t\alpha(H) = tp$ because $p \geq 2$ and $t \geq 3$, which imply that $0 \leq (t-2)(p-1) - 1$. □

Proof of Corollary 8 If $c > a$ then $K_{a:b}$ is an induced subgraph of $K_{c:b}$, so, by Compendium (2) and (3), it suffices to prove the claim with $a = 3b + \lceil \frac{b}{p} \rceil$. The idea of the proof is to show that W_{2p+1} is an induced subgraph of $K_{a:b}$; by Theorem 2 and Compendium (2) and (3), again, this will suffice.

First we properly $(2p+1, p)$-color C_{2p+1} with the colors $\{1, \ldots, 2p+1\}$ by setting $\varphi(v_i) = \{(i-1)p + 1, \ldots, ip\} \pmod{2p+1}$, $i = 1, \ldots, 2p+1$. This was the coloring used by Stahl [6] to embed C_{2p+1} as a subgraph of $K_{2p+1:p}$. What is important here is that this coloring embeds C_{2p+1} as an *induced* subgraph of $K_{2p+1:p}$; that is, not only are $\varphi(v_i)$ and $\varphi(v_{i\pm1})$ disjoint, $i = 1, \ldots, 2p+1$, but, also, if v_i and v_j are not adjacent in C_{2p+1}, ($j \not\equiv i \pm 1 \pmod{2p+1}$) then $\varphi(v_i) \cap \varphi(v_j) \neq \emptyset$. (Verifications omitted.)

If $b = p$ set $\theta = \varphi$. If $b > p$ we will eventually define θ by first setting $b_1 = b - p$. It is well known that $\chi^{(b_1)}(C_{2p+1}) = \lceil b_1 \rho(C_{2p+1}) \rceil = 2b_1 + \lceil \frac{b_1}{p} \rceil$; this also follows from Corollary 1, with $k = b_1$ and $G = C_{2p+1}$, and Compendium (1), for odd cycles. Let ψ be a proper $(2b_1 + \lceil \frac{b_1}{p} \rceil, b_1)$-coloring of C_{2p+1} using colors none of which are among $1, \ldots, 2p+1$, and let $\theta = \varphi \cup \psi$. (That is, $\theta(v_i) = \varphi(v_i) \cup \psi(v_i), i = 1, \ldots, 2p+1$.) Then θ is a proper $(2b_1 + \lceil \frac{b_1}{p} \rceil + 2p + 1, b_1 + p)$-coloring (i.e., a proper $(2b + \lceil \frac{b}{p} \rceil, b)$-coloring) of C_{2p+1} such that for non-adjacent vertices v, w on the cycle, $\theta(v) \cap \theta(w) \neq \emptyset$. Now bring in b new colors to color u, and we see that

W_{2p+1} is an induced subgraph of $K_{a:b}, a = 3b + \lceil \frac{b}{p} \rceil$. □

Proof of Corollary 9 Suppose that $b \geq \frac{1}{\epsilon}$ and that $a \geq (3+\epsilon)b$. Let $p = \lceil \frac{1}{\epsilon} \rceil \geq 2$. Then $b \geq p$, because b is an integer, and $\frac{1}{p} \leq \epsilon$, so $\frac{b}{p} \leq \epsilon b$. Therefore $a \geq 3b + \epsilon b \geq 3b + \frac{b}{p}$ implies that $a \geq 3b + \lceil \frac{b}{p} \rceil$, because a and b are integers. Now the conclusion follows from Corollary 8. □

Proof of Corollary 10 If $G = W_{2p+1}$, $p \geq 2$, then $\rho(G) = 3$ and $\chi^{(k)}(G) = k + \chi^{(k)}(C_{2p+1}) = k + \lceil k\rho(C_{2p+1}) \rceil = 3k + \lceil \frac{k}{p} \rceil$, because $W_{2p+1} = K_1 \vee C_{2p+1}$, and by previous remarks about $\chi^{(k)}(C_{2p+1})$. Now, $\{3, 4, \dots\}$ is the set of t such that W_{2p+1} is not Hall t-chromatic, by Theorem 2, and it is straightforward to verify directly that $\bigcup_{k=1}^{\infty}\{\lceil k\rho(W_{2p+1})\rceil, \dots, \chi^{(k)}(W_{2p+1}) - 1\} = \bigcup_{k=1}^{\infty}\{3k, \dots, 3k-1+\lceil \frac{k}{p} \rceil\}$ misses some of the values in this set (although the union covers a tail of $\mathbb{N}$, because $\rho(W_{2p+1}) = 3 < \chi_f(W_{2p+1}) = 3 + 1/p$). For instance, taking $p = 2$, we see that the claim of the corollary is verified with $G = W_5$ (previously noted, as have all the $W_{2p+1}, p \geq 2$, to be critically not Hall t-chromatic , for all $t \geq 3$) and $t = 4, 5, 7, 8, 11$, or 14.
□

Proof of Theorem 3 Suppose that G is critically not Hall t-chromatic, and $t < \rho(G)$. Because G is not Hall t-chromatic, there is a color demand function $\kappa : V(G) \to \mathbb{N}$ which satisfies Hall's condition, with G and t, but such that there is no proper (t, κ)-coloring of G. Because $t < \rho(G)$, for some subgraph H of G, $t < n(H)/\alpha(H)$. Applying $(*)$, we have $\sum_{v \in V(H)} \kappa(v) \leq t\alpha(H) < n(H)$. Therefore, $\kappa(v) = 0$ for some $v \in V(H)$. But $G - v$ is Hall t-chromatic; therefore, there is a proper (t, κ)-coloring φ of $G - v$, which we can extend to a proper (t, κ)-coloring of G by setting $\varphi(v) = \emptyset$. This contradiction establishes the claim of the theorem. □

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