Hamiltonian Circuits in Sparse Circulant Digraphs

Z. Bogdanowicz
US Army Armament R&D Center
Picatinny Arsenal, New Jersey 07806, USA

ABSTRACT

A circulant digraph $G(a_1, a_2,...,a_k)$ where $0 < a_1 < a_2 <...< a_k < |V(G)| = n$ is the vertex transitive directed graph that has vertices $i+a_1$, $i+a_2$, ..., $i+a_k \pmod n$ adjacent to each vertex i. We give the necessary and sufficient conditions for $G(a_1, a_2)$ to be hamiltonian, and we prove that $G(a, n-a, b)$ is hamiltonian. In addition, we identify the explicit hamiltonian circuits for a few special cases of sparse circulant digraphs.

KEY WORDS
Hamiltonian Cycles, Circulants, Digraphs, Hamiltonian Circuits

1. Introduction

Circulants have been widely studied in the literature [1,3,4,6,7]. An undirected graph G of order n is *circulant* if its automorphism group contains an *n-cycle*. This in turn means that G has vertices $i\pm a_1, i\pm a_2, ..., i\pm a_k \pmod n$ adjacent to each vertex i for $0 < a_1 < a_2 < ... < a_k \leq n/2$. The integers a_is are called the *jumps* [3]. A lot is known about hamiltonian properties of circulants. It has been shown that all connected circulants are hamiltonian [8], none-bipartite circulants of degree at least three are hamilton-connected, and bipartite circulants of degree at least three are hamilton-laceable [5]. Furthermore, it has also bean established that connected circulants of girth three are pancyclic, and connected circulants with at least two jumps are edge-bipancyclic [4]. However, not too much has been done to identify hamiltonian properties of circulant digraphs. In this work we partly address this issue and identify several properties of sparse circulant diagraphs (i.e., circulant diagraphs with two and three jumps).

In order to present the results we first extend the definition of circulant to a digraph. That is, circulant digraph $G_n(a_1, a_2, ..., a_k)$ is a directed graph of order n with vertices $i+a_1, i+a_2, ..., i+a_k \pmod n$ adjacent to each vertex i, where $0 < a_1 < a_2 < ... < a_k < n$ (see Figure 1). One can easily verify that not all circulant digraphs are hamiltonian. For example, $G_{12}(2,3)$ illustrated in Figure 1 is not hamiltonian.

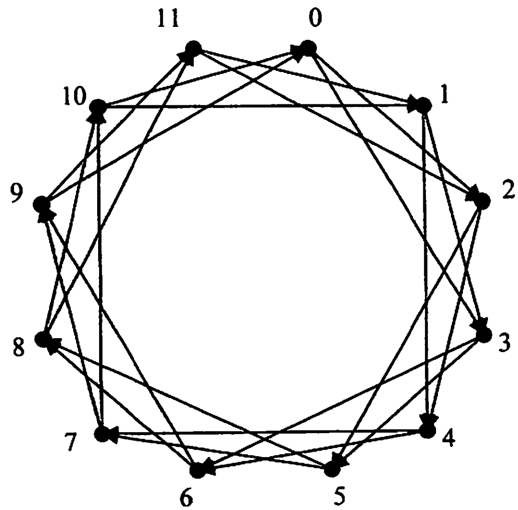

Figure 1 – Circulant diagraph $G_{12}(2,3)$

Throughout this paper our terminology and notation is based on the book by C. Berge except as indicated below [2]. Unless explicitly stated otherwise throughout the proofs in this paper, *gcd(x,y)* will be denoted by *(x,y)*. We give hamiltonian properties for circulant digraphs restricted to either two or three jumps. For convenience, $G_n(a,b)$ (or $G_n(a,b,c)$) will denote circulant diagraph of order n with two (or three) distinct jumps, so $a<b(<c)$ is not enforced. The paper is organized in such a way that in Section 2 we focus on results for $G_n(a,b)$, and in Section 3 we give results for $G_n(a,n-a,b)$. Our main result (Theorem 2.3) gives the necessary and sufficient conditions for $G_n(a,b)$ to have a hamiltonian circuit. We then show that connected $G_n(a,n-a,b)$ that contains edges corresponding to jump a is always hamiltonian (Theorem 3.2). In addition, we identify explicit hamiltonian cycles for a few special cases of sparse circulant digraphs in both sections.

2. Hamiltonian Circuits in Circulant Digraphs with Two Jumps

From this point on by circulant we mean a circulant digraph and by path we mean a directed path. To present our main result we first prove the following two lemmas.

Lemma 2.1 Let $G_n(a,b)$ be a hamiltonian circulant digraph without a hamiltonian circuit formed by a single jump. Let q be the number of circuits $x_i^j x_{i+1}^j \dots x_{i+r\,(mod\,r)}^j$ formed by jump a in $G_n(a,b)$, and r be the size of such a circuit. Then every hamiltonian circuit of G has associated integer p $(r \geq p > 0)$ and is of the form $C = P^0 P^1 \dots P^{k-1} x_0^0$, where $P^0 = x_0^0 x_1^0 \dots x_{p-1}^0$, $P^j = x_{i\,(mod\,r)}^{j\,(mod\,q)} x_{i+1\,(mod\,r)}^{j\,(mod\,q)} \dots x_{i+p-1\,(mod\,r)}^{j\,(mod\,q)}$, and $0<j<k$.

Proof: Clearly, C must include at least one arc of the form $x_i^j x_{i'}^{j+1\,(mod\,q)}$ and at least one arc of the form $x_i^j x_{i+1\,(mod\,r)}^j$. Thus, one of the vertices in C must be incident to both types of arc. By vertex transitivity assume without loss of generality that C contains $x_i^{q-1} x_0^0$ and $x_0^0 x_1^0$. Thus, C contains path $P^0\, x_{(p-1)'}^1$ for some integer p (where $r \geq p > 1$).
Suppose that C contains path $P^j\, x_{(i+p-1\,(mod\,r))'}^{j+1\,(mod\,q)}$ (Figure 2).

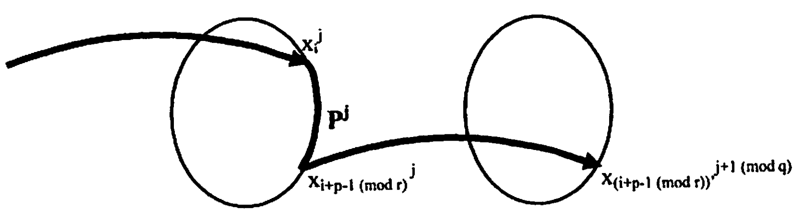

Figure 2 – P^j

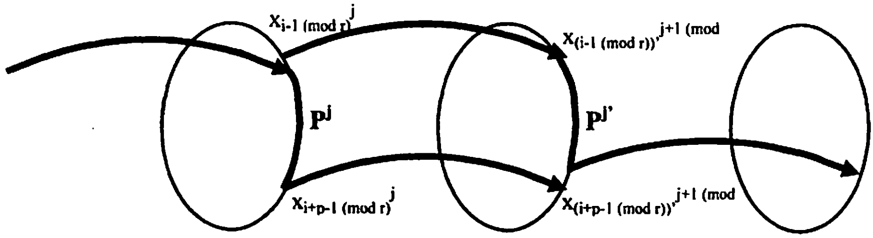

Figure 3 – P^j and its implied $P^{j'}$

If C doesn't contain arc $(x_{i-1}^j, x_{(i-1)'}^{j+1 \ (mod\ q)})$ then p is equal to the size of circuit formed by a single jump, k equals to the number of such circuits (i.e., $k=q$), and we are done. Suppose that C contains arc $(x_{i-1}^j, x_{(i-1)'}^{j+1 \ (mod\ q)})$. In this case, there are exactly p vertices between vertices $x_{(i-1\ (mod\ r))'}^{j+1\ (mod\ q)}$ and $x_{(i+p-1\ (mod\ r))'}^{j+1\ (mod\ q)}$ that immediately follow vertex $x_{(i-1\ (mod\ r))'}^{j+1\ (mod\ q)}$ in C. So, C must also contain $P^{j'}$ consisting of exactly p vertices (Figure 3). Then, by induction every path $P^0, P^1 \ldots, P^{k-1}$ must consist of exactly p vertices and each one must be followed by exactly one arc of the form $x_i^j x_{i'}^{j+1\ (mod\ q)}$ in C.

□

Lemma 2.2 Let $G_n(a,b)$ be a connected circulant digraph with two relatively prime jumps a,b and without a hamiltonian circuit formed by a single jump. Let r,s,t be positive integers such that $r \geq 1$, $n/\gcd(n,a) > s \geq 1$ and $n/\gcd(n,b) > t \geq 1$. Then $G_n(a,b)$ is hamiltonian if and only if either $n = r(s+1) \cdot \gcd(n,a) \cdot \gcd(n,b)$ and $\gcd(n, sa+b) = s+1$, or $n = r(t+1) \cdot \gcd(n,a) \cdot \gcd(n,b)$ and $\gcd(n, a+tb) = t+1$.

Proof: Consider a circulant $H_n(a)$ with $c = sa+b$ or $c = a+tb$. It must have either $s+1$ or $t+1$ circuits formed by c respectively. Without loss of generality consider c to be one of the above, say $c = sa+b$. Each arc $(i, i+a)$ in H corresponds then to a path $P = (P^i, i+sa+b)$ in G, where P^i is of the form $P^i = (i, i+a, i+2a, \ldots, i+sa)$. Because $(n, sa+b) = s+1$ then $i \equiv i+sa+b \ (mod\ s+1)$. In addition, vertices $i, i+a, i+2a, \ldots, i+sa$ in P^i must be all distinct $(mod\ s+1)$. Otherwise, $i \equiv i+s'a \ (mod\ s+1)$ for some $s' < s+1$. That in turn would mean that a can be divided by one of the factors, say f, of $s+1$. Since $sa+b$ is also divisible by f then it would further imply that b must be divisible by f, which would be a contradiction because a and b are relatively prime (i.e., $(a,b) = 1$). So, $i \equiv i+sa+b \ (mod\ s+1)$ and vertices $i, i+a, i+2a, \ldots, i+sa \ (mod\ s+1)$ in P^i are all distinct.

Consider now a circuit $(0, c, 2c, \ldots, (n/(s+1))c) \pmod n$ in $H_n(a)$ and traverse a corresponding circuit $P^0P^a \ldots P^{(n/(s+1)-1)a}0 \pmod n$ in $G_n(a,b)$. In P^0 all vertices $0, a, 2a, \ldots, sa$ are distinct $(\mathrm{mod}\ s+1)$, and hence they are visited exactly once. In addition, the next vertex c satisfies $sa+b \equiv 0\ (\mathrm{mod}\ s+1)$ and $sa+b > 0$, so it is visited once as well. Suppose that all vertices in $P^0P^c \ldots P^{ic}$ are visited exactly once for $i<(n/(s+1)-1)$. Then unvisited vertex $(i+1)c$, where $(i+1)c \equiv 0$ $(\mathrm{mod}\ s+1)$, will be visited next. Furthermore, the next $s+1$ vertices will be distinct $(\mathrm{mod}\ s+1)$, uniquely implied by current vertex $(i+1)c$, and hence visited exactly once. So, all the vertices in $P^0P^c \ldots P^{ic}\ P^{(i+1)c}$ must be visited exactly once for $i+1 \leq (n/(s+1)-1)$. Thus, by induction there exists a circuit of n vertices corresponding to circuit $(0, c, 2c, \ldots, (n/(s+1))c) \pmod n$ of $n/(s+1)$ arcs in $H_n(c)$. This completes the sufficient conditions.

If $G_n(a,b)$ is hamiltonian then by Lemma 2.1 a hamiltonian circuit must be of form either $(0, sa, sa+b, 2sa+b, 2sa+2b, 3sa+2b \ldots) \pmod n$ or $(0, tb, a+tb, a+2tb, 2a+2tb, 2a+3tb \ldots) \pmod n$. This induces $H_n(c)$ with $c = sa+b$ or $c = a+tb$ respectively. If $c = sa+b$ then the size of a circuit formed by c in $H_n(c)$ is $s+1$ times smaller than n. So, in this case there must be $s+1$ such circuits in $H_n(c)$ (i.e., $(n,c) = s+1$). If $c=a+tb$ then the size of a circuit formed by c in $H_n(c)$ is $t+1$ times smaller than n. So, in this case there must be $t+1$ such circuits in $H_n(c)$ (i.e., $(n,c) = t+1$). Since there is no hamiltonian circuit in $G_n(a,b)$ formed by a single jump, and a and b are relatively prime then (n,a) and (n,b) can not have a common factor f $(f>1)$. These terms however represent the number of circuits in $G_n(a,b)$ generated by a and b respectively. In addition, n must be divisible by either $(s+1)$ or $(t+1)$ since one of these terms (depending on which one is applied) must be relatively prime with respect to a and b. Thus, the number of vertices in $G_n(a,b)$ must be either $n = r(s+1)(n,a)(n,b)$ or $n = r(t+1)(n,a)(n,b)$ for some $r \geq 1$. This completes the necessary conditions.

□

Theorem 2.3 Let $G_n(a,b)$ be a connected circulant digraph and let r,s, and t be positive integers such that $r \geq 1$, $n/\gcd(n,a) > s \geq 1$, and $n/\gcd(n,b) > t \geq 1$. Then $G_n(a,b)$ is hamiltonian if and only if any of the following holds:

(1) $\gcd(n,a) = 1$
(2) $\gcd(n,b) = 1$
(3) $n = r(s+1) \cdot \gcd(n,a) \cdot \gcd(n,b)$ and $\gcd(n, sa+b) = s+1$
(4) $n = r(t+1) \cdot \gcd(n,a) \cdot \gcd(n,b)$ and $\gcd(n, a+tb) = t+1$.

Proof: If either $(n,a) = 1$ or $(n,b) = 1$ holds then the case is obvious where a hamiltonian circuit is formed by a single jump. Suppose that (1) and (2) do not hold. Since no single jump forms a hamiltonian circuit in G then jumps a,b can be represented by Ca', Cb', where $(n,C) = 1$ and $(a',b') = 1$. Note, that for G to be connected there must exist jumps a', b' that are relatively prime. If a and b are relatively prime then $C = 1$ (or otherwise G would be disconnected) and proof follows from Lemma 2.2. Otherwise, $G = G_n(Ca',Cb')$ for some $C>1$. In this case isomorphism $C*i \pmod n \rightarrow i$ induces a circulant H with jumps a' and b' (i.e., $H = H_n(a',b')$) that is isomorphic with G (i.e., $H = G$). Furthermore, the following relations are satisfied: $(n,a') = (n,Ca')$, $(n, b') = (n,Cb)$, $(n,sa'+b') = (n,sCa'+Cb')$, and $(n,a'+tb') = (n,Ca'+tCb')$. So, the proof follows from Lemma 2.2.

□

Based on Theorem 2.3 a circulant digraph with two jumps and nontrivial hamiltonian circuit requires at least 2*3*5 vertices. By the above theorem $G_{30}(3,5)$ is hamiltonian.

Corollary 2.4 Let G_n be a circulant digraph with at least two jumps. Define $H_n(a,b)$ to be a connected subgraph of G_n. Let s,t be positive integers such that $n/\gcd(n,a)>s\geq 1$ and $n/\gcd(n,b)>t\geq 1$. Then G_n is hamiltonian if it contains $H_n(a,b)$ that satisfies at least one of the following conditions:

(1) a forms a hamiltonian circuit, or
(2) $\gcd(n,sa+b) = s+1$, or
(3) $\gcd(n, a+tb) = t+1$.

Proof: Follows directly from Theorem 2.3.

□

Corollary 2.5 Let $G_{2(n,\ a)\ (n,\ b)}(a,b)$ be a connected circulant digraph with two odd jumps a,b and without a hamiltonian circuit formed by a single jump. Then G_n (where $n = 2\cdot\gcd(n, a)\cdot\gcd(n, b)$) has a hamiltonian circuit of form $(0, a, a+b, 2a+b, , 2a+2b, 3a+2b, 3a+3b\ldots) \pmod n$.

Proof: It is sufficient to show that $n = 2(n,a)(n,b)$ implies $(n,a+b) = 2$. Since jumps a,b are odd then among other factors n must have exactly one factor f, where $f = 2$. Suppose that there exists an odd factor f' that satisfies $(n,a+b)$, (i.e., $n\equiv 0 \pmod{f'}$ and $a+b\equiv 0 \pmod{f'}$). Such a factor could not satisfy at the same time a and b (i.e., $a\equiv 0 \pmod{f'}$ and $b\equiv 0 \pmod{f'}$) because that would mean that G is disconnected. On the other hand, if it just satisfied only one jump then it could not satisfy $a+b$ - a contradiction. In addition $a+b$ must be even, so $(n,a+b) = 2$ is satisfied. Rest of the proof follows directly from Lemma 2.1 and Theorem 2.3 as a special case.

□

Note, we cannot extend Corollary 2.5 to apply it to $G_{k(n,\ a)\ (n,\ b)}(a,b)$ for $k>2$, since $\gcd(n,sa+b)$ and $\gcd(n,a+tb)$ cannot be satisfied in general.

3. Hamiltonian Circuits in Circulant Digraphs containing Edges Defined by a Single Jump

Throughout this section we assume without loss of generality that a defines edges in circulant G_n (i.e., G_n contains jumps a and $n-a$, or briefly a and $-a$). Note, case $a = n/2$ was already implicitly covered as a special case in the previous section. So, we assume now that $a \neq n/2$.

Lemma 3.1 Let $G_n(a,-a,b)$ be a connected circulant digraph. Let n satisfy $n = k \cdot \gcd(n,a) \cdot \gcd(n,b)$ for $k \geq 1$. Then $G_n(a,-a,b)$ is hamiltonian if any of the following holds:

(1) $\gcd(n,a) = 1$
(2) $\gcd(n,b) = 1$
(3) $\gcd(n,a)$ even
(4) $\gcd(n,a) > \gcd(n,b)$,
$\gcd(k \cdot \gcd(n,b), \gcd(n,b)((\gcd(n,b)-1)\, a+b)+(\gcd(n,a)-\gcd(n,b))\, b) = \gcd(n,b)$, and $\gcd(n,b)$ odd.

Proof: If either $(n,a)=1$ or $(n,b)=1$ holds then the case is obvious where a hamiltonian circuit is formed by a single jump. Suppose that (1) and (2) do not hold. We will show that an explicit hamiltonian circuit can be formed for (n,a) being even and for $(n,a)>(n,b)$ if (n,a), (n,b) are odd and $(n,b)((n,b)-1)\, a+((n,b)-1)b \equiv 0 \pmod{k(n,b)}$. The relationship between circuits formed by a and b is illustrated in Figure 4 for $k=1$, and in Figure 5 for $k>1$.

Case 1. (n,a) even. Then number of circuits formed by a is even and we first traverse (n,a) circuits as follows:

$0, a, 2a, \ldots, ((n,b)-1)\, a,\ ((n,b)-1)\, a+b,$
$((n,b)-1)\, a+b-a,\ ((n,b)-1)\, a+b-2a, \ldots, a+b,$
$b,\ b+a,\ b+2a, \ldots, b+((n,b)-1)\, a,\ ((n,b)-1)\, a+2b,$
$((n,b)-1)\, a+2b-a,\ ((n,b)-1)\, a+2b-2a, \ldots, a+2b,$
....
$((n,a)-2)b,\ ((n,a)-2)b+a,\ ((n,a)-2)b+2a, \ldots, ((n,b)-1)\, a+((n,a)-1)b,$
$((n,b)-1)\, a+((n,a)-1)b-a,\ ((n,b)-1)\, a+((n,a)-1)b-2a, \ldots, a+((n,a)-1)b, \pmod n.$

If $k=1$ then we terminate hamiltonian circuit with vertices $((n,a)-1)b,\ (n,a)b, \pmod n$, where $(n,a)b \equiv 0 \pmod n$. Otherwise, we follow with vertex $((n,a)-1)b \pmod n$, (see Figures 4,5), and for every consecutive i iteration, $1 \leq i < k$ we continue traversing (n,a) circuits in $G_n(a,-a,b)$ as follows:

$(n,a)bi,\ (n,a)bi+a,\ (n,a)bi+2a, \ldots, (n,a)bi+((n,b)-1)\, a,\ (n,a)bi+((n,b)-1)\, a+b,$
$(n,a)bi+((n,b)-1)\, a+b-a,\ (n,a)bi+((n,b)-1)\, a+b-2a, \ldots, (n,a)bi+a+b,$

$(n, a)bi + b, b+a, (n, a)bi + b+2a, \ldots, (n, a)bi + b+((n,b)-1) a, ((n,b)-1) a+ 2b,$
$(n, a)bi + ((n,b)-1) a+2b-a, (n, a)bi + ((n,b)-1) a+2b-2a, \ldots, (n, a)bi + a+2b,$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$(n, a)bi+((n, a)-2)b, (n, a)bi+((n, a)-2)b+a, \ldots, (n, a)bi+((n,b)-1) a+ ((n, a)-1)b,$
$(n, a)bi+((n,b)-1) a+((n, a)-1)b-a, (n, a)bi+((n,b)-1) a+((n, a)-1)b-2a, \ldots,$
$(n, a)bi+a+((n, a)-1)b, (n, a)bi+((n, a)-1)b \pmod n$

After the last iteration i, we complete a hamiltonian circuit with vertex $(n, a)bk \pmod n$, where $(n,a)bk \pmod n \equiv 0 \pmod n$.

Case 2. $(n,a)>(n,b)$, $(k(n,b),(n,b)(((n,b)-1) a+b)+((n,a)-(n, b)) b)=(n,b)$, and (n,b) odd. In this case we consider the number of circuits formed by a to be odd (otherwise *Case 1* is satisfied) and we traverse first j, $j=(n,b)$ circuits in the first iteration i, $i=1$ as follows:

$0, a, 2a, \ldots, ((n,b)-1) a,$
$((n,b)-1) a+ b, ((n,b)-1) a+ b+ a, ((n,b)-1) a+ b+ 2a \ldots, 2((n,b)-1) a+ b,$
$(((n,b)-1) a+ b,)2, (((n,b)-1) a+ b,)2+ a, (((n,b)-1) a+ b,)2+ 2a, \ldots,$
$3((n,b)-1) a+2b,$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$(((n,b)-1) a+ b,)(j-1), (((n,b)-1) a+ b,)(j-1)+ a, (((n,b)-1) a+ b,)(j-1)+ 2a, \ldots,$
$j((n,b)-1) a+(j-1)b \pmod n$

At this point the last vertex satisfies $j((n,b)-1)a+(j-1)b = (n,b)((n,b)-1) a+((n,b)-1)b$ and consequently satisfies $(n,b)((n,b)-1) a+((n,b)-1)b \equiv 0 \pmod{(n,b)}$. This vertex corresponds to Y in Figures 4 and 5 when $k=1$ or $k=2$ respectively. Since $(n,a)-(n, b)$ is even, the remaining $(n,a)-n,b)$ circuits formed by a can follow a pattern as in *Case 1* for i-th iteration with starting vertex $(n,b)(((n,b)-1) a+b)i \pmod n$ instead of $(n, a)bi \pmod n$. If $k=1$ then we terminate hamiltonian circuit with vertex $(n,b)(((n,b)-1) a+b)+((n,a)-(n, b)) b \pmod n$. Otherwise, we continue with subsequent iterations for $i=2,3,..,k$. After the last iteration i, $i=k$ we complete a hamiltonian circuit with vertex $((n,b)(((n,b)-1) a+b)+((n,a)-(n, b)) b)k \pmod n$, where $((n,b)(((n,b)-1) a+b)+((n,a)-(n, b)) b)k \equiv 0 \pmod n$.

□

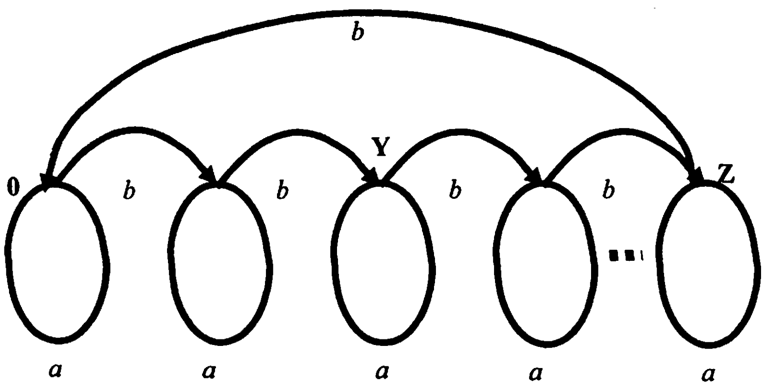

Figure 4 – Relationship between circuits formed by *a* and *b* for $k=1$

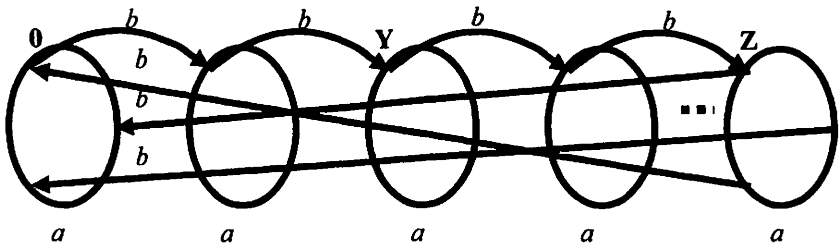

Figure 5 – Relationship between circuits formed by *a* and *b* for $k=3$

Theorem 3.2 The connected circulant digraph $G_n(a,-a,b)$, $(a\neq n/2)$ is hamiltonian.

Proof: If either $(n,a)=1$ or $(n,b)=1$ holds then the case is trivial. Suppose that there is no hamiltonian circuit formed by a single jump. Order of G_n must satisfy $n=k(n, a)(n, b)$ for $k\geq 1$. By *Case1* in Lemma 3.1 $G_n(a,-a,b)$ is hamiltonian for (n,a) even. *Case 2* in Lemma 3.1 is always satisfied for (n,a) odd, and for $k=1$ or 2. Hence it remains to show that for (n,a) odd, and for $k>2$ or (n,b) even, $G_n(a,-a,b)$ is hamiltonian. If $k>2$ then the size of circuit formed by *b* must be at least 4 (Figure 5). If (n,b) is even then it would have to be at least 4 because 2 would imply $a=n/2$. So, we must only consider odd number of circuits formed by jump *a*, where the size of each circuit is at least 4. We show that an explicit hamiltonian circuit can be constructed in this case.

First we visit each circuit formed by jump a exactly once (i.e., (n,a) circuits) and terminate on vertex $(n-1)a \pmod n$ in Pass 1 as follows:

$0, b, b-a, b-2a, \ldots b+(k-1)(n,b)a-2a,$
$2b+(k-1)(n,b)a-2a,\ 2b+(k-1)(n,b)a-a,$
$3b+(k-1)(n,b)a-a,\ 3b+(k-1)(n,b)a-2a,$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$((n,a)-2)b+(k-1)(n,b)a-a,\ ((n,a)-2)b+(k-1)(n,b)a-2a,$
$((n,a)-1)b+(k-1)(n,b)a-2a,\ ((n,a)-1)b+(k-1)(n,b)a-a,\ (n-1)a \pmod n$

In Pass 2 we traverse through all circuits formed by a again visiting each circuit once, visiting all the remaining vertices, and terminating on vertex 0 (refer to Figures 3,4).

$(n-2)a,\ (n-3)a,\ \ldots,\ 2a, a,$
$b+a,\ b+2a,\ \ldots,\ b+(k-1)(n,b)a-3a,$
$2b+(k-1)(n,b)a-3a,\ 2b+(k-1)(n,b)a-4a,\ \ldots, 2b+(k-1)(n,b)a,$
$3b+(k-1)(n,b)a,\ 3b+(k-1)(n,b)a+a,\ \ldots,\ 3b+(k-1)(n,b)a-3a,$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$((n,a)-2)b+(k-1)(n,b)a,\ ((n,a)-2)b+(k-1)(n,b)a+a,\ \ldots,\ ((n,a)-2)b+(k-1)(n,b)a-3a,$
$((n,a)-1)b+(k-1)(n,b)a-3a,\ ((n,a)-1)b+(k-1)(n,b)a-4a,\ \ldots,\ ((n,a)-1)b+(k-1)(n,b)a,$
$0 \pmod n$

□

Based on Theorem 3.2 we can state the following result.

Corollary 3.3 Let G_n be a circulant digraph with at least three jumps and $H_n(a,-a,b)$), $(a \neq n/2)$, be a connected spanning subgraph of G_n. Then G_n is hamiltonian.

Finally, we give a miscelaneous special case result derived from the proof of Lemma 3.1.

Corollary 3.4 Let $G_n(a,2a,b)$ be a connected circulant digraph without a hamiltonian circuit formed by a single jump. Let n satisfy $n = \gcd(n,a) \cdot \gcd(n,b)$ and at least one of the following cases be also satisfied:

(1) $\gcd(n\ a)$ even, or
(2) $\gcd(n,a) > \gcd(n,b)$ and $\gcd(n,b)$ odd.

Then $G_n(a,2a,b)$ is hamiltonian.

Proof: Consider first circulant $H_n(a,-a,b)$ induced by $G_n(a,2a,b)$. By Lemma 3.1, $k = 1$ and hence equation $(k(n,b),(n,b)(((n,b)-1)\ a+b)+((n,a)-(n,\ b))\ b) = (n,b)$ is satisfied. So, by Lemma 3.1 circulant $H_n(a,-a,b)$ is hamiltonian. In addition, for $k = 1$ each hamiltonian circuit C identified in proof of Lemma 3.1 can be represented as $C = P^0P^1...P^{k-1}x_0^0$, where $P^j = x_i^jx_{i+1}^j...x_{i+(n,b)-1}^j$ or $P^j = x_{i+(n,b)-1}^j x_{i+(n,b)-2}^j...x_i^j$, and $0 \leq j < (n,b)$. But each P^j can be realized in $G_n(a,2a,b)$ as follows:

(1) $P^j = x_i^jx_{i+1}^j...x_{i+(n,b)-1}^j = x_i^j, x_i^j+a, x_i^j+2a, x_i^j+3a, ...,$
$x_i^j+((n,b)-1)a\ (mod\ (n,b))$ or

(2) $P^j = x_{i+(n,b)-1}^jx_{i+(n,b)-2}^j...x_i^j = x_i^j+((n,b)-1)a, x_i^j+((n,b)+1)a,$
$x_i^j+((n,b)+2)a,\ ...,\ x_i^j+((n,b)-2)a, x_i^j\ (mod\ (n,b)).$

□

Finally we conclude that it would be a natural extension to this paper to solve the following problem(s). Define G to be *minimally connected* if the removal of arcs associated with any single jump in G would result in disconnected graph. Identify the necessary and/or sufficient conditions for the minimally connected graph G to be hamiltonian.

REFERENCES

1. B. Alspach, S. Locke, and D. Witte, The Hamilton spaces of Cayley graphs on Abelian groups. Discrete Math. **82** (1990) 113-126.
2. C. Berge, *Graphs and Hypergraphs.* North-Holland, Reading, 1979.
3. F.T. Boesch and R. Tindell, *Circulants and their connectivities.* J. Graph Theory **8** (1984) 129-138.
4. Z. R. Bogdanowicz, *Pancyclicity of connected circulant graphs.* J. Graph Theory **22** (1996) 167-174.
5. C. Chen and N. Quimpo, *On strongly hamiltonian abelian group graphs.* Combinatorial Mathematics VIII, Lecture Notes in Mathematics 884, Springer-Verlag, Berlin (1981).
6. P.J. Davis, Circulant Matrices, Wiley, New York (1979)
7. B. Elspas and J. Turner, Graphs with circulant adjacency matrices. *J. Combin. Theory* **9** (1970) 297-307.
8. L. Lovasz, Combinatorial Problems and Exercises, North-Holland, Amsterdam (1979).
9. D. Marusic and T. D. Parsons, Hamiltonian paths in vertex-symmetric graphs of order 4p. Discrete Math. **43** (1983) 91-96