A σ_k type condition for heavy cycles in weighted graphs

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Abstract

A weighted graph is one in which every edge e is assigned a non-negative number, called the weight of e. For a vertex v of a weighted graph, $d^w(v)$ is the sum of the weights of the edges incident with v. For a subgraph H of a weighted graph G, the weight of H is the sum of the weights of the edges belonging to H. In this paper, we give a new sufficient condition for a weighted graph to have a heavy cycle. Let G be a k-connected weighted graph where $2 \le k$. Then G contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$, if G satisfies the following conditions: (1) The weighted degree sum of any k independent vertices is at least m, (2) $w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x,y) = 2$, and (3) In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.

Keywords. Weighted graph, heavy cycle, weighted degree sum

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1 Terminology and notation

We only consider finite undirected graphs without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the set of vertices and edges of a graph G, respectively. A *weighted graph* is one in which each edge e is assigned a nonnegative number $w(e)$, called the *weight* of e. For a subgraph H of G, the *weight* of H is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

For each vertex $v \in V(G)$, $N_G(v)$ is the set, and $d_G(v)$ the number, of neighbors of v in G. We define the *weighted degree* of v in G by

$$d_G^w(v) = \sum_{u \in N_G(v)} w(uv).$$

When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$, and $d_G^w(v)$ by $N(v)$, $d(v)$, and $d^w(v)$, respectively. An *(x,y)-path* is a path whose end vertices are x and y. The distance between two vertices x and y, denoted by $d(x,y)$, is the minimum length of an (x,y)-path.

The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$. For a positive integer $k \leq \alpha(G)$, $\sigma_k(G)$ denotes the minimum value of the degree sum of any k independent vertices, and $\sigma_k^w(G)$ denotes the minimum value of the weighted degree sum of any k independent vertices.

2 Results

There have been many results on the existence of long cycles in graphs. The following results are well-known.

Theorem A (Dirac [4]) *Let G be a 2-connected graph such that $d(v) \geq d$ for every vertex v in $V(G)$. Then G contains either a Hamilton cycle or a cycle of length at least $2d$.*

Theorem B (Pósa [6]) *Let G be a 2-connected graph such that $d(u) + d(v) \geq c$ for each pair of nonadjacent vertices u and v in $V(G)$. Then G contains either a Hamilton cycle or a cycle of length at least c.*

Theorem C (Fournier and Fraisse [5]) *Let G be a k-connected graph where $2 \le k < \alpha(G)$, such that $\sigma_{k+1}(G) \ge m$. Then G contains either a Hamilton cycle or a cycle of length at least $2m/(k+1)$.*

An unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every vertex v, and the weight of a cycle is simply the length of the cycle.

Theorem A and Theorem B were extended to weighted graphs by the following two theorems, respectively.

Theorem 1 (Bondy and Fan [2]) *Let G be a 2-connected weighted graph such that $d^w(v) \ge d$ for every vertex v in $V(G)$. Then G contains a cycle of weight at least $2d$ or every heaviest cycle in G is a Hamilton cycle.*

Theorem 2 (Bondy et al. [1]) *Let G be a 2-connected weighted graph such that $d^w(u) + d^w(v) \ge c$ for each pair of nonadjacent vertices u and v in $V(G)$. Then G contains either a Hamilton cycle or a cycle of weight at least c.*

And, Theorem C was extended to weighted graphs by the following theorem in the case $k = 2$.

Theorem 3 (Zhang et al. [7]) *Let G be a 2-connected weighted graph which satisfies the following conditions:*

(1) $\sigma_3^w(G) \ge m$.

(2) $w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x, y) = 2$.

(3) In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.

Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.

In this paper, we extend Theorem C to weighted graphs for all k. Our main result is the following theorem.

Theorem 4 *Let G be a k-connected weighted graph where $k \ge 2$. Suppose that G satisfies the following conditions.*

(1) $\sigma^w_{k+1}(G) \geq m$.

(2) $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x, y) = 2$.

(3) In every triangle T *of* G*, either all edges of* T *have different weights or all edges of* T *have the same weight.*

Then G *contains either a Hamilton cycle or a cycle of weight at least* $2m/(k+1)$.

In our proof of Theorem 4, we need the following lemma.

Lemma 1 *Let* G *be a connected weighted graph satisfying Conditions (2) and (3) of Theorem 4. Then* G *satisfies one of the following:*

(a) all edges of G *have the same weight, or*

(b) G *is a complete multi-partite graph.*

3 Proof of Lemma 1

Let G be a connected weighted graph satisfying Conditions (2) and (3) of Theorem 4. Suppose that there exists $e_1, e_2 \in E(G)$ such that $w(e_1) \neq w(e_2)$. We need to prove that G is a complete multi-partite graph.

Since G is connected, we can choose a vertex x so that there exist $u, v \in N(x)$ such that $w(ux) \neq w(vx)$. Let $\bigcup_{i=1}^{n} V_i$ be a partition of $N(x)$ such that for $u \in V_i$ and $v \in V_j$, $w(ux) = w(vx)$ if and only if $i = j$. Now we denote the weight of the edges joining x and V_i by w_i for $1 \leq i \leq n$.

Claim 1 *Let* $1 \leq i, j \leq n$ *and* $v_i \in V_i$, $v_j \in V_j$. *If* $i \neq j$, $v_i v_j \in E(G)$.

Proof. Since $w(xv_i) \neq w(xv_j)$, Condition (2) of Theorem 4 implies $d(v_i, v_j) \neq 2$. Hence $v_i v_j \in E(G)$. □

Claim 2 *If there exists a vertex* y *such that* $d(x, y) = 2$, *then* $vy \in E(G)$ *for all* $v \in N(x)$.

Proof. The fact $d(x, y) = 2$ shows that there is a neighborhood v_1 of y in $N(x)$. Without loss of generality, we may assume $v_1 \in V_1$. And Condition (2) of Theorem 4 implies $w(yv_1) = w_1$.

Now suppose that there exists a vertex $v \in \bigcup_{i=2}^{n} V_i$ with $yv \notin E(G)$. Then Claim 1 implies $v_1 v \in E(G)$, and Condition (2) of Theorem 4 shows $w(v_1 v) = w(yv_1) = w_1$. Hence, applying Condition (3) of Theorem 4 to the triangle xv_1v, we have $w(xv) = w_1$. This contradicts the definition of the partition $\bigcup V_i$. So we must have $yv \in E(G)$ for all $v \in \bigcup_{i=2}^{n} V_i$.

Applying the same argument to $v_2 \in V_2 \cap N(y)$ and $v \in V_1$, we have $yv \in E(G)$ for every $v \in V_1$. □

If there exists a vertex y such that $d(x, y) = 2$, Condition (2) of Theorem 4 implies $w(v_i y) = w_i$ for all $v_i \in V_i$.

Claim 3 *There is no vertex z such that $d(x, z) = 3$.*

Proof. Suppose that there exists a vertex z such that $d(x, z) = 3$. Then z has a neighbor y such that $d(x, y) = 2$. Now Claim 2 implies that we have $v_1 \in N(y) \cap V_1$ and $v_2 \in N(y) \cap V_2$ with $w(yv_1) = w_1$ and $w(yv_2) = w_2$.

Since $d(z, v_1) = d(z, v_2) = 2$, Condition (2) of Theorem 4 shows $w(zy) = w(yv_1) = w_1$ and $w(zy) = w(yv_2) = w_2$. So we have $w_1 = w_2$, a contradiction. □

Let $V_0 = \{x\} \cup \{y : d(x, y) = 2\}$. Then $\bigcup_{i=0}^{n} V_i$ is a partition of $V(G)$.

Claim 4 *Let $0 \le i < j \le n$ and $v_i \in V_i$, $v_j \in V_j$. Then $v_i v_j \in E(G)$.*

Proof. If $i \ne 0$ and $j \ne 0$, Claim 1 implies $v_i v_j \in E(G)$. So we may assume $i = 0$. If $v_i = x$, the definition of $\bigcup_{i=1}^{n} V_i$ shows $v_i v_j \in E(G)$, and if $v_i \ne x$, Claim 2 implies $v_i v_j \in E(G)$. □

Note that for all $v_0 \in V_0$, $v_0 = x$ or $d(x, v_0) = 2$. Hence for all $v_i \in V_i (1 \le i \le n)$, $w(v_0 v_i) = w_i$.

Claim 5 *$v_0 v_0' \notin E(G)$ for all $v_0, v_0' \in V_0$.*

Proof. If $v_0 = x$, $d(x, v_0') = 2$ for all vertices $v_0' \in V_0 \backslash \{v_0\}$. Hence $v_0 v_0' \notin E(G)$. So we may assume $v_0, v_0' \ne x$. Now we suppose $v_0 v_0' \in E(G)$. Claim 2 implies that there exists $v_1 \in V_1, v_2 \in V_2$ such that $v_1, v_2 \in N(v_0) \cap N(v_0')$. Now we have $w(v_0 v_1) = w(v_0' v_1) = w_1$, $w(v_0 v_2) = w(v_0' v_2) = w_2$. So applying Condition (3) of Theorem 4 to the triangles $v_0 v_0' v_1$ and $v_0 v_0' v_2$, we have $w_1 = w_2$, a contradiction. □

Claim 6 *Let $0 \le i \le n$ and $t, u, v \in V_i$. If $tu, uv \notin E(G)$, then $tv \notin E(G)$.*

Proof. If $i = 0$, Claim 5 implies that $tv \notin E(G)$. So we assume that $1 \leq i \leq n$. Suppose $tv \in E(G)$. Without loss of generality, we may assume $i = 1$. Let $v_2 \in V_2$. Now, since $t, u, v \in V_1$, $w(xt) = w(xu) = w(xv) = w_1$. Then applying Condition (3) of Theorem 4 to the triangle xtv, we have $w(tv) = w(xt) = w_1$. On the other hand, Claim 4 implies $v_2t, v_2u, v_2v \in E(G)$. Since tu and $uv \notin E(G)$, Condition (2) of Theorem 4 shows $w(v_2t) = w(v_2u) = w(v_2v)$. Then applying Condition (3) of Theorem 4 to the triangle v_2tv, we have $w(tv) = w(v_2t)$. Hence, $w(v_2t) = w(tv) = w(xt) = w_1$. So applying Condition (3) of Theorem 4 to the triangle xtv_2, we have $w_2 = w(xv_2) = w(xt) = w_1$, a contradiction. □

Now on every V_i $(0 \leq i \leq n)$, nonadjacency is an equivalence relation. Let $V_{i,1}, \ldots, V_{i,m_i}$ be the equivalence classes of V_i. Then, for all vertices $u \in V_{i,j}$ and $v \in V_{i',j'}$, $uv \in E(G)$ if and only if $(i,j) \neq (i',j')$. Hence, G is a complete multi-partite graph with partite sets $V_0, V_{i,j}(1 \leq i \leq n, 1 \leq j \leq m_i)$. This completes the proof of Lemma 1. □

4 Proof of Theorem 4

Let G be a weighted graph satisfying the conditions of Theorem 4. If $k \geq \alpha(G)$, the following theorem implies the assertion.

Theorem D (Chvátal and Erdős [3]) *Let G be a k-connected graph with at least three vertices. If $k \geq \alpha(G)$, then G contains a Hamilton cycle.*

So we may assume $2 \leq k < \alpha(G)$. Now Lemma 1 implies that all edges of G have the same weight or G is a complete multi-partite graph.

Assume that all edges of G have the same weight w_1. If $w_1 = 0$, the assertion is obvious. If $w_1 \neq 0$, $d^w(v) = w_1 d(v)$ for all $v \in V(G)$. Hence $\sigma_{k+1}(G) = \sigma^w_{k+1}(G)/w_1 \geq m/w_1$. Then Theorem C implies that G contains either a Hamilton cycle or a cycle C of length at least $2m/(k+1)w_1$. Now $w(C) = w_1|E(C)| \geq 2m/(k+1)$.

Hence, we may assume that G is a complete multi-partite graph. Let $n = |G|$ and $V_1, \cdots, V_l$ be the partite sets of G.

Claim 1 *If $x, y \in V_i$, then $w(xz) = w(yz)$ for every $z \in V(G)\backslash V_i$. In particular, $d^w(x) = d^w(y)$.*

Proof. Since x and y are in the same partite set V_i, $xy \notin E(G)$. Hence, Condition (2) implies $w(xz) = w(yz)$. And hence, the assertion $d^w(x) = d^w(y)$ is obvious. □

Claim 2 *If G is not hamiltonian, then $|V_i| > n/2$ for some i such that $1 \leq i \leq l$.*

Proof. Suppose that $|V_i| \leq n/2$ for all i $(1 \leq i \leq l)$. Then for each $v \in V_j$ $(1 \leq j \leq l)$,

$$d(v) = \sum_{r \neq j} |V_r| = n - |V_r| \geq n/2.$$

Hence, Theorem A implies that G has a Hamilton cycle, a contradiction.
□

Without loss of generality, we can assume that $|V_1| > n/2$. Let $p = |V_1|$ and $q = n - p$. Then, since G is k-connected, it is obvious that $k \leq q < p$. And let $V_1 = \{v_1, v_2, \ldots, v_p\}$, $V(G) \backslash V_1 = \{u_1, u_2, \ldots u_q\}$.

Claim 3 *$d^w(v) \geq m/(k+1)$ for all $v \in V_1$.*

Proof. Since $k < p$, we can choose $v_1, v_2, \ldots, v_{k+1}$ in V_1. Now, $\{v_1, v_2, \ldots, v_{k+1}$ is independent, hence $\sum_{i=1}^{k+1} d^w(v_i) \geq m$. Then Claim 1 implies $d^w(v_1) = d^w(v_2) = \cdots = d^w(v_{k+1})$, so $d^w(v_1) \geq m/(k+1)$. Using Claim 1 again, we have $d^w(v) \geq m/(k+1)$ for all $v \in V_1$. □

Now we consider the cycle $C = v_1u_1v_2u_2 \cdots v_{q-1}u_{q-1}v_qu_qv_1$. Then Claim 1 implies

$$\begin{aligned} w(C) &= w(v_1u_1) + w(u_1v_2) + w(v_2u_2) + \cdots \\ &\quad + w(v_{q-1}u_{q-1}) + w(u_{q-1}v_q) + w(v_qu_q) + w(u_qv_1) \\ &= w(v_1u_1) + w(u_1v_1) + w(v_1u_2) + \cdots \\ &\quad + w(v_1u_{q-1}) + w(u_{q-1}v_1) + w(v_1u_q) + w(u_qv_1) \\ &= 2\sum_{i=1}^{q} w(v_1u_i) \\ &= 2d^w(v_1). \end{aligned}$$

Hence, by Claim 3, $w(C) \geq 2m/(k+1)$. This completes the proof of Theorem 4.

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