ON LINE GRAPHS WHICH ARE WEAKLY MAXIMAL CLIQUE IRREDUCIBLE

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ABSTRACT. The maximal clique that contains an edge which is not contained in any other maximal cliques is called **essential**. A graph in which each maximal clique is essential is said to be **maximal clique irreducible**. Maximal clique irreducible graphs were introduced and studied by W.D. Wallis and G.-H. Zhang in 1990 [6]. We extend the concept and define a graph to be **weakly maximal clique irreducible** if the set of all essential maximal cliques is a set of least number of maximal cliques that contains every edge. We characterized the graphs for which each induced subgraph is weakly maximal clique irreducible in[4]. In this article, we characterize the line graphs which are weakly maximal clique irreducible and also the line graphs which are maximal clique irreducible.

1. INTRODUCTION

In this paper, all graphs considered are finite, undirected, and without multiple edges or loops. If G and H are two graphs, then G is said to be H-free if it contains no induced subgraph isomorphic to H. For terminology not defined here, please see [5].

A clique in a graph G is a subset Q of the vertex set $V(G)$ such that every two vertices in Q are adjacent. The complete subgraph induced by a clique is also called a clique. A clique is maximal if it is not properly contained in any other clique. An edge (maximal resp.) clique covering of a graph G is a set of (maximal respectively) cliques that contains every edge in G. We denote by ECC (EMCC respectively) an edge (maximal respectively) clique covering, and mECC (mEMCC respectively) an edge (maximal respectively) clique covering of the minimum size, and $cc(G)$ the number of cliques in an mECC of a graph G.

Definition 1.1. *For a graph G,*

(1) *$M(G)$ = the set of all maximal cliques in G, and $m(G) = |M(G)|$ = the number of maximal cliques in G.*
(2) *A maximal clique $Q \in M(G)$ is called* **essential** *if there exists an edge e in Q that is not in any other maximal clique in G, and* **inessential** *otherwise.*
(3) *$EM(G)$ = the set of all essential maximal cliques in G, and $em(G) = |EM(G)|$ = the number of essential maximal cliques in G.*

In [2] we have the following observations for the relationships among parameters $cc(G)$, $m(G)$, and $em(G)$ of a graph G:

Propersition 1.1. *For any graph G,*

(1) *$EM(G) \subseteq q \subseteq M(G)$, where q is an EMCC of G;*
(2) *$em(G) \leq cc(G) \leq m(G)$;*
(3) *$cc(G) = m(G) \Rightarrow em(G) = cc(G)$.*

Definition 1.2. *For a graph G,*

(1) *G is said to be* **maximal clique irreducible** *if $cc(G) = m(G)$.*
(2) *G is said to be* **weakly maximal clique irreducible** *if $cc(G) = em(G)$.*

Please see Figure 1 for examples of weakly maximal clique irreducible and maximal clique irreducible graphs.

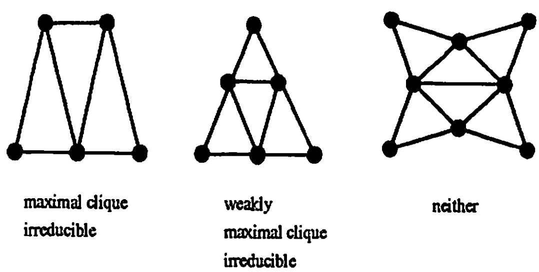

FIGURE 1. Examples

The notion of maximal clique irreducible graphs was introduced and studied by W.D. Wallis and G.-H. Zhang in 1990 [6]. It was noticed by Opsut and Roberts [3] in 1981 that any interval graph is maximal clique irreducible. In [4] we extended the concept to weakly maximal clique irreducible graphs and characterized the graphs for which each induced subgraph is weakly maximal clique irreducible. It is interesting to consider the problems of characterizing or giving efficient algorithms to recognize maximal clique irreducible graphs and weakly maximal clique irreducible graphs. In this paper, we characterize the line graphs which are maximal clique irreducible and the line graphs which are weakly maximal clique irreducible.

2. CHARACTERIZATIONS

Theorem 2.1. *(Wallis and Zhang [6]) For a graph G, for each $A \subseteq V(G)$ the induced subgraph G_A satisfies $cc(G_A) = m(G_A)$ if and only if G is F_1-free, F_2-free, F_3-free and F_4-free. (please see Figure 2)*

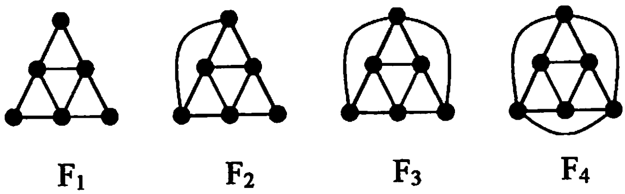

FIGURE 2. Four forbidden subgraphs for maximal clique irreducible graphs

More generally, we have the following characterization of hereditary weakly maximal clique irreducible graphs(in [4]):

Theorem 2.2. *For a graph G, for each $A \subseteq V(G)$, the induced subgraph G_A satisfies $em(G_A) = cc(G_A)$ if and only if G is G_1-free, G_2-free,......, and G_{19}-free. (please see Figure 3)*

In particular, forbidding these 4 subgraphs F_1, F_2, F_3, and F_4 gives a necessary and sufficient condition for recognizing hereditary maximal clique irreducible graphs, and hence a polynomial time algorithm for recognizing

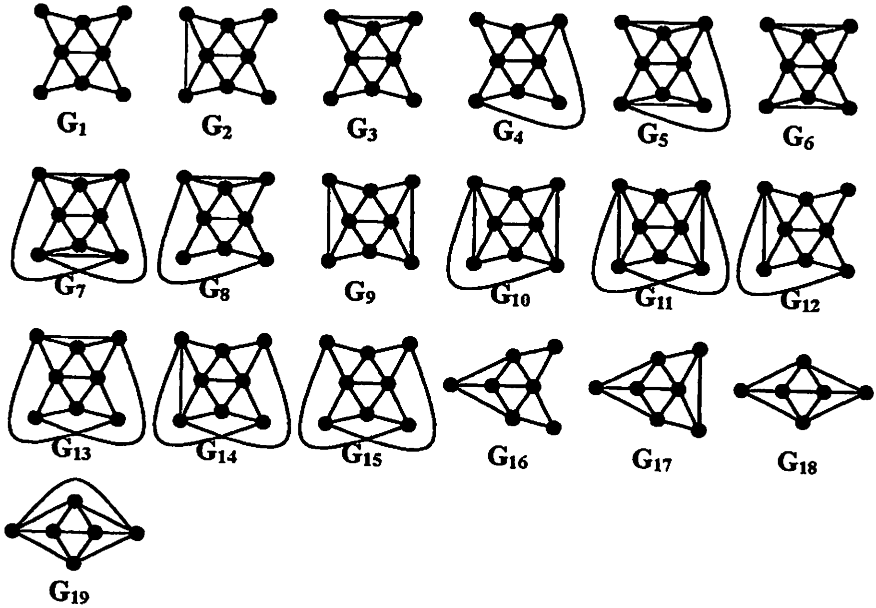

FIGURE 3. Nineteen forbidden subgraphs for weakly maximal clique irreducible graphs

such class of graphs as well. On the other hand, forbidding these 19 subgraphs $G_1, G_2, \ldots\ldots, G_{19}$ gives a necessary and sufficient condition for recognizing hereditary weakly maximal clique irreducible graphs, and hence a polynomial time algorithm for recognizing them. However, it may be interesting to find what these conditions imply when applied to special classes of graphs. Here we give the following characterizations for line graphs which are maximal clique irreducible and line graphs which are weakly maximal clique irreducible:

Theorem 2.3. *Let $L(G)$ be the line graph of a graph G, then $L(G)$ is maximal clique irreducible if and only if G is K_4-free, net-free and pendant-diamond-free.(please see Figure 4)*

Proof. If $cc(L(G)) \neq m(L(G))$, then $L(G)$ contains one of F_1, F_2, F_3 and F_4 as in Theorem 2.1. However by the well-known nine forbidden subgraph characterization for line graphs(see [1]), we see that F_3 is forbidden. Hence G contains a net, a pendant-diamond, or a K_4, since their corresponding line graphs are F_1, F_2, and F_4.

Conversely, if G contains a K_4, a net or a pendant-diamond, then in each case there is an in-essential maximal clique K_3 in $L(G)$ and hence $cc(L(G)) \neq m(L(G))$. Q.E.D.

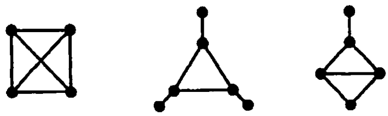

FIGURE 4

Theorem 2.4. *Let $L(G)$ be the line graph of a graph G, then $L(G)$ is weakly maximal clique irreducible if and only if G has no simplicial vertex of degree more than two, where a vertex v in G is simplicial if the open neighborhood $N(v)$ of v in G forms a clique.*

Proof.

We denote by Q_v the maximal clique in $L(G)$ induced by the maximal star centered at v, for a vertex v of degree at least three in G. Note that if v is simplicial and has degree at least three, then Q_v is inessential.

If $cc(L(G)) \neq em(L(G))$, then $L(G)$ has an edge which is in in-essential maximal cliques only and $L(G)$ contains one of $G_1,\ldots,G_{19}$ as in Theorem 2.2. However by the well-known nine forbidden subgraph characterization for line graphs, we have that $L(G)$ contains $G_{19} \cong F_4$ only (see Figure 2 and Figure 3). Hence G contains a K_4 since $L(K_4) \cong G_{19}$. Suppose this K_4 has vertices v_1, v_2, v_3 and v_4. Then we have the following situations:

CASE 1: If some v_i has degree three, then G has a simplicial vertex of degree three.

CASE 2: If every v_i has degree more than three, then we claim there must be some v_i which is simplicial. Since otherwise every corresponding maximal clique Q_{v_i} of v_i in $L(G)$ is essential, and G_{19} in $L(G)$ is edge-covered by these essential Q_{v_i}'s. This is a contradiction because G_{19} contains an edge which is in in-essential maximal cliques of $L(G)$ only.

Conversely, if G has some simplicial vertex v of degree at least three, then in $L(G)$ the maximal clique Q_v corresponding to v is in-essential. Note that an edge of Q_v belongs to either Q_v or another in-essential maximal clique, thus $cc(L(G)) > em(L(G))$. Q.E.D.

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