A Note on Neighborhood Unions and Independent Cycles

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Abstract

We prove that if G is a simple graph of order $n \geq 3k$ such that $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices x and y, then G contains k vertex independent cycles.

1 Introduction

In this paper, we only consider simple graphs. Other than the following instances, notation used is standard. We define $N(x_1, x_2, \cdots, x_r) = N(x_1) \cup N(x_2) \cup \cdots \cup N(x_r)$ and $N_H(x) = N(x) \cap V(H)$. Also, when we refer to k independent cycles we will always mean k vertex independent cycles.

In 1963 Corradi and Hajnal in [1] produced the following result which proved a conjecture of Erdos:

Theorem 1 *If G is a graph of order $n \geq 3k$, $k \geq 1$, with $\delta(G) \geq 2k$, then G contains k independent cycles.*

In 1989, Justesen in [2] generalized this result to degree sums of nonadjacent pairs and in 1999 Justesen's result was improved by Wang in [4] with the following sharp result:

Theorem 2 *If G is a graph of order $n \geq 3k$ such that $deg(u) + deg(v) \geq 4k - 1$ for all pairs u, v of nonadjacent vertices, then G contains k independent cycles.*

A summary of results on independent cycles in graphs can be found in [3].

In this paper, we look at neighborhood unions that imply the existence of k independent cycles. Specifically we prove the following result:

Theorem 3 *If G is a graph of order $n \geq 3k$ such that $|N(x,y)| \geq 3k$ for all nonadjacent pairs of vertices x and y, then G contains k vertex independent cycles.*

The result above is sharp in the sense that for any k the graph $G = K_{3k-1} \cup K_2$ has $|N(x,y)| = 3k-1$ for all nonadjacent vertices x and y and does not have k independent cycles. Also, note that $|N(x,y)| = 2$ for all nonadjacent vertices x and y does not guarantee the existence of single cycle. Thus, for $k = 1$ and for any n, we need $|N(x) \cup N(y)| \geq 3k$.

2 Proof of Theorem 3

The proof will proceed by double induction on n and k.

The theorem is clearly true for small values of n. That is, for a graph G of order $n = 3, 4$ or 5, we have $k = 1$ and so we assume $|N(x,y)| \geq 3$. Thus, G must contain a cycle. Thus, we assume the statement of the theorem is true for graphs of order less than n.

Let G be a graph of order n satisfying the hypothesis of the theorem. Let $k = 1$. Then $|N(x,y)| \geq 3$ for all nonadjacent pairs of vertices. Thus G must contain a cycle.

Assume G does not contain k independent cycles for $k \leq n/3$. If G contains a triangle, T, then $G - T$ clearly contains $k-1$ independent cycles by the inductive hypothesis. Thus, G contains k independent cycles. So we assume $g(G) \geq 4$.

Let $\mathcal{C} = \{C_1, C_2, C_3, ..., C_{k-1}\}$ be a collection of $k-1$ vertex disjoint cycles which exist by the inductive hypothesis. Choose $\mathcal{C}$ so that $|V(\mathcal{C})|$ is minimized. Let $L = G - V(\mathcal{C})$. Note that our choice of $\mathcal{C}$ implies that $|V(L)| \geq 3$ since $G - \{v_1, v_2, v_3\}$ contains $k-1$ independent cycles for any choice of v_1, v_2, v_3.

Of all collections $\mathcal{C}$ such that $|V(\mathcal{C})|$ is minimized, choose one such that L has a minimum number of connected components. Finally, of all collections $\mathcal{C}$ with a minimum number of connected components, pick one such that the order of a maximum component of L is maximized. Note that each component of L must be a tree or a kth cycle can be found. Also, all cycles of $\mathcal{C}$ are induced.

Claim 1: L has exactly one connected component.

Assume L has two or more components. Let v and w be end vertices of distinct trees in L such that w is in a component of maximum order. Then $|N_{\mathcal{C}}(v,w)| \geq 3k-2$. So there exists $C_i \in \mathcal{C}$ such that $|N_{C_i}(v,w)| \geq 4$. By the minimality of $|V(\mathcal{C})|$, we know that C_i must be a 4-cycle with vertices (in order), $u_1u_2u_3u_4$, such that without loss of generality $vu_1, vu_3, wu_2, wu_4 \in E(G)$.

Let C_i' be the cycle $u_1vu_3u_4u_1$. Let $C' = C - C_i \cup \{C_i'\}$. Now $L' = G - V(C')$ has a larger maximum connected component than L. This contradicts our choice of C. Thus, L has at most one component.

Claim 2: We can assume L is a path.

If L is not a path, pick a path P of maximum length in L. Let w be an end vertex of this path. Let v be an end vertex of L not on this path. As in the proof of claim 1, we can simultaneously insert v into C and append u_2 to P. Continue this process until L is a path.

Claim 3: We can assume that at least one penultimate vertex on the path L has degree in G at least $3k/2$. Note that if L has only three vertices there is only one penultimate vertex.

Pick v, w to be end vertices of L. Without loss of generality, we assume $d(w) \geq 3k/2$. If there does not exist a penultimate vertex with degree at least $3k/2$, then, as in the proof of Claim 1, we can simultaneously insert v into C and append u_2 to L. Now w is a penultimate vertex with degree at least $3k/2$.

Label the vertices of the path $L: x_1x_2...x_m$ and, without loss of generality, assume x_2 is the penultimate vertex of degree at least $3k/2$. Now, $|N_C(x_1, x_2, x_3)| = |N_C(x_1, x_3)| + |N_C(x_2)| \geq 3k-2+\frac{3k}{2}-2 = \frac{9k}{2}-4 > 4(k-1)$ for $k \geq 1$. But this means there exists $C_i \in C$ such that $|N_{C_i}(x_1, x_2, x_3)| \geq 5$ which contradicts the minimality of $|V(C)|$. Thus, G has k independent cycles completing the proof.

3 Conjecture

We conjecture that if the order of the graph n is larger relative to k (perhaps $n \geq 4k$) then the neighborhood condition can be lowered (perhaps to $2k$). This conjecture is motivated by the complete balanced bipartite graph of order $4k$. This conjecture, of course, requires that $k \geq 2$.

References

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