Comment on "Properly Coloured Hamiltonian Paths in Edge–coloured Complete Graphs without MonochromaticTriangles"

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Fix a positive integer c. Let K_n^ϕ denote the complete graph K_n of order n whose edges have been colored by ϕ: $E(K_n) \to C$ with # C = c. A subgraph H of K_n^ϕ is *properly colored* if any two adjacent edges in H have distinct colors; and, is *monochromatic* if all edges in H have the same color. Olof Barr [1] recently proved the following result (here $c \geq 3$).

Theorem [1; Theorem 2.1] If K_n^ϕ contains no monochromatic triangle, then K_n^ϕ contains a properly colored hamiltonian path.

While this is an aesthetically-pleasing result it, unfortunately, applies to at most $1 \leq n < (c+1)!$ for each $c \geq 3$.

For $c \geq 1$, let f(c) = min{n: each K_n^ϕ contains a monochromatic triangle}. Observe that if f(c) exists, then each K_n^ϕ contains a monochromatic triangle whenever $n \geq f(c)$.

Observation. For $c \geq 2$, f(c) exists and is at most (c+1)!.

Proof. (Induction on c) Clearly f(2) exists and equals 6. Assume f(c-1) exists and is at most c! where $c \geq 3$. Set n = (c+1)!. We assume ϕ:$E(K_n) \to [c]$ = {1,...,c} and fix a vertex $v \in V(K_n^\phi)$. Let $V_i = \{w \in V(K_n^\phi): \phi\{v,w\} = i\}$ for $i \in [c]$. Then some $m = \# V_i \geq \lceil ((c+1)!-1)/c \rceil \geq c!$. If $\phi\{w_j, w_k\} = i$ for some $w_j, w_k \in V_i$, K_n^ϕ contains a monochromatic triangle. Otherwise, the edges of $K_n^\phi[V_i] \cong K_m^\phi$ are colored using the c-1 colors $[c] - \{i\}$. By induction, $K_n^\phi[V_i]$ contains a monochromatic triangle using colors $[c] - \{i\}$. ∎

Our bound for f(c) is not best possible but suffices here. Consequently, Theorem 2.1 of [1] is of limited use, since it applies to at most $1 \leq n < (c+1)!$ for each $c \geq 3$.

Reference

1. Olof Barr, Properly Coloured Hamiltonian Paths in Edge-coloured Complete Graphs without Monochromatic Triangles, *Ars Combinatoria* **50** (1998), 316-318.