Pseudograceful Labelings of Cycles

MAGED Z. YOUSSEF [1]

Department of Mathematics, Faculty of Science,
Ain Shams University, Abbassia, Cairo, Egypt.

Abstract

In this paper, we give a complete characterization of the pseudogracefulness of cycles.

Keywords : Graceful labeling; Pseudograceful labeling.

1. Introduction

We follow the basic notations and terminology of graph theory as in [1].

A vertex labeling of a graph G is a function f of labels to the vertices of G that induces for each edge xy a label depending on the vertex labels $f(x)$ and $f(y)$. The most known kinds of vertex labelings are the graceful labelings [6] and the harmonious labelings [5]. Several authors introduced variations on these labelings. For a survey on the vertex labelings, readers are referred to Gallian [4].

Recall that a (p,q) graph G is called graceful if there exists an injective function f, called a graceful labeling of G, $f: V(G) \to \{0,1,\cdots,q\}$ such that the induced function $f^*: E(G) \to \{1,2,\cdots,q\}$ defined by $f^*(xy) = |f(x) - f(y)|$, for all edge $xy \in E(G)$ is an injection. The concept of graceful labeling was introduced by Rosa [6] in 1967, who has shown that if G is a graceful Eulerian graph with q edges, then $q \equiv 0$ or $3(mod\, 4)$. We call this condition the graceful parity condition. Rosa [6] also proved that this necessary condition is sufficient for cycles by showing that the cycle C_n is graceful if and only if $n \equiv 0$ or $3(mod\, 4)$.

Frucht [3] give a slight variation to the definition of graceful graph by calling a (p,q) graph with $p = q+1$ (i.e., graphs that are trees or the disjoint union of a tree and uncyclic graphs) is pseudograceful if there exists an injective function f, called a pseudograceful labeling of G, $f: V(G) \to \{0,1,\cdots,q-1,q+1\}$ such that the induced function $f^*: E(G) \to \{1,2,\cdots,q\}$ defined by $f^*(xy) = |f(x) - f(y)|$ for all edge $xy \in E(G)$ is an

[1] E-mail address : myoussef11566@yahoo.com

injection. Frucht [3] showed that some families of graphs are pseudograceful.

Seoud and Youssef [7] extended the definition of pseudograceful to all graphs with $p \leq q+1$. Extending the definition enable them to obtain a large families of graceful disconnected graphs as well as of pseudograceful disconnected graphs. They proved that if G is a pseudograceful graphs, then $G \cup K_{m,n}$ is graceful for $m, n \geq 2$ and $G \cup K_{m,n}$ is pseudograceful for $m, n \geq 2$ and $(m, n) \neq (2, 2)$. Youssef [8] proved that if G is a pseudograceful graph and if H is an α-labeled graph, then $G \cup H$ can be graceful or pseudograceful under some conditions on the α-labeling function of H. See Gallian [4] for the definition of α-lebeled graphs.

In this paper, we complete the characterization of the pseudogracefulness of cycles by showing that the cycle C_n is pseudograceful if and only if C_n is graceful.

2. Pseudogracefulness of Cycles

Seoud and Youssef [7] observed that if G is a pseudograceful Eulerian graph of q edges, then $q \equiv 0$ or $3(mod\ 4)$. We call this condition the pseudograceful parity condition. They also have completely setteled the pseudogracefulness of the graphs K_n, $K_{m,n}$ and $P_m + \overline{K}_n$ while for the cycle C_n, they gave pseudograceful labelings for $n = 3, 4, 7$ and 8. In the following theorem, we give all values of n for which C_n is pseudograceful.

Theorem 1.

C_n is pseudograceful if and only if $n \equiv 0$ or $3(mod\ 4)$.

Proof.

Necessity follows from the pseudograceful parity condition [7]. For sufficiency, let $V(C_n) = \{u_1, u_2, \cdots, u_n\}$, $n \geq 3$ whewre $u_i u_j \in E(C_n)$ if and only if $i - j \equiv \pm 1(mod\ n)$. The pseudograceful labelings of C_n, $n = 3, 4, 7$ and 8 are in [7]. For $n > 8$, we have the following four cases :
Case 1 : $n = 8k$, $k \geq 2$. We define the labeling function

$$f : V(C_n) \to \{0, 1, \cdots, n-1, n+1\}$$

as follows

$$f(u_1) = n+1, \qquad f(u_3) = n-1,$$

$$f(u_{4i}) = \begin{cases} 2(i-1), & 1 \leq i \leq k-1 \\ 2i, & k \leq i \leq 2k \end{cases}$$

$$f(u_{4i+1}) = \begin{cases} n-1-2i, & 1 \le i \le k-1 \\ n-2i, & k \le i \le 2k-1 \end{cases}$$

$$f(u_{4i-2}) = \begin{cases} 2i-1, & 1 \le i \le k-1 \\ 2(i-1), & i = k \\ 2i-1, & k+1 \le i \le 2k \end{cases}$$

$$f(u_{4i+3}) = \begin{cases} n-2i, & 1 \le i \le k-1 \\ n-1-2i, & k \le i \le 2k-1 \ . \end{cases}$$

Observe that f is injective with $2k-1 \notin f(V(C_n))$. We have to show that f^* is injective as well. We abbreviate $f^*(u_iu_j)$ to $f^*(i,j)$.

$$\begin{aligned}
f^*(E(C_n)) &= \{f^*(1,2), f^*(2,3), f^*(3,4), f^*(n,1)\} \cup \\
&\quad \{f^*(4i,4i+1) : 1 \le i \le 2k-1\} \cup \\
&\quad \{f^*(4i+1,4i+2) : 1 \le i \le 2k-1\} \cup \\
&\quad \{f^*(4i+2,4i+3) : 1 \le i \le 2k-1\} \cup \\
&\quad \{f^*(4i+3,4i+4) : 1 \le i \le 2k-1\} \\
&= \{n,\ n-2,\ n-1,\ n-4k+1\} \cup \\
&\quad (\{f^*(4i,4i+1) : 1 \le i \le k-1\} \cup \\
&\quad \{f^*(4i,4i+1) : k \le i \le 2k-1\}) \cup \\
&\quad (\{f^*(4i+1,4i+2) : 1 \le i \le k-2\} \cup \\
&\quad \{f^*(4i+1,4i+2) : i = k-1\} \cup \\
&\quad \{f^*(4i+1,4i+2) : k \le i \le 2k-1\}) \cup \\
&\quad (\{f^*(4i+2,4i+3) : 1 \le i \le k-2\} \cup \\
&\quad \{f^*(4i+2,4i+3) : i = k-1\} \cup \\
&\quad \{f^*(4i+2,4i+3) : k \le i \le 2k-1\}) \cup \\
&\quad (\{f^*(4i+3,4i+4) : 1 \le i \le k-2\} \cup \\
&\quad \{f^*(4i+3,4i+4) : i = k-1\} \cup \\
&\quad \{f^*(4i+3,4i+4) : k \le i \le 2k-1\}) \\
&= \{n,\ n-1,\ n-2,\ n-4k+1\} \cup (\{n-4i+1 : 1 \le i \le k-1\} \\
&\quad \{n-4i : k \le i \le 2k-1\}) \cup (\{n-4i-2 : 1 \le i \le k-2\} \cup \\
&\quad \{n-4k+3\} \cup \{n-4i-1 : k \le i \le 2k-1\}) \cup \\
&\quad (\{n-4i-1 : 1 \le i \le k-2\} \cup \{n-4k+4\} \cup \\
&\quad \{n-4i-2 : k \le i \le 2k-1\}) \cup (\{n-4i : 1 \le i \le k-2\} \cup \\
&\quad \{n-4k+2\} \cup \{n-4i-3 : k \le i \le 2k-1\}) \ .
\end{aligned}$$

If we group the edge labels according to the congruence class modulo 4, then

$$
\begin{aligned}
f^*(E(C_n)) &= (\{n\} \cup \{n-4i : 1 \le i \le k-2\} \cup \\
&\quad \{n-4k+4\} \cup \{n-4i : k \le i \le 2k-1\}) \cup \\
&\quad (\{n-4i+1 : 1 \le i \le k-1\} \cup \{n-4k+1\} \cup \\
&\quad \{n-4i-3 : k \le i \le 2k-1\}) \cup (\{n-2\} \cup \\
&\quad \{n-4i-2 : 1 \le i \le k-2\} \cup \{n-4k+2\} \cup \\
&\quad \{n-4i-2 : k \le i \le 2k-1\}) \cup (\{n-1\} \cup \\
&\quad \{n-4i-1 : 1 \le i \le k-2\} \cup \{n-4k+3\} \cup \\
&\quad \{n-4i-1 : k \le i \le 2k-1\}) \\
&= \{1,\, 2,\, \cdots,\, n\}\,.
\end{aligned}
$$

Hence f is a pseudograceful labeling of C_n .
Case 2 : $n = 8k+3$, $k \ge 1$. As in Case 1, we define a labeling function f as follows

$$f(u_1) = n+1, \qquad f(u_3) = n-1,$$

$$f(u_{4i}) = \begin{cases} 2(i-1), & 1 \le i \le k \\ 2i-1, & k+1 \le i \le 2k \end{cases}$$

$$f(u_{4i+1}) = n-1-2i, \qquad 1 \le i \le 2k$$

$$f(u_{4i-2}) = \begin{cases} 2i-1, & 1 \le i \le k \\ 2(i-1), & k+1 \le i \le 2k+1 \end{cases}$$

$$f(u_{4i+3}) = \begin{cases} n-2i, & 1 \le i \le k-1 \\ n-2-2i, & k \le i \le 2k\,. \end{cases}$$

Also, observe that f is injective with $6k+3 \notin f(V(C_n))$.

$$\begin{aligned} f^*(E(C_n)) &= \{f^*(1,2), f^*(2,3), f^*(3,4), f^*(n,1)\}\cup \\ &\quad (\{f^*(4i,4i+1) : 1 \le i \le k\}\cup \\ &\quad \{f^*(4i,4i+1) : k+1 \le i \le 2k\})\cup \\ &\quad (\{f^*(4i+1,4i+2) : 1 \le i \le k-1\}\cup \\ &\quad \{f^*(4i+1,4i+2) : k \le i \le 2k\})\cup \\ &\quad (\{f^*(4i+2,4i+3) : 1 \le i \le k-1\}\cup \\ &\quad \{f^*(4i+2,4i+3) : k \le i \le 2k\})\cup \\ &\quad (\{f^*(4i+3,4i+4) : 1 \le i \le k-1\}\cup \\ &\quad \{f^*(4i+3,4i+4) : k \le i \le 2k-1\}) \\ &= \{n,\ n-2,\ n-1,\ n-4k\} \cup (\{n-4i+1 : 1 \le i \le k\}\cup \\ &\quad \{n-4i : k+1 \le i \le 2k\}) \cup (\{n-4i-2 : 1 \le i \le k-1\}\cup \\ &\quad \{n-4i-1 : k \le i \le 2k\}) \cup (\{n-4i-1 : 1 \le i \le k-1\}\cup \\ &\quad \{n-4i-2 : k \le i \le 2k\}) \cup (\{n-4i : 1 \le i \le k-1\}\cup \\ &\quad \{n-4i-3 : k \le i \le 2k-1\})\,. \end{aligned}$$

If we group the edge labels according to the congruence class modulo 4, then

$$\begin{aligned} f^*(E(C_n)) &= (\{n-4i+1 : 1 \le i \le k\} \cup \{n-4i-3 : k \le i \le 2k-1\})\cup \\ &\quad (\{n-2\} \cup \{n-4i-2 : 1 \le i \le k-1\}\cup \\ &\quad \{n-4i-2 : k \le i \le 2k\})\cup \\ &\quad (\{n-1\} \cup \{n-4i-1 : 1 \le i \le k-1\}\cup \\ &\quad \{n-4i-1 : k \le i \le 2k\})\cup \\ &\quad (\{n\} \cup \{n-4i : 1 \le i \le k-1\} \cup \{n-4k\})\cup \\ &\quad \{n-4i : k+1 \le i \le 2k\}) \\ &= \{1,\ 2,\ \cdots,\ n\}\,. \end{aligned}$$

Hence f is a pseudegraceful labeling of C_n .

In the next two cases, we give the labeling functions and with an argument similar to that in Cases 1 and 2, the reader can show that the labeling is pseudograceful.

Case 3 : $n = 8k+4$, $k \ge 1$. We define a labeling function as follows

$$f(u_1) = n+1, \qquad f(u_3) = n-1,$$

$$f(u_{4i}) = \begin{cases} 2(i-1), & 1 \le i \le k \\ 2i-1, & k+1 \le i \le 2k+1 \end{cases}$$

$$f(u_{4i+1}) = n-1-2i, \qquad 1 \le i \le 2k$$

$$f(u_{4i-2}) = \begin{cases} 2i-1, & 1 \le i \le k \\ 2(i-1), & k+1 \le i \le 2k+1 \end{cases}$$

$$f(u_{4i+3}) = \begin{cases} n-2i, & 1 \le i \le k-1 \\ n-2-2i, & k \le i \le 2k\ . \end{cases}$$

Observe that f is injective with $6k+4 \notin f(V(C_n))$.

Case 4 : $n = 8k+7$, $k \ge 1$. We define a labeling function as follows

$$f(u_1) = n+1, \qquad f(u_3) = n-1,$$

$$f(u_{4i}) = \begin{cases} 2(i-1), & 1 \le i \le k \\ 2i, & k+1 \le i \le 2k+1 \end{cases}$$

$$f(u_{4i+1}) = \begin{cases} n-1-2i, & 1 \le i \le k \\ n-2i, & k+1 \le i \le 2k+1 \end{cases}$$

$$f(u_{4i-2}) = \begin{cases} 2i-1, & 1 \le i \le k \\ 2(i-1), & i = k+1 \\ 2i-1, & k+2 \le i \le 2k+2 \end{cases}$$

$$f(u_{4i+3}) = \begin{cases} n-2i, & 1 \le i \le k \\ n-1-2i, & k+1 \le i \le 2k+1\ . \end{cases}$$

Observe that f is injective with $2k+1 \notin f(V(C_n))$. $\diamond$

Finally, we complete the characterization of the graceful graphs in the family $C_n \cup K_{p,q}$, $n \equiv 0$ or $3\ (mod\ 4)$. Seoud and Youssef [7] gave the complete characterization of the gracefulness of $C_n \cup K_{p,q}$ when $n = 3, 4, 7$ and 8. We have $C_n \cup S_m$ is graceful for all $n \ge 7$ and $m \ge 1$ by [2]. Combining the result of this paper with the result of [7, Theorem 2.1] and the result of [2] mentioned above, we have the following corollary.

Corollary 2.

If $n > 8$ and $n \equiv 0$ or $3\ (mod\ 4)$, then $C_n \cup K_{p,q}$ is graceful for all p and q and is pseudograceful for all $p, q \ge 2$ and $(p,q) \ne (2,2)$.

Remark.

Note that, for $n \equiv 1$ or $2\ (mod\ 4)$, the graph $C_n \cup K_{p,q}$ may not be graceful by using the graceful parity condition [6].

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