On constrained 2-Partitions of monochromatic sets and generalizations in the sense of Erdős-Ginzburg-Ziv

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Abstract

Let $m \geq 4$ be a positive integer and let Z_m denote the cyclic group of residues modulo m. For a system L of inequalities in m variables, let $R(L;2)$ ($R(L;Z_m)$) denote the minimum integer N such that every function $\Delta : \{1,2,\dots,N\} \to \{0,1\}$ ($\Delta : \{1,2,\dots,N\} \to Z_m$) admits a solution of L, say $(x_1,\dots,x_m)$, such that $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m)$ (such that $\sum_{i=1}^{m} \Delta(x_i) = 0$). Define the system $L_1(m)$ to consist of the inequality $x_2 - x_1 \leq x_m - x_3$, and the system $L_2(m)$ to consist of the inequality $x_{m-2} - x_1 \leq x_m - x_{m-1}$, where $x_1 < x_2 < \cdots < x_m$ in both $L_1(m)$ and $L_2(m)$. The main result of this paper is that $R(L_1(m);2) = R(L_1(m);Z_m) = 2m$, and $R(L_2(m);2) = 6m - 15$. Furthermore, we support the conjecture that $R(L_2(m);2) = R(L_2(m);Z_m)$ by proving it for $m = 5$.

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1. Introduction

Denote by $[a, b]$ the set of integers x such that $a \leq x \leq b$. For a set S, an S-coloring of $[1, N]$ is a function $\Delta : [1, N] \to S$. If $S = \{0, 1\}$ then we call Δ a 2-coloring, and if S is the set of elements of the cyclic group Z_m, then we call Δ a Z_m-coloring. The following is a rephrasing of the Erdős-Ginzburg-Ziv (EGZ) theorem, proved in [11].

Theorem 0: *Let $m \geq 2$ be an integer. If $\Delta : [1, 2m-1] \to Z_m$, then there exist integers $x_1 < x_2 < \cdots < x_m \in [1, 2m-1]$ such that $\sum_{i=1}^{m} \Delta(x_i) = 0$. Moreover, $2m - 1$ is the smallest number for which the above assertion holds.*

If the Z_m-coloring in Theorem 0 is replaced by a 2-coloring, then its conclusion follows from the pigeonhole (PH) principle. Thus, the EGZ theorem can be viewed as a PH generalization for 2 boxes and m pigeons. Following the Erdős-Ginzburg-Ziv theorem, several theorems of Ramsey-type have been generalized by considering Z_m-colorings and zero-sum configurations rather than 2-colorings and monochromatic configurations. We call such theorems generalizations in the sense of Erdős-Ginzburg-Ziv (EGZ). Generalizations dealing with colorings of graphs or the positive integers appear in [5],[12], and [15], or [4], [7], [8], and [14], respectively.

For a system L of inequalities and equations in m variables, let $R(L; 2)$ ($R(L; Z_m)$) denote the minimum integer N such that every 2-coloring (Z_m-coloring) of $[1, N]$ implies a solution of L which is monochromatic (zero-sum); if such an integer does not exist, then set $R(L; 2) = \infty$ ($R(L; Z_m) = \infty$). It is clear that $R(L; 2) \leq R(L; Z_m)$ for every system L of inequalities and equations in m variables. When the reverse inequality holds, we say that L admits a generalization in the sense of EGZ.

At present there is no general theory which addresses generalizations of systems of inequalities and equations in the sense of EGZ. By computer one can check that several classical systems of equations such as the van der Waerden system for $m = 3, 4$ (i.e. $x_2 - x_1 = x_3 - x_2$ and $x_2 - x_1 = x_3 - x_2 = x_4 - x_3$) do generalize in the sense of EGZ. On the other hand, the generalized Schur equation $\sum_{i=1}^{m-1} x_i = x_m$ [2] does not generalize for $m = 3, 4$. Indeed, when $m = 3$ and L is the Schur equation $x_1 + x_2 = x_3$, then $R(L; 2) = 5$ while $R(L; Z_3) = 10$. Investigating EGZ generalizations for systems of equations seems very difficult. However, the study of systems of inequalities is more manageable with the present tools.

This paper was motivated by the idea of ascending waves [1], [3], [10] and a problem regarding nondecreasing diameter dealt with in [7] and [13]. For positive integers m and i satisfying $2 \leq i \leq m - 2$, let $L(m, i)$ denote the following system of inequalities:

$$\begin{aligned} x_i - x_1 &\leq x_m - x_{i+1} \\ x_1 < x_2 &< \cdots < x_m. \end{aligned}$$

The system $L(m, i)$ is a partition of an m-subset at position i into two disjoint sets with nondecreasing diameter. We conjecture the following with regard to the systems $L(m, i)$.

Conjecture: For a given $m, i \in \mathbb{N}$, where $2 \leq i \leq m-2$,

$$R(L(m,i);2) = R(L(m,i);Z_m).$$

In this paper we prove the conjecture for $i = 2$ and support the conjecture for $i = m-2$. In particular, the paper is organized as follows. In Section 2 we show that $R(L(m,2);2) = R(L(m,2);Z_m) = 2m$, thus confirming the conjecture for $i = 2$. In Section 3 we show that $R(L(m,m-2);2) = 6m-15$ and give support to the claim that $R(L(m,m-2);2) = R(L(m,m-2);Z_m)$.

2. The System $L(m, 2)$

For notational convenience, let $L_1(m)$ denote the system $L(m, 2)$. Further, for the sake of expediency, define the distance function $d_1 : \mathbb{N}^2 \to \mathbb{Z}$ by

$$d_1(x_1, x_2) = x_2 - x_1 - 1.$$

If $x_1 \leq x_2$, then $d_1(x_1, x_2)$ is the number of integers between x_1 and x_2. Notice that an m-tuple $(x_1, x_2, \ldots, x_m)$ is a solution of $L_1(m)$ if and only if $d_1(x_1, x_2) \leq d_1(x_3, x_m)$.

Theorem 2.1: *For $m \geq 4$, $R(L_1(m); 2) = 2m$.*

Proof. Let an integer $m \geq 4$ be given. The coloring given by

$$010^{m-1}1^{m-2}$$

establishes the lower bound $R(L_1(m); 2) \geq 2m$. To prove the reverse inequality, let $\Delta : [1, 2m] \to \{0, 1\}$ be an arbitrary coloring. We will assume that $m \geq 6$, as the cases $m = 4, 5$ can be verified separately. Define the set $S = [1, 2m] \setminus \{m+3\}$. Then without loss of generality there exist $x_1, \ldots, x_m \in S$ such that $x_1 < x_2 < \cdots < x_m$ and $\Delta(x_i) = 1$ for $i = 1, 2, \ldots, m$. If $x_2 \leq 4$, then $d_1(x_1, x_2) \leq 4 - 1 - 1 = 2$, whence $m \geq 6$ and $d_1(x_3, x_m) \geq m - 4$ imply that $(x_1, \ldots, x_m)$ is a monochromatic solution of $L_1(m)$. Therefore, we may assume that $x_2 \geq 5$. It follows that $x_m \geq m+3$, and hence the definition of S implies that $d_1(x_3, x_m) \geq m-3$. Therefore, we may assume that $d_1(x_1, x_2) \geq m - 2$. But then it follows that $x_1 \in \{1, 2\}$, $x_2 \in \{m, m+1\}$, and

$$x_3 = m+2,\ x_4 = m+4,\ x_5 = m+5, \ldots,\ x_m = 2m.$$

Further, we deduce that $\Delta(m+3) = 0$, as otherwise $(x_2, m+3, x_3, \ldots, x_m)$ is a monochromatic solution of $L_1(m)$. Analogously, we may assume that every $j \in [3, m-1]$ is such that $\Delta(j) = 0$, as otherwise $(j, x_2, x_3, \ldots, x_m)$ is a monochromatic solution of $L_1(m)$. Now, if there exists $j \in \{1, 2\}$ such that $\Delta(j) = 0$, then $(j, 2, 3, \ldots, m-1, m+3)$ is a monochromatic solution of $L_1(m)$. Therefore, we may assume that $\Delta(1) = \Delta(2) = 1$. But then $(1, 2, x_2, x_4, x_5, \ldots, x_m)$ is a monochromatic solution of $L_1(m)$.

Thus, the proof of the theorem is complete. □

In order to prove the next theorem, we need the following result.

Theorem 2.2: [6] *Let $m \geq 3$ be an integer, let S be a set such that $|S| = 2m-2$, and let $\Delta : S \to Z_m$ be a coloring. If $|\Delta(S)| \geq 3$, then there exist integers $x_1, \ldots, x_m \in S$ such that $\sum_{i=1}^{m} \Delta(x_i) = 0$.*

Theorem 2.3: *For $m \geq 4$, $R(L_1(m); Z_m) = R(L_1(m); 2) = 2m$.*

Proof. In view of Theorem 2.1, in order to prove the theorem it suffices to prove the upper bound $R(L_1(m); Z_m) \leq 2m$. Thus, let $\Delta : [1, 2m] \to Z_m$ be an arbitrary coloring. We will assume that $m \geq 6$, as the cases $m = 4, 5$ can be verified separately. Define $S_1 = [1, 2m] \setminus \{m+3, m+4\}$ and $S_2 = [1, 2m] \setminus \{m+2, 2m-1\}$. We claim the following.

Claim 1: Any m distinct elements of S_1 form a solution of $L_1(m)$.

To prove the claim, let $x_1 < x_2 < \cdots < x_m \in S_1$ be arbitrary. As in the proof of Theorem 2.1, we may assume that $x_2 \geq 5$. But then $x_m \geq m+3$, whence the definition of S_1 implies that $d_1(x_3, x_m) \geq m-2$. Therefore, we may assume that $d_1(x_1, x_2) \geq m-1$. However, this implies that

$$d_1(x_1, x_m) \geq d_1(x_1, x_2) + d_1(x_3, x_m) + 2 \geq 2m-1,$$

whence $x_m \geq 2m+1$, a contradiction. Thus, the proof of the claim is complete.

Claim 2: Any m distinct elements of S_2 form a solution of $L_1(m)$.

The proof of Claim 2 is analogous to that of Claim 1, and so we omit it.

To prove the theorem, suppose first that $|\Delta(S_1)| \geq 3$. Then since $|S_1| = 2m-2$, it follows from Theorem 2.2 that there exist $x_1 < x_2 < \cdots < x_m \in S_1$ such that $\sum_{i=1}^{m} \Delta(x_i) = 0$. But then Claim 1 implies that $(x_1, x_2, \ldots, x_m)$ is a zero-sum solution of $L_1(m)$. Therefore, we may suppose that $|\Delta(S_1)| \leq 2$. Analogously, applying Claim 2 yields that $|\Delta(S_2)| \leq 2$. Thus, since $[1, 2m] = S_1 \cup S_2$ and since $S_1 \cap S_2 \neq \emptyset$, it follows that $|\Delta([1, 2m])| \leq 2$. Thus, by Theorem 2.1, Δ admits a solution of $L_1(m)$ which is monochromatic. Since a monochromatic m-tuple is zero-sum in Z_m, the proof of the theorem is complete. □

3. The System $L(m, m-2)$

For notational convenience, let $L_2(m)$ denote the system $L(m, m-2)$. Further, for the sake of expediency, define the distance function $d_2 : \mathbb{N}^2 \to \mathbb{Z}$ by

$$d_2(x_1, x_2) = x_2 - x_1.$$

Notice that an m-tuple $(x_1, x_2, \ldots, x_m)$ is a solution of $L_2(m)$ if and only if $d_2(x_1, x_{m-2}) \le d_2(x_{m-1}, x_m)$.

Theorem 3.1: *For $m \ge 4$, $R(L_2(m); 2) = 6m - 15$.*

Proof. It is easy to check that the coloring given by

$$0^{m-3}1^{m-3}0^{2m-5}1^{2m-5}$$

admits no monochromatic solution of $L_2(m)$, and thus we conclude $R(L_2(m); 2) \ge 6m - 15$. To prove the reverse inequality, let $\Delta : [1, 6m-15] \to \{0, 1\}$ be an arbitrary coloring. We begin with the following easy observation.

$$\text{In a monochromatic interval of } 2m-4 \text{ elements there is a monochromatic solution of } L_2(m). \tag{1}$$

Now, without loss of generality we may assume that

$$\left|\Delta^{-1}(1) \cap [1, 2m-3]\right| \ge m - 1.$$

Consider the set S of all $(m-1)$-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_{m-1})$ such that $x_i \in [1, 2m-3]$, $x_i < x_{i+1}$ and $\Delta(x_i) = 1$ for $1 \le i \le m-1$. Define the function $f : S \to \mathbb{N}$ by $f(\mathbf{x}) = x_{m-1} + x_{m-2} - x_1$, and let

$$\mathbf{z} = \min_{\mathbf{x} \in S} f(\mathbf{x}).$$

It is clear that $\mathbf{z} \le (2m-3) + (2m-4) - 1 = 4m - 8$. We consider three cases.

Case 1: $\mathbf{z} \le 4m - 10$.

It follows that $\Delta(i) = 0$ for $i \ge 4m - 10$, whence $[4m-10, 6m-15]$ is a monochromatic interval of $2m-4$ elements. The case is complete by (1).

Case 2: $\mathbf{z} = 4m - 9$.

There are two possibilities.

Case 2a: $x_{m-1} = 2m-3$, $x_{m-2} = 2m-4$, and $x_1 = 2$.

If $\Delta(1) = \Delta(2)$, then define $y_1 = 1$, and $y_{i+1} = x_i$ for $1 \le i \le m-2$. Set $\mathbf{y} = (y_1, y_2, \ldots, y_{m-1})$. Then $\mathbf{y} \in S$ and $f(\mathbf{y}) < \mathbf{z}$, a contradiction by the definition of $\mathbf{z}$. Therefore, we may assume that $\Delta(1) = 0$. Furthermore, if $\left|\Delta^{-1}(1) \cap [2, 2m-3]\right| \ge m$, then it is easy to see that there exists $\mathbf{y} \in S$

such that $f(\mathbf{y}) < \mathbf{z}$, a contradiction. Therefore, we may assume

$$\left|\Delta^{-1}(1) \cap [2, 2m-3]\right| \le m-1,$$

whence there exist $(m-3)$ integers $y_2, y_3, \ldots, y_{m-2} \in [2, 2m-3]$ such that $y_2 < y_3 < \cdots < y_{m-2}$ and $\Delta(y_i) = 0$ for $2 \le i \le m-2$. Next, set $y_1 = 1$. Finally, since $\mathbf{z} = 4m-9$, it follows that $\Delta(i) = 0$ for $i \ge 4m-9$, and so we define $y_{m-1} = 4m-9$ and $y_m = 6m+15$. Then $d_2(y_m, y_{m-1}) = (6m+15)-(4m-9) = 2m-6$, while $d_2(y_1, y_{m-2}) \le 2m-5-1 = 2m-6$, whence $(y_1, \ldots, y_m)$ is a monochromatic solution of $L_2(m)$.

Case 2b: $x_{m-1} = 2m-3$, $x_{m-2} = 2m-5$, $x_1 = 1$.

If $\left|\Delta^{-1}(1) \cap [1, 2m-6]\right| \ge m-2$, then it is easy to see that there exists $\mathbf{y} \in S$ such that $f(\mathbf{y}) < \mathbf{z}$. Thus, we may assume

$$\left|\Delta^{-1}(1) \cap [1, 2m-6]\right| \le m-3,$$

whence there exist $(m-3)$ integers $y_1, y_2, \ldots, y_{m-3} \in [1, 2m-6]$ such that $y_1 < y_2 < \cdots < y_{m-3}$ and $\Delta(y_i) = 0$ for $1 \le i \le m-3$. Furthermore, if $\Delta(2m-4) = 1$, then setting $r_i = x_i$ for $1 \le i \le m-2$, and $r_{m-1} = 2m-4$, we find that $\mathbf{r} = (r_1, \ldots, r_{m-1}) \in S$ and $f(\mathbf{r}) < \mathbf{z}$, a contradiction. Thus, we may assume $\Delta(2m-4) = 0$. Set $y_{m-2} = 2m-4$. As in Case 2a, we may assume $\Delta(i) = 0$ for $i \ge 4m-9$. Thus, setting $y_{m-1} = 4m-9$ and $y_m = 6m-15$, we obtain a monochromatic solution $(y_1, \ldots, y_m)$ of $L_2(m)$. This completes Case 2b, and thus Case 2.

Case 3: $\mathbf{z} = 4m-8$.

It follows that $x_1 = 1$, $x_{m-2} = 2m-4$, and $x_{m-1} = 2m-3$. It is easy to see that if $\left|\Delta^{-1}(1) \cap [1, 2m-3]\right| \ge m-1$, then there exists $\mathbf{y} \in S$ such that $f(\mathbf{y}) < \mathbf{z}$, a contradiction. Thus, we may assume that there exist $(m-2)$ integers $y_1, \ldots, y_{m-2} \in [1, 2m-3]$ such that $y_1 < y_2 < \cdots < y_m$ and $\Delta(y_i) = 0$ for $1 \le i \le m-2$. Furthermore, we may assume that $\Delta(i) = 0$ for $i \ge 4m-8$. Thus, define $y_{m-1} = 4m-8$, and $y_m = 6m-15$. Then $d_2(y_m, y_{m-1}) = (6m-15)-(4m-8) = 2m-7$, while $d_2(y_1, y_2) \le (2m-5)-2 = 2m-7$. Therefore, $(y_1, \ldots, y_m)$ is a monochromatic solution of $L_2(m)$. Hence, Case 3 is complete.

Thus, the proof of the theorem is complete.

□

Finally, we will support the conjecture for $i = m-2$ by proving that $R(L_2(5); 2) = R(L_2(5); Z_5)$. In fact, it was confirmed that $R(L_2(7); 2) = R(L_2(7); Z_7)$. However, we do not include the proof since the number of branches in the case analysis for $m = 7$ is significantly larger than that for $m = 5$. In order to confirm the conjecture for $i = m-2$, where m is a larger prime, one should find a more sophisticated method of case analysis.

We first introduce some terminology. Let S_1 be a set, and let $\Delta : S_1 \to Z_5$ be a coloring. If $S \subset S_1$, then an *n-transversal* of S is a collection of $n \geq 2$ sets $T_1, \ldots, T_n \subset S$ such that $|T_i| = 2$ for $i = 1, 2 \ldots, n$, $T_i \cap T_j = \emptyset$ for $1 \leq i < j \leq n$, and if $\alpha_i, \beta_i \in T_i$ are distinct, then $\Delta(\alpha_i) \neq \Delta(\beta_i)$ for $i = 1, 2, \ldots, n$.

The basic method behind the proof of Theorem 3.2 is the construction of a 4-transversal followed by application of the Cauchy-Davenport theorem. For the sake of expediency, let $h : \mathbb{N}^5 \to \mathbb{N}^5$ denote the function that reorders the components of its input so that the components of the resulting 5-tuple are in increasing order.

Theorem 3.2: $R(L_2(5); 2) = R(L_2(5); Z_5) = 15$.

Proof. The lower bound $R(L_2(5); 2) > 14$ is established by Theorem 3.1. To prove the upper bound $R(L_2(5); Z_5) \leq 15$, let $\Delta : [1, 15] \to Z_5$ be an arbitrary coloring. Consider the set $S = [1, 8]$. If $|\Delta(S)| = 1$, then $(1, 2, 3, 4, 6)$ is a zero-sum solution of $L_2(5)$. Therefore, we may suppose that $|\Delta(S)| \geq 1$. However, we claim that $|\Delta(S)| \leq 4$. Indeed, suppose $|\Delta(S)| = 5$. Take $\beta_{i+1} \in \{\Delta^{-1}(i) \cap S\}$ for $i = 0, \ldots, 5$, and let $T_i = [1, 5] \setminus \{i\}$ for $i = 1, 2, \ldots, 5$. Then

$$\sum_{r \in T_i} \Delta(\beta_r)$$

for $i = 1, 2, \ldots, 5$, form five distinct residues of Z_5. It follows that there exists $i \in \{1, 2, \ldots, 5\}$ such that

$$\sum_{r \in T_i} \Delta(\beta_r) = -\Delta(15).$$

Letting $x_1 < x_2 < x_3 < x_4$ represent the elements of T_i, and defining $x_5 = 15$, it follows that $(x_1, x_2, \ldots, x_5)$ is a zero-sum solution of $L_2(5)$ because

$$-x_1 + x_3 + x_4 - x_5 \leq -1 + 7 + 8 - 15 < 0.$$

Therefore, we may assume that $|\Delta(S)| \in \{2, 3, 4\}$. We consider each case separately.

Case 1: $\Delta(S) = \{\eta_1, \eta_2, \eta_3, \eta_4\}$.

We consider two subcases.

Case 1a: $|\Delta^{-1}(\eta_i) \cap S| \geq 2$ and $|\Delta^{-1}(\eta_j) \cap S| \geq 2$ for some integers $i, j \in [1, 4]$.

It is easy to see that we may construct a 4-transversal $T_1, \ldots, T_4 \subset S$. From the Cauchy-Davenport theorem it follows that there exist integers $x_i \in T_i$ for $i = 1, 2, 3, 4$ such that $\sum_{i=1}^{4} \Delta(x_i) = -\Delta(15)$. Setting $x_5 = 15$, we obtain the zero sum solution $h(x_1,\dots,x_5) = (y_1,\dots,y_5)$ of $L_2(5)$, as

$$-y_1 + y_3 + y_4 - y_5 \leq -1 + 7 + 8 - 15 < 0.$$

Thus, Case 1a is complete. This leads to one final subcase of Case 1.

Case 1b: $|\Delta^{-1}(\eta_1) \cap S| = 5$, and $|\Delta^{-1}(\eta_i) \cap S| = 1$ for $j = 2,3,4$.

Let $i_j \in \{\Delta^{-1}(\eta_j) \cap S\}$ for $j = 2,3,4$, and let $\beta_1 < \beta_2 < \cdots < \beta_5 \in \{\Delta^{-1}(\eta_1) \cap S\}$. It is easy to see that if $\beta_1 \neq 1$, then $(\beta_1,\beta_2,\dots,\beta_5)$ is a zero-sum solution of $L_2(5)$. Therefore, we may assume that $\beta_1 = 1$. Further, if $\Delta(9) = \eta_1$, then $(\beta_2,\beta_3,\dots,\beta_5,9)$ is a zero-sum solution of $L_2(5)$. Therefore, we may assume that $\Delta(9) \neq \eta_1$. Suppose without loss of generality that $i_2 < i_3 < i_4$, and define the sets

$$T_1 = \{\beta_2, i_2\} \quad T_2 = \{\beta_3, i_3\} \quad T_3 = \{\beta_4, i_4\}, \quad T_4 = \{\beta_5, 9\}.$$

The sets T_i form a 4-transversal of $S' = [1,9]$, and thus by the Cauchy-Davenport theorem there exist integers $x_i \in T_i$ for $i = 1,2,3,4$ such that $\sum_{i=1}^{4} \Delta(x_i) = -\Delta(15)$. Setting $x_5 = 15$, we find that $h(x_1,\dots,x_5) = (y_1,\dots,y_5)$ is a zero sum solution of $L_2(5)$, as

$$-y_1 + y_3 + y_4 - y_5 \leq -2 + 8 + 9 - 15 = 0.$$

Thus, Case 1 is complete.

Case 2: $|\Delta(S)| = \{\eta_1, \eta_2, \eta_3\}$.

We consider three subcases.

Case 2a: $|\Delta^{-1}(\eta_i) \cap S| \geq 2$ for $i = 1,2,3$.

As in Case 1a, it is easy to see that we may construct a 4-transversal $T_1,\dots,T_4 \subset S$ and form a solution of $L_2(5)$.

Case 2b: There exists $i,j \in [1,3]$ such that $|\Delta^{-1}(\eta_i) \cap S| = 1$ and $|\Delta^{-1}(\eta_j) \cap S| = 2$.

Argument analogous to that of Case 1b completes Case 2a. Thus, we are left with one subcase of Case 2.

Case 2c: $|\Delta^{-1}(\eta_1) \cap S| = 6$, $|\Delta^{-1}(\eta_2) \cap S| = 1$, $|\Delta^{-1}(\eta_3) \cap S| = 1$.

Let $\beta_1 < \beta_2 < \cdots < \beta_6 \in \{\Delta^{-1}(\eta_1)\}$. It is not difficult to see that if $\beta_6 = 8$, then $(\beta_2,\dots,\beta_6)$ is a zero-sum solution of $L_2(5)$. Therefore, we may assume that $\beta_6 \leq 7$. Furthermore, if $\beta_1 \neq 1$, then $(\beta_2,\beta_3,\dots,\beta_6)$ is a zero sum solution of $L_2(5)$. Therefore, we may assume that $\beta_1 = 1$. But then if $\beta_3 \leq 3$, it follows that $(\beta_1,\beta_2,\beta_3,\beta_4,\beta_6)$ is a zero-sum solution of $L_2(m)$. Therefore, we assume that $\beta_3 \geq 4$, whence we find

$$\beta_1 = 1, \quad \beta_3 = 4, \quad \beta_4 = 5, \quad \beta_5 = 6, \quad \beta_6 = 7.$$

Now, if there exists $i \in [9,11]$ such that $\Delta(i) = \eta_1$, then $(\beta_3,\beta_4,\dots,\beta_6,i)$ is a zero-sum solution of $L_2(5)$. Hence we may suppose that $\Delta(i) \neq \eta_1$ for

$i \in [9, 11]$. Define the sets

$$T_1 = \{\beta_3, 11\}, \quad T_2 = \{\beta_4, 10\}, \quad T_3 = \{\beta_5, 9\}, \quad T_4 = \{\beta_6, 8\}.$$

The sets T_i form a 4-transversal of $S' = [1, 11]$, and thus, by the Cauchy-davenport theorem, there exist $x_i \in T_i$ such that $\sum_{i=1}^{4} \Delta(x_i) = -\Delta(15)$. Setting $x_5 = 15$, we find that $h(x_1, \ldots, x_5) = (y_1, \ldots, y_5)$ is a zero-sum solution of $L_2(5)$, since

$$-y_1 + y_3 + y_4 - y_5 \leq -4 + 9 + 10 - 15 = 0.$$

Thus, Case 2 is complete.

Case 3: $\Delta(S) = \{\eta_1, \eta_2\}$.

Case 3a: $|\Delta^{-1}(\eta_1) \cap S| = |\Delta^{-1}(\eta_2) \cap S| = 4$.

Argument analogous to that of Case 1a shows that there exists zero-sum solution of $L_2(5)$.

Case 3b: $|\Delta^{-1}(\eta_i) \cap S| = 5$, $|\Delta^{-1}(\eta_j)| = 3$ for some $i, j \in \{1, 2\}$.

Argument analogous to that of Case 1b shows that there exists a zero sum solution of $L_2(5)$.

Case 3c: $|\Delta^{-1}(\eta_i) \cap S| = 6$, $|\Delta^{-1}(\eta_j)| = 2$ for some $i, j \in \{1, 2\}$.

Argument analogous to that of Case 2c show that there exists a zero-sum solution of $L_2(5)$.

Case 3d: $|\Delta^{-1}(\eta_1)| = 7$, $|\Delta^{-1}(\eta_2)| = 1$.

Let $\beta_1 < \beta_2 < \cdots < \beta_7 \in \{\Delta^{-1}(\eta_1) \cap S\}$. It is not difficult to see that regardless of the values of the β_i, it follows that $(\beta_3, \beta_4, \ldots, \beta_7)$ is a zero-sum solution of $L_2(5)$. Thus, Case 3 is complete.

Thus, the proof of the theorem is complete. □

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