COMPOSITIONS WITH PARTS CONSTRAINED BY THE LEADING SUMMAND

ARNOLD KNOPFMACHER † AND NEVILLE ROBBINS

ABSTRACT. We consider compositions or ordered partitions of the natural number n for which the largest (resp. smallest) summand occurs in the first position of the composition.

1. INTRODUCTION

Let $a_1 + a_2 + \cdots + a_k = n$, with $a_i \geq 1$ for $i \geq 1$ be a composition of n. It is well known that there are 2^{n-1} compositions of n. Composition counting problems become more interesting when constraints relating to the parts a_i or their relative order are introduced. For example, in [2] compositions with no two adjacent equal parts are studied, and in [5] compositions with all parts distinct are considered. Binary compositions (all parts powers of two) are treated in [3]. Of course, the best known class of compositions are those with nonincreasing parts, (better known as *partitions* of n)!

We consider here compositions of the natural number n for which the largest (resp. smallest) summand occurs in the first position of the composition. That is $a_1 + a_2 + \cdots + a_k = n$, with $a_i \geq 1$ for $i \geq 1$ and $a_i < a_1$ for $i \geq 2$, or $a_i > a_1$ for $i \geq 2$. As a variation we consider also the larger classes of compositions for which $a_i < a_1$ is replaced by $a_i \leq a_1$, and similarly $a_i > a_1$ is replaced by $a_i \geq a_1$ for $i \geq 2$.

2. COMPOSITIONS WITH LARGEST PART IN THE FIRST POSITION

We consider first the case of strict inequality. That is $a_1 + a_2 + \cdots + a_k = n$, with $a_i \geq 1$ for $i \geq 1$ and $a_i < a_1$ for for $i \geq 2$. If the first part is the number $k \geq 2$, then the ordinary generating function for such compositions with leading part k is just

$$\frac{z^k}{1 - z - z^2 - \cdots - z^{k-1}} = \frac{z^k}{1 - z\frac{(1-z^{k-1})}{1-z}} = \frac{(1-z)z^k}{1 - 2z + z^k}. \tag{1}$$

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Summing over all $k \geq 2$ gives the generating function for compositions with strictly greatest part in the first position,

$$f(z) = \sum_{k \geq 1} \frac{(1-z)z^k}{1-2z+z^k}. \tag{2}$$

The term z from $k = 1$ above represents the unique composition of 1.

To estimate the coefficient f_n of z^n in $f(z)$ we follow the approach of Flajolet, Gourdon and Dumas [1], who considered a similar generating function relating to the longest run of 1's in a random binary string of length n.

Theorem 1. *As $n \to \infty$,*

$$f_n \sim \frac{2^{n-1}}{n \log 2} (1 + \delta(\log_2 n))$$

where $\delta(x)$ is a continuous periodic function of period 1, mean zero, small amplitude and Fourier expansion

$$\delta(x) = \sum_{k \neq 0} \Gamma(1 + \chi_k) e^{-2k\pi i x}. \quad \square$$

Proof. Let ρ_k be the smallest positive root of the denominator of (1) that lies between 1/2 and 1. An application of the principle of the argument shows such a root to exist with all other roots of larger modulus. By dominant pole analysis,

$$q_{n,k} := [z^n] \frac{(1-z)z^k}{1-2z+z^k} \sim c_k \rho_k^{-n} \quad \text{with} \quad c_k = \frac{(1-\rho_k)\rho_k^k}{\rho_k(2 - k\rho_k^{k-1})},$$

for large n but fixed k.

The denominators of the terms in (2) behave like a perturbation of $1 - 2z$ near $z = 1/2$, so one expects ρ_k to be approximated by $\frac{1}{2}$ as $k \to \infty$. By "bootstrapping" we find that

$$\rho_k = \frac{1}{2}(1 + 2^{-k} + O(k2^{-k})) \tag{3}$$

and hence $c_k = \frac{1}{2^{k+1}}(1 + O(k2^{-k}))$. The use of this approximation can be justified for a wide range of values of k and n (see Knuth [4]). We thus obtain the approximate formula

$$q_{n,k} \approx 2^{n-k-1}(1 - 2^{-k})^n \approx 2^{n-k-1} e^{-n/2^k}.$$

Then as in Knuth [4]

$$f_n := [z^n] f(z) = 2^{n-1} \left(\sum_{k=2}^{\infty} \frac{1}{2^k} e^{-n/2^k} + o(1) \right).$$

Let

$$g(x) := \sum_{k=2}^{\infty} \frac{1}{2^k} e^{-x/2^k}.$$

For $x \in \mathbb{R}$, the Mellin transform of the function $g(x)$ is

$$g^*(s) = \int_0^\infty g(x) x^{s-1} dx = \left(\sum_{k \geq 2} 2^{(s-1)k} \right) \Gamma(s) = \frac{2^{2s-2}}{1 - 2^{s-1}} \Gamma(s), \quad 0 < \mathrm{Re}(s) < 1.$$

Here we have used the fact that the Mellin transform of e^{-x} is $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$, for $\mathrm{Re}(s) > 0$. To estimate the sum $g(n)$ and hence f_n we use the Mellin inversion formula,

$$g(x) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} g^*(s) x^{-s} ds.$$

We move the contour of integration to the right and must compute some residues as a compensation.

Let $\chi_k = 2k\pi i / \log 2$. There are simple poles of the integrand at $s = 1 + \chi_k$ for each $k \in \mathbb{Z}$, with residue

$$\frac{1}{\log 2} 2^{2\chi_k} \Gamma(1 + \chi_k) x^{-1-\chi_k} = \frac{1}{x \log 2} \Gamma(1 + \chi_k) e^{-2k\pi i \log_2 x}.$$

Combining the contributions for all $k \in \mathbb{Z}$, we find that

$$g(n) \sim \frac{1}{n \log 2} \sum_{k \in \mathbb{Z}} \Gamma(1 + \chi_k) e^{-2k\pi i \log_2 n}.$$

The result follows after separating out the largest term of the above sum, which comes from $k = 0$.

□

Computations show that $|\delta(x)| < 10^{-5}$, as a result of the fast decrease of the gamma function with imaginary argument. Consequently, the proportion of compositions of n with largest part first satisfies

$$0.99999 \frac{1}{n \log 2} < \frac{f_n}{2^{n-1}} < 1.00001 \frac{1}{n \log 2}$$

for large n.

For numerical purposes $|\delta(x)|$ is well approximated using the terms for $k = 1$ and $k = -1$:

Now consider the variation where we allow parts equal in size to the first part, that is $a_1 + a_2 + \cdots + a_k = n$, with $a_i \geq 1$ for $i \geq 1$ and $a_i \leq a_1$ for $i \geq 2$.

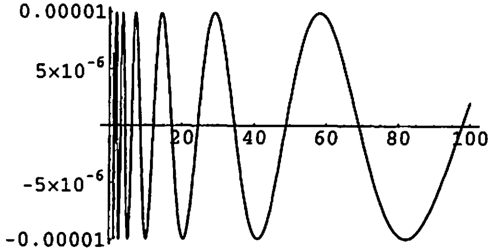

FIGURE 1. *Plot of* $\delta(\log_2 n)$ *for* $1 \le n \le 100$.

If the first part is the number $k \ge 1$, then the ordinary generating function for such compositions with leading part k is just

$$\frac{z^k}{1-z-z^2-\cdots-z^k} = \frac{z^k}{1-z\frac{(1-z^k)}{1-z}} = \frac{(1-z)z^k}{1-2z+z^{k+1}}. \tag{4}$$

Summing over all $k \ge 1$ gives the generating function for compositions with all parts less than or equal to the part in the first position,

$$\hat{f}(z) = \sum_{k\ge1} \frac{(1-z)z^k}{1-2z+z^{k+1}}. \tag{5}$$

From this we see that $z\hat{f}(z) = f(z) - z$ and hence for $n \ge 2$ the coefficients $\hat{f}_n$ of $\hat{f}(z)$ satisfy $\hat{f}_n = f_{n+1}$. Consequently we have immediately

Theorem 2. *As* $n \to \infty$,

$$\hat{f}_n = f_{n+1} \sim \frac{2^n}{n \log 2}(1 + \delta(\log_2 n))$$

where $\delta(x)$ *is defined as in Theorem 1.* □

Remark By symmetry, Theorems 1 and 2 hold also for counting the number of compositions of n with *largest part in the* last *position.*

The intial terms of the f_n sequence for $n \ge 1$ are

1, 1, 2, 3, 5, 8, 14, 24, 43, 77, 140, 256, 472, 874, 1628, 3045, 5719, 10780, 20388, 38674, 73562, ⋯ .

The sequence exhibits an interesting pattern modulo 2.

Theorem 3. *The coefficient f_n is odd if and only if $n = m^2$ or $n = m^2+1$.*

Proof. By taking the generating function (2) for $f(z)$ modulo 2 we obtain

$$\sum_{n=1}^{\infty} f_n z^n \equiv \sum_{k\geq 1} \frac{(1+z)z^k}{1-z^k} \pmod{2}.$$

or equivalently,

$$\sum_{n=1}^{\infty} f_n z^n \equiv \sum_{k\geq 1} \frac{z^k}{1-z^k} + \sum_{k\geq 1} \frac{z^{k+1}}{1-z^k} \pmod{2}.$$

Now

$$\sum_{k=1}^{\infty} \frac{z^k}{1-z^k} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} z^{kj}.$$

If we let $r = kj$ and define the divisor function $\tau(r) = \sum_{d|r} 1$, we can rewrite the double sum in the form

$$\sum_{k=1}^{\infty} \frac{z^k}{1-z^k} = \sum_{r=1}^{\infty} \tau(r) z^r.$$

It is easy to see that $\tau(r) \equiv 1 \pmod{2}$ if and only if $n = m^2$. Therefore we have

$$\sum_{k=1}^{\infty} \frac{z^k}{1-z^k} \equiv \sum_{i=1}^{\infty} z^{i^2} \pmod{2},$$

and therefore

$$\sum_{k=1}^{\infty} \frac{z^{k+1}}{1-z^k} \equiv \sum_{i=1}^{\infty} z^{i^2+1} \pmod{2}.$$

The conclusion now follows.

□

To compute the exact values of the f_n sequence we can employ a recursive approach. Let

$$f_k(z) := \sum_{n\geq 0} f_{k,n} z^n = \frac{(1-z)z^k}{1-2z+z^k}, \tag{6}$$

be the generating function for compositions with largest part k in the first position. By equating coefficients of z^n in the relation

$$(1-2z+z^k) \sum_{n\geq 0} f_{k,n} z^n = (1-z)z^k$$

we obtain the recurrence relation

$$f_{k,n} = 2f_{k,n-1} - f_{k,n-k} + \delta_{k,n} - \delta_{k+1,n}$$

where $\delta_{m,n} = 1$ if $m = n$ and is zero otherwise, and with initial conditions $f_{k,n} = 0$, for $n < k$.

We may then compute f_n using

$$f_n = \sum_{k=2}^{n} f_{k,n}.$$

3. Compositions with smallest part in the first position

We consider first the case of strict inequality. That is $a_1 + a_2 + \cdots + a_k = n$, with $a_i \geq 1$ for $i \geq 1$ and $a_i > a_1$ for for $i \geq 2$. If the first part is the number $k \geq 1$, then the ordinary generating function for such compositions with leading part k is just

$$\frac{z^k}{1 - z^{k+1} - z^{k+2} - \dots} = \frac{z^k}{1 - \frac{z^{k+1}}{1-z}} = \frac{(1-z)z^k}{1-z-z^{k+1}}. \tag{7}$$

Summing over all $k \geq 1$ gives the generating function for compositions with strictly smallest part in the first position,

$$h(z) = \sum_{k\geq 1} \frac{(1-z)z^k}{1-z-z^{k+1}}. \tag{8}$$

Theorem 4. *As $n \to \infty$,*

$$h_n := [z^n]h(z) = \frac{3-\sqrt{5}}{2\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n + O(\rho_2^{-n})$$

where $\rho_2 \approx 0.682328$ is the smallest positive root of $1 - z - z^3 = 0$. □

Proof. Let ρ_k be the smallest positive root of the denominator of (7) that lies between 0 and 1. By dominant pole analysis,

$$q_{n,k} := [z^n]\frac{(1-z)z^k}{1-z-z^{k+1}} \sim c_k\rho_k^{-n} \quad \text{with} \quad c_k = \frac{(1-\rho_k)\rho_k^k}{\rho_k(1+(k+1)\rho_k^k)},$$

for large n but fixed k.

As $\rho_1 < \rho_2 < \rho_3 \ldots$, the largest contribution $q_{n,k}$ comes from the case $k = 1$, with the other terms bounded by the size of $q_{n,2}$. Since $\rho_1 = \frac{\sqrt{5}-1}{2}$ and $c_1 = \frac{1-\rho_1}{1+2\rho_1} = \frac{3-\sqrt{5}}{2\sqrt{5}}$ and the smallest positive root of $1 - z - z^3 = 0$ is $\rho_2 \approx 0.682328$, the result follows. □

We see that asymptotically almost all compositions counted by h_n have leading part 1 and subsequent parts greater than 1, with generating function

$$\frac{(1-z)z}{1-z-z^2}.$$

This case is equivalent to counting compositions of $n-1$ with all parts ≥ 2.

Now consider the variation where we allow parts equal in size to the first part, that is $a_1 + a_2 + \cdots + a_k = n$, with $a_i \geq 1$ for $i \geq 1$ and $a_i \geq a_1$ for for $i \geq 2$.

If the first part is the number $k \geq 1$, then the ordinary generating function for such compositions with leading part k is now

$$\frac{z^k}{1-z^k-z^{k+1}-\ldots} = \frac{z^k}{1-\frac{z^k}{1-z}} = \frac{(1-z)z^k}{1-z-z^k}. \tag{9}$$

Hence the generating function for compositions with all parts greater than or equal to the part in the first position is,

$$\hat{h}(z) = \sum_{k\geq 1}\frac{(1-z)z^k}{1-z-z^k}. \tag{10}$$

From this we see that

$$z\hat{h}(z) = \frac{z-z^2}{1-2z} + zh(z)$$

and hence for $n \geq 2$ the coefficients $\hat{h}_n$ of $\hat{h}(z)$ satisfy

$$\hat{h}_n = 2^{n-2} + h_{n-1}.$$

Consequently

Theorem 5. *As $n \to \infty$,*

$$\hat{h}_n \sim 2^{n-2} + O\left(\frac{\sqrt{5}+1}{2}\right)^n. \quad \square$$

The case $k = 1$ in (9) corresponds to counting unrestricted compositions of $n-1$, which explains the contribution 2^{n-2} above.

The initial terms of the h_n sequence for $n \geq 1$ are

1, 1, 2, 2, 4, 5, 8, 12, 19, 28, 45, 70, 110, 173, 275, 436, 695, 1107, 1769, 2831, 4537, 7276, 11683, $\cdots$.

To compute the exact values of the h_n sequence we employ an approach akin to that for f_n. Let

$$h_k(z) := \sum_{n\geq 0} h_{k,n} z^n = \frac{(1-z)z^k}{1-z-z^{k+1}}, \tag{11}$$

be the generating function for compositions with smallest part k in the first position. Equating coefficients of z^n in the relation

$$(1-z-z^{k+1}) \sum_{n\geq 0} h_{k,n} z^n = (1-z)z^k$$

gives the recurrence relation

$$h_{k,n} = h_{k,n-1} + h_{k,n-k-1} + \delta_{k,n} - \delta_{k+1,n}$$

where $\delta_{m,n}$ as before, and with initial conditions $h_{k,n} = 0$, for $n < k$.

Then

$$h_n = \sum_{k=1}^{n} h_{k,n}.$$

We remark finally that Theorems 4 and 5 hold also for counting the number of compositions of n with *smallest part in the* last *position.*

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ARNOLD KNOPFMACHER, THE JOHN KNOPFMACHER CENTRE FOR APPLICABLE ANALYSIS AND NUMBER THEORY, UNIVERSITY OF THE WITWATERSRAND, P. O. WITS, 2050 JOHANNESBURG, SOUTH AFRICA, EMAIL: arnoldk@cam.wits.ac.za.

NEVILLE ROBBINS, MATHEMATICS DEPARTMENT, SAN FRANCISCO STATE UNIVERSITY, CA 94132, U.S.A., EMAIL: robbins@math.sfsu.edu.