PACKING OF ANY SET OF GRAPHS INTO A GRACEFUL/ HARMONIOUS / ELEGANT GRAPH

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Abstract

Balakrishnan et al. [1, 2] have shown that every graph is a subgraph of a graceful graph and an elegant graph. Also Liu and Zhang [4] have shown that every graph is a subgraph of a harmonious graph. In this paper we prove a generalization of these two results that any given set of graphs $G_1, G_2, \dots, G_t$ can be packed into a graceful/harmonious / elegant graph.

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1 Introduction

A function f is called a *graceful labeling* of a graph G with m edges if f is an injection from the vertex set of G to the set $\{0, 1, 2, \ldots, m\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph which admits a graceful labeling is called a *graceful graph.*

A function f is called *harmonious labeling* of a graph G with m edges if f is an injection from the vertex set of G to the group of integers modulo m such that, when each edge xy is assigned the label $f(x) + f(y)(\text{mod} m)$, the resulting edge labels are distinct. A graph which admits a harmonious labeling is called a *harmonious graph.*

An *elegant labeling* f of a graph G with m edges is an injective function from the vertex set of G to the set $\{0, 1, 2, \ldots, m\}$ such that when each edge xy is assigned the label $f(x) + f(y)(\text{mod}(m + 1))$ the resulting edge labels are distinct and nonzero. A graph which admits an elegant labeling is called an elegant graph.

A sequence of graphs $G_1, G_2, \ldots, G_t$ is said to be *packed* into a graph G if G has edge disjoint subgraphs $F_1, F_2, \ldots, F_t$ such that $F_j \simeq G_j$, for $j = 1, 2, \ldots, t$.

For $m \geqslant 1$, mK_1 denotes m disjoint copies of K_1. For a given set of graphs $G_1, G_2, \ldots, G_t$, $t \geqslant 1$, we denote $\tilde{G} = \bigcup_{i=1}^{k} G_i$ and $H_m = \tilde{G} \bigcup mK_1$.

Balakrishnan et al. [1, 2] have shown that every graph is a subgraph of a graceful graph and an elegant graph. Also Liu and Zhang [4] have shown that every graph is a subgraph of a harmonious graph. In this paper we prove a generalization of these two results, that any given set of graphs $G_1, G_2, \ldots, G_t$ can be packed into a graceful, harmonious and elegant graph.

More precisely, In this paper we prove the following results.

For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geqslant 1$, the graph $G = H_m + K_1$ is

(i) graceful, for $m \geqslant 2^{|V(\tilde{G})|} - (|V(\tilde{G})| + |E(\tilde{G})| + 1)$;

(ii) harmonious and elegant, for $m = 2^{|V(\tilde{G})|} - (|V(\tilde{G})| + |E(\tilde{G})|)$.

2 The Graph $H_m + K_1$ is Graceful, Harmonious and Elegant

Theorem 1: For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geqslant 1$, the graph $G = H_m + K_1$ is graceful, for $m \geqslant 2^{|V(\tilde{G})|} - (|V(\tilde{G})| + |E(\tilde{G})| + 1)$.

Proof: For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geqslant 1$, consider the graph $G = H_m + K_1$ with $m \geqslant 2^{|V(\tilde{G})|} - (|V(\tilde{G})| + |E(\tilde{G})| + 1)$. Then G has $|V(\tilde{G})| + m + 1$ vertices and $M = |E(\tilde{G})| + |V(\tilde{G})| + m$ edges. Let $N = |V(\tilde{G})|$. Then, for convenience we label the vertices of G as $v_0, v_1, v_2, v_3, \ldots, v_N$, $u_1, u_2, \ldots, u_m$, where v_0 is the unique vertex of G with maximum degree $N + m$ and $v_1, v_2, \ldots, v_N$ are the vertices of $\tilde{G}$ and $u_1, u_2, \ldots, u_m$ are the remaining m pendant vertices adjacent to v_0 of G.

Define
$$\phi(v_0) = 0$$
$$\phi(v_i) = 2^i - 1, \; 1 \leqslant i \leqslant N$$

For each i, $1 \leqslant i \leqslant m$, define $\phi(u_i)$ to be any one of the (unassigned) distinct labels from the set

$$A = \{1, 2, \ldots, M\} \setminus \left\{ 2^i - 1 \middle| 1 \leqslant i \leqslant N \right\} \setminus \bigcup_{i=1}^{N-1} \left\{ 2^i(2^{j-i} - 1) \middle| i + 1 \leqslant j \leqslant N \text{ and } v_i v_j \in E(\tilde{G}) \right\}$$

Note that any bijective mapping from the set $\{u_1, u_2, \ldots, u_m\}$ onto the set A can be chosen for defining the function ϕ on the vertices u_i's. Thus ϕ is injective and the edge values of G are distinct and range from 1 to M. Hence G is Graceful. □

Following corollary is an immediate consequence of Theorem 1.

Corollary 1: For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geqslant 1$, there exists a graceful graph G such that $G_1, G_2, \ldots, G_t$ can be packed into G.

Remark 1: Corollary 1 is a generalization of the result of Balakrishnan et al. [1] that every graph is a subgraph of a graceful graph.

Theorem 2: For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geqslant 1$, the graph $G = H_m + K_1$ is harmonious, for $m = 2^{|V(\tilde{G})|} - (|V(\tilde{G})| + |E(\tilde{G})|)$.

Proof: Let $v_0, v_1, v_2, \ldots, v_N, u_1, u_2, \ldots, u_m$ and M be as in the proof of Theorem 1.

$$\text{Define} \qquad \begin{aligned} &\phi(v_0) = 0 \\ &\phi(v_i) = 2^i - 1,\ 1 \leqslant i \leqslant N-1 \\ &\phi(v_N) = 2^N - 2 \end{aligned}$$

For each i, $1 \leqslant i \leqslant m$, define $\phi(u_i)$ to be any one of the (unassigned) distinct labels from the set

$$B = \{1, 2, \ldots, M-1\} \setminus \left\{ 2^i - 1 \middle| 1 \leqslant i \leqslant N-1 \right\} \setminus \{2^N - 2\} \setminus \bigcup_{i=1}^{N-1} \left\{ \begin{array}{r} (2^i + 2^j - \beta(\text{mod} M)) \Big| i+1 \leqslant j \leqslant N \text{ and} \\ v_i v_j \in E(\tilde{G}) \\ \text{where } \beta = 2 \text{ if } j \neq N \\ \beta = 3 \text{ if } j = N \end{array} \right\}$$

Note that any bijective mappaing from the set $\{u_1, u_2, \ldots, u_m\}$ onto the set B can be chosen for defining the function ϕ on the vertices u_i's. Thus ϕ is injective and the edge values of G are distinct and range from 0 to $M-1$. Hence G is harmonious. □

The following corollary is an immediate consequence of Theorem 2.

Corollary 2: For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geqslant 1$, there exists a harmonious graph G such that $G_1, G_2, \ldots, G_t$ can be packed into G.

Remark 2: Corollary 2 is a generalization of the result of Liu and Zhang [4] that every graph is a subgraph of a harmonious graph.

Theorem 3: For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geqslant 1$, the graph $G = H_m + K_1$ is elegant, for $m = 2^{|V(\tilde{G})|} - (|V(\tilde{G})| + |E(\tilde{G})|$.

Proof: Let $v_0, v_1, v_2, v_3, \ldots, v_N$, $u_1, u_2, \ldots, u_m$ and M be as in the proof of Theorem 1.

$$\text{Define} \qquad \begin{aligned} \phi(v_0) &= 0 \\ \phi(v_i) &= 2^i, \ 1 \leqslant i \leqslant N \end{aligned}$$

For each i, $1 \leqslant i \leqslant m$, define $\phi(u_i)$ to be any one of the (unassigned) distinct labels from the set

$$C = \{1, 2, \ldots, M\} \setminus \left\{ 2^i \middle| 1 \leqslant i \leqslant N \right\} \setminus \bigcup_{i=1}^{N-1} \left\{ 2^i + 2^j \pmod{M+1} \middle| i+1 \leqslant j \leqslant N \text{ and } v_i v_j \in E(\tilde{G}) \right\}$$

Note that any bijective mapping from the set $\{u_1, u_2, \ldots, u_m\}$ onto the set C can be chosen for defining the funtion ϕ on the vertices u_i's. Thus ϕ is injective and the edge values of G are distinct and range from 1 to M. Hence G is elegant. □

The following corollary is an immediate consequence of Theorem 3.

Corollary 3: For any set of graphs $G_1, G_2, \ldots, G_t$, $t \geqslant 1$, there exists an elegant graph G such that $G_1, G_2, \ldots, G_t$ can be packed into G.

Remark 3: Corollary 3 is a generalization of the result of Balakrishnan et al. [2] that every graph is a subgraph of an elegant graph.

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