Two sufficient conditions for a graph to be type 1

Jian-Liang Wu *
School of Mathematics, Shandong University
Jinan, Shandong, 250100, P. R. China

Ping Wang †
Department of Mathematics, Statistics and Computer Science
St. Francis Xavier University, Antigonish, Nova Scotia, Canada
e-mail: pwang@stfx.ca

Anthony Hilton
Department of Mathematics, University of Reading,
Whiteknights, P.O. Box 220, Reading, RG6 2AX, Great Britain
e-mail: j.w.hilton@reading.ac.uk

Abstract

In this paper, we give two sufficient conditions for a graph to be type 1 with respect to the total chromatic number and prove the following results.

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*Research supported by the James Chair from St. Francis Xavier University

†Research supported by The Natural Sciences and Engineering Council of Canada.

(i) if G and H are of type 1, then $G \times H$ is of type 1;
(ii) if $\varepsilon(G) \leq v(G) + \frac{3}{2}\Delta(G) - 4$, then G is of type 1.

1 Introduction

We consider finite simple graphs with at least two vertices. Any undefined notation follows that of Bondy and Murty [3]. The number of vertices, the number of edges, the maximum degree, the minimum degree, the edge chromatic number, and the chromatic number of a graph G are denoted by $v(G)$, $\varepsilon(G)$, $\Delta(G)$, $\delta(G)$, $\chi'(G)$, and $\chi(G)$ respectively. A *total colouring* of a graph G is a function assigning colours to the elements of $VE(G) = V(G) \cup E(G)$ in such a way that no two adjacent or incident elements of $VE(G)$ are assigned the same colour. A graph is *totally k-colourable* if it has a total colouring of k colours. The *total chromatic number* $\chi_T(G)$ is the minimum number k for which G is totally k-colourable. The *Cartesian product* of G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. For convenience, we use the following notations for the Cartesian product $G \times H$. Let $V(G) = \{u_1, u_2, \dots, u_m\}$, $V(H) = \{v_1, v_2, \dots, v_n\}$, and

$V(G \times H) = \{w_{ij} | w_{ij} = (u_i, v_j) : i = 1, 2, \dots, m,\ j = 1, 2, \dots, n\}$;

$E(G \times H) = \{w_{ij}w_{st} | i = s \text{ and } v_jv_t \in E(H), \text{ or } j = t \text{ and } u_iu_s \in E(G)\}$.

Behzad and Vizing (see page 86 in [13]) conjectured independently in 1965 that any graph G is totally $(\Delta(G)+2)$-colourable.

Various colouring techniques have been introduced in order to prove this conjecture for some special classes graphs(see survey papers [11] and [16]). The simple graphs G satisfying the total colouring conjecture (with $\chi_T \leq \Delta(G)+2$) fall into two classes. A graph G is of *type* 1 if $\chi_T(G) = \Delta(G) + 1$, and is of *type* 2 if $\chi_T(G) = \Delta(G)+2$. Sanchez-Arroyo [14] proved that for a graph G it is **NP**-complete to decide if $\chi_T(G) = \Delta(G) + 1$. A necessary (but not sufficient condition) for a graph G to be type 1 is given by Chetwynd and Hilton [4] in 1988. The classification problem of whether the graph is of type 1 or type 2 was widely studied (see [5], [6], [7], [8], [9],[10], and [12]). For example, the following families of graphs are classified: the complete graphs, the complete r-partite graphs, graphs with very large maximum degree ($\Delta(G) = v(G) - 1, v(G) - 2$ and some partial results on $\Delta(G) = v(G) - 3$) and graphs of regular degree k where $k \geq \frac{\sqrt{7}}{3}v(G)$(see [16]). In this paper we provide two sufficient conditions for a graph to be type 1. First, we prove that if two graphs are of type 1 then the Cartesian product graph of these two graphs is also of type 1. Second, we prove that for any graph with $\varepsilon(G) \leq v(G) + \frac{3}{2}\Delta(G) - 4$, then G is of type 1.

2 Total colouring of $G \times H$

In 1968, Behzad and Mahmoodian [2] first explored different types of chromatic numbers for Cartesian product graphs. They prove the following results for various chromatic numbers of Cartesian product graphs.

Theorem 2.1. $\chi(G \times H) = max\ \{\chi(G), \chi(H)\}$.

Theorem 2.2. *If* $\chi'(G) = \Delta(G)$ *and* $\chi'(H) = \Delta(H)$, *then* $\chi'(G \times H) = \Delta(G \times H)$.

Theorem 2.3. *If* $\chi(G) \leq \chi_T(H)$,*then* $\Delta(G) + \Delta(H) + 1 \leq \chi_T(G \times H) \leq \chi_T(H) + \chi'(G)$.

For the edge chromatic number, Theorem 2.2 asserts that if both factors in the Cartesian product are of class 1, then the product is also of class 1. Anderson and Lipman [1] prove a similar type of result on "lexicographic product" (sometimes referred to as the "composition", denoted by G[H]). That is, if G is of class 1, then $G[H]$ is also of class 1. In the case of the total chromatic number, we can prove that the Cartesian product is a closed operation for type 1 graphs. Note that this is not true for type 2 graphs. For example, type 2 $\times$ type 2 can be either type 1 ($C_4 \times K_2$) or type 2 ($K_2 \times K_2$).

First we give general lower and upper bounds on the total chromatic number for the Cartesian product of two graphs.

Theorem 2.4. $\Delta(G\times H)+1 \leq \chi_T(G\times H) \leq \chi_T(G)+\chi_T(H)-1$.

Proof. The lower bound on $\chi_T(G\times H)$ is trivial. Let $|V(G)| = n$ and $|V(H)| = m$, $\chi_T(G) = a$ and $\chi_T(H) = b$. Without loss of generality, we may assume that $a \geq b > 1$. Let $\alpha : VE(G) \rightarrow \{0, 1, 2, \cdots, a-1\}$ be a total colouring of G, and $\beta : VE(H) \rightarrow \{0, a, a+1, \cdots, a+b-2\}$ be a total colouring of H. Now we give a total colouring $\gamma : VE(G \times H) \rightarrow \{0, 1, \cdots, a+b-2\}$ of $G \times H$ as follows.

First, we colour each edge in a copy of H the same colour as in H, that is, $\gamma(w_{ki}w_{kj}) = \beta(v_iv_j)$ for $v_iv_j \in E(H)$ and

$k = 1, 2, \cdots, m$. Then we colour the vertices and the remaining edges of $G \times H$ as follows. If $\beta(v_k) = 0$ where $v_k \in V(H)$, then $\gamma(w_{ik}) = \alpha(u_i)$ and $\gamma(w_{ik}w_{jk}) = \alpha(u_iu_j)$; otherwise,

$$\gamma(w_{ik}) = \begin{cases} \beta(v_k), & \text{if } \alpha(u_i) + \beta(v_k) + 1 \equiv 0 \mod a, \\ (\alpha(u_i) + \beta(v_k) + 1) \mod a, & \text{otherwise;} \end{cases}$$

and

$$\gamma(w_{ik}w_{jk}) = \begin{cases} \beta(v_k), & \text{if } \alpha(u_iu_j) + \beta(v_k) + 1 \equiv 0 \mod a, \\ (\alpha(u_iu_j) + \beta(v_k) + 1) \mod a, & \text{otherwise.} \end{cases}$$

Here, and elsewhere, $x \bmod a$ is taken to be the integer x' such that $x' \equiv x \mod a$ and $0 \leq x' < a$. In the following, we verify that γ is a proper $(a + b + 1)$-total-colouring of G. Clearly, $\beta(v_k) > (\alpha(u_i) + \beta(v_k) + 1) \bmod a$ and $\beta(v_k) > (\alpha(u_iu_j)+\beta(v_k)+1) \bmod a$ unless $\beta(v_k) = 0$ since $\alpha : VE(G) \rightarrow \{0, 1, 2, \cdots, a-1\}$ is a total colouring of G, and $\beta : VE(H) \rightarrow \{0, a, a+1, \cdots, a+b-2\}$ is a total colouring of H. This means that there cannot be a conflict when one element has a non-zero β-value and the other has a value taken mod a.

Case 1. Any two adjacent edges are coloured properly.

Clearly, the edges in each copy of H are coloured properly. Now consider the colouring of the edges in each copy of G. For the copies of G corresponding to the vertex v_k where $\beta(v_k) = 0$, their edges are coloured properly by α. If there exist two incident edges $w_{ik}w_{jk}$ and $w_{jk}w_{sk}$ assigned the same colouring, say $\beta(v_k)$, in $G \times H$, then $\alpha(u_iu_j)+\beta(v_k)+1 \equiv \alpha(u_ju_s)+\beta(v_k)+1 \equiv 0 \mod a$ which implies that $\alpha(u_iu_j) = \alpha(u_ju_s)$, a contradiction as

α is an edge-colouring of G. Similarly, we can show that mod a $\gamma(w_{ik}w_{jk}) = (\alpha(u_iu_j) + \beta(v_k) + 1) \neq (\alpha(u_ju_s) + \beta(v_k) + 1) = \gamma(w_{jk}w_{sk})$. Hence, the edges in each copy of G are coloured properly. Finally, we have to check that two incident edges (one from G and one from H) received different colours under γ. Suppose to the contrary, that $\gamma(w_{sj}w_{st}) = \gamma(w_{it}w_{st})$ where $u_iu_s \in E(G)$ and $v_jv_t \in E(H)$. If $\beta(v_t) = 0$ then it follows that $\beta(v_jv_t) = \gamma(w_{sj}w_{st}) = \gamma(w_{it}w_{st}) = \alpha(u_iu_s)$, and this is only possible if $\beta(v_jv_t) = 0$. But then $\beta(v_t) = \beta(v_jv_t) = 0$, contradicting the fact that β is a total colouring of H. If $\beta(v_t) \neq 0$ then similarly $\beta(v_jv_t) = \gamma(w_{it}w_{st}) = \beta(v_t)$ if $\alpha(u_iu_j) + \beta(v_t) + 1 = 0$ mod a, contradicting β being a total colouring of H, or $\beta(v_jv_t) = \alpha(u_iu_j) + \beta(v_t) + 1$ mod a, which is impossible as $\beta(v_jv_t) \geq a$.

Case 2. Any two adjacent vertices are coloured properly.

Suppose $\gamma(w_{ik}) = \gamma(w_{jk})$. If $\beta(v_k) = 0$ then $\alpha(u_i) = \gamma(w_{ik}) = \gamma(w_jk) = \alpha(u_j)$, contradicting α being a total colouring of G.

If $\beta(v_k) \neq 0$ then $\gamma(w_{ik}) = \beta(v_k)$ or $\alpha(u_i) + \beta(v_k) + 1$ mod a and similarly $\gamma(w_{jk}) = \beta(v_k)$ or $\big(\alpha(u_j) + \beta(v_i) + 1\big)$ mod a. If $\beta(v_k) = \gamma(w_{ik}) = \gamma(w_{jk}) = \beta(v_k)$ then $(\alpha(u_i) + \beta(v_k) + 1) \equiv 0 \pmod{a} \equiv \big(\alpha(u_j) + \beta(v_k) + 1\big) \pmod{a}$, so $\alpha(u_i) = \alpha(u_j)$, contradicting α being a total colouring of G. If $\beta(v_k) = \gamma(w_{ik}) = \alpha(w_{jk}) = \big(\alpha(u_j) + \beta(v_k) + 1\big)$ mod a, we have an impossibility as $\beta(v_k) \geq a$. If $\big(\alpha(u_i) + \beta(v_k) + 1\big)$ mod $a = \gamma(w_{ik}) = \gamma(w_{jk}) = \big(\alpha(u_j) + \beta(v_k) + 1\big)$ mod a then $\alpha(u_i) = \alpha(u_j)$, contradicting the fact that α is a total colouring of G.

Now suppose that $\alpha(w_{ki}) = \gamma(w_{kj})$. Then $v_iv_j \in E(H)$, so it is not possible that $\beta(v_i) = \beta(v_j) = 0$. If $\beta(v_i) = 0$ then $\alpha(u_k) =$

$\gamma(w_{ki}) = \gamma(w_{kj}) = \beta(v_j)$ or $\big(\alpha(u_k)+\beta(v_j)+1\big)$ mod a. But it is impossible that $\alpha(u_k) = \beta(v_j)$ since $\beta(v_j) \geq a > \alpha(u_k)$. Therefore $\alpha(u_k) = \big(\alpha(u_k)+\beta(v_j)+1\big)$ mod a, so $\beta(v_j)+1 \equiv 0 \pmod{a}$. But this is impossible as we are assuming that $b \geq a$. A similar argument applies if $\beta(v_j) = 0$. If $\beta(v_i) \neq 0$ and $\beta(v_j) \neq 0$ then $\beta(v_i)$ or $\big(\alpha(u_k) + \beta(v_i) + 1\big)$ mod $a = \gamma(w_{ki}) = \gamma(w_{kj}) = \beta(v_j)$ or $\big(\alpha(u_k) + \beta(v_j) + 1\big)$ mod a. Since β is a total colouring of H, $\beta(v_i) \neq \beta(v_j)$. Since $\beta(v_i) \geq a \geq \big(\alpha(u_k)+\beta(v_j)+1\big)$ mod a, it is not possible for $\beta(v_i) = \big(\alpha(u_k)+\beta(v_j)+1\big)$ mod a, and similarly $\beta(v_j) \neq \big(\alpha(u_k) + \beta(v_j) + 1\big)$ mod a. If $\big(\alpha(u_k) + \beta(v_i) + 1\big)$ mod $a = \big(\alpha(u_k) + \beta(v_j) + 1\big)$ mod a, then $\beta(v_i) = \beta(v_j)$, which is an impossibility.

Case 3. Any incident vertex and edge are coloured properly.

Let w_{st} be any vertex of $G \times H$. Suppose first that $\gamma(w_{st}) = \gamma(w_{sj}w_{st})$ for some j. Then $v_jv_t \in E(H)$, and so $\beta(v_j), \beta(v_t)$ and $\beta(v_jv_t)$ are distinct and $\gamma(w_{sj}w_{st}) = \beta(v_jv_t)$. If $\beta(v_t) = 0$ then $\alpha(u_s) = \gamma(w_{st}) = \gamma(w_{sj}w_{st}) = \beta(v_jv_t)$, so $\alpha(u_s) = 0 = \beta(v_jv_t)$. But then $\alpha(u_s) = \beta(v_t) = 0$ so $\alpha(u_s) + \beta(v_t) + 1 \not\equiv 0$ (mod a) so $\gamma(w_{st}) = \big(\alpha(u_s) + \beta(v_t) + 1\big)$ mod $a = 1$, a contradiction against $\gamma(w_{st}) = 0$. If $\beta(v_t) \neq 0$, then $\beta(v_iv_t) = \gamma(w_{sj}w_{st}) = \gamma(w_{st}) = \beta(v_t)$ or $\big(\alpha(u_j) + \beta(v_t) + 1\big)$ mod a. Since β is a total colouring of H it is impossible that $\beta(v_jv_t) = \beta(v_t)$, so we have $\beta(v_jv_t) = \big(\alpha(u_j) + \beta(v_t) + 1\big)$ mod a, which is impossible as $\beta(v_jv_t) \geq a > \big(\alpha(u_j) + \beta(v_t) + 1\big)$ mod a.

Now suppose that $\gamma(w_{st}) = \gamma(w_{it}w_{st})$. Then $u_iu_s \in E(G)$. If $\beta(v_t) = 0$ then $\alpha(u_i) = \gamma(w_{ik}) = \gamma(w_{it}w_{st}) = \alpha(u_iu_s)$, which contradicts α being a total colouring of G. If $\beta(v_t) \neq 0$

then one of $\beta(v_t)$, $\big(\alpha(u_i) + \beta(v_t) + 1\big)$ mod a equals $\alpha(w_{ik}) = \gamma(w_{it}w_{st})$ which equals one of $\beta(v_t)$ and $\big(\alpha(u_tu_j) + \beta(v_t) + 1\big)$ mod a. If $\beta(v_t) = \gamma(w_{ik}) = \gamma(w_{it}w_{st})$, then $(\alpha(u_i)+\beta(v_t)+1) \equiv 0 \pmod a$ and $\alpha(u_iu_j) + \beta(v_k) + 1 \equiv 0 (\text{mod } a)$ so $\alpha(u_i) = \alpha(u_iu_j)$, a contradiction against α being a total colouring of G. Therefore we must have $\big(\alpha(u_i) + \beta(v_t) + 1\big)$ mod $a = \gamma(w_{ik}) = \gamma(w_{it}w_{st}) = \big(\alpha(u_tu_j) + \beta(v_t) + 1\big)$ mod a, and we get the same contradiction. □

From the above theorem and the fact that $\Delta(G \times H) = \Delta(G) + \Delta(H)$, we can obtain the following sufficient condition for a graph to be a type 1.

Corollary 2.5. *Let G and H be two graphs of type* 1, *then* $G \times H$ *is of type* 1.

3 Another sufficient condition for graphs to be type one

We will make use of the following well-known theorem (see [13])by Kostochka.

Theorem 3.1. $\chi_T(G) \leq \Delta(G) + 2$ *if* $\Delta(G) \leq 5$.

Theorem 3.2 gives a sufficient condition for graphs to be type 1. It also tells us a number of edges that can always be added to a tree in any way such that the maximum degree is not increased without producing a type 2 graph. This number cannot be increased; for example, P_4 is of type 1 and C_4 is of type 2.

Theorem 3.2. *If* $\varepsilon(G) \le v(G) + \frac{3}{2}\Delta(G) - 4$ *and* G *is connected, then* G *is of type* 1.

Proof. If $\Delta(G) = 2$, then $\varepsilon(G) \le v(G) - 1$ and G is a path. Clearly, $\chi_T(G) = \Delta(G) + 1$. We may assume from now that $\Delta(G) \ge 3$. Let G be a counterexample to Theorem 3.2 such that, for $\Delta(G)$ fixed, $v(G)$ is as small as possible, and, given that, $\varepsilon(G)$ is as small as possible. We first show the following claims on the structural properties of G.

Claim 3.3. *Let* uv *be any edge of* G. *If* $d(v) = 1$, *then* $d(u) = \Delta(G)$ *and* u *is the unique vertex with the maximum degree.*

Proof. Suppose either $d(u) < \Delta(G)$ or u' is another vertex with $d(u') = \Delta(G)$. Let $G' = G - v$. Then $\Delta(G') = \Delta(G)$, $\varepsilon(G') = \varepsilon(G) - 1$, $v(G') = v(G) - 1$ and $\varepsilon(G') = \varepsilon(G) - 1 \le v(G) + \frac{3}{2}\Delta(G) - 5 = v(G') + \frac{3}{2}\Delta(G') - 4$. Moreover, G' is connected. By the minimality of G, G' is totally $(\Delta(G') + 1)$-colourable. Since $d(v) = 1$ and $\Delta(G') = \Delta(G)$, it is easy to extend this total colouring with $\Delta(G) + 1$ colours from G' to G, a contradiction to the fact that G is not totally $(\Delta(G) + 1)$-colourable. □

Claim 3.4. *G contains at most two vertices of maximum degree.*

Proof. Suppose G has t $(t \ge 3)$ vertices of maximum degree. By Claim 3.3, $\delta(G) \ge 2$ and thus $2(v(G) + \frac{3}{2}\Delta(G) - 4) \ge 2\varepsilon(G) = \sum_{v \in V(G)} d(v) \ge t\Delta(G) + 2(v(G) - t)$. It follows that $t \le 3 - \frac{2}{\Delta(G)-2} < 3 \le t$, a contradiction. □

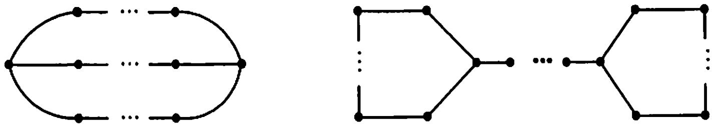

Figure 1

If $\Delta(G) = 3$ and $\delta(G) = 1$, then G has a vertex u such that $d(u) = 1$ and in turn $\Delta(G-u) = 2$ by Claim 3.3. By Theorem 3.1, $\chi_T(G-u) \le 4$ and any total colouring with ≤ 4 colours can be easily extended to G. If $\Delta(G) = 3$ and $\delta(G) = 2$, then G has at most two vertices of degree 3 by Claim 3.4. Thus if G has exactly two vertices of degree 3 then G is either a graph consisted of three paths starting at a vertex of degree 3 and terminating at another vertex of degree 3 or two cycles connected by a path (see Figure 1). It is not hard to see that $\chi_T(G) = 4$. Since every graph has an even number of vertices of odd degrees, we may assume $\Delta(G) \ge 4$.

Claim 3.5. *Let uv be any edge of G. If $d(v) \le \lfloor \frac{\Delta(G)}{2} \rfloor$ and $d(u) + d(v) \le \Delta(G) + 1$, then $d(u) = \Delta(G)$.*

Proof. Suppose, to the contrary, that $d(u) \ne \Delta(G)$. We first show that uv is a cut edge of G. Suppose $G - uv$ is connected. Let $G' = G - uv$. Then $\Delta(G') = \Delta(G)$, $v(G') = v(G)$ and $\varepsilon(G') = \varepsilon(G) - 1 \le (v(G) + \frac{3}{2}\Delta(G) - 4) - 1 = v(G') + \frac{3}{2}\Delta(G') - 5$. By the minimality of G, G' is totally $(\Delta(G') + 1)$-colourable. We erase the colour of v. Since $d(u) + d(v) \le \Delta(G) + 1$, there is at least one colour available to colour uv. Now we obtain a total colouring with $(\Delta(G) + 1)$ colours for all edges and all vertices except v of G. Since $d(v) \le \lfloor \frac{\Delta(G)}{2} \rfloor$, there is also a colour available to colour v. Thus G is totally $(\Delta(G)+1)$-colourable, a contradiction.

From the above we know that uv is a cut edge of G. Let $N_{G-u}(v) = \{v_1, v_2, ..., v_k\}$ and $G' = G - v + \{uv_1, uv_2, ..., uv_k\}$. Clearly, we have $d_{G'}(u) < \Delta(G)$, $\varepsilon(G') = \varepsilon(G) - 1$, $v(G') = v(G) - 1$, $\Delta(G') = \Delta(G)$ and $\varepsilon(G') = \varepsilon(G) - 1 \leq (v(G)-1) + \frac{3}{2}\Delta(G) - 4 = v(G') + \frac{3}{2}\Delta(G') - 4$. Also G' is connected. By the minimality of G, G' is totally $(\Delta(G')+1)$-colourable. Use the colour of uv_i to colour vv_i in G for $i = 1, ..., k$ and then colour uv and v as above. Now we are able to totally colour G with $\Delta(G)+1$ colours, a contradiction. □

Note that Claim 3.5 implies that for any edge uv, $d(u) + d(v) \geq \Delta(G) + 2$ if $d(u) \leq \frac{\Delta(G)}{2}$ or $d(v) \leq \frac{\Delta(G)}{2}$ and neither u nor v is a vertex of maximum degree. By Claim 3.4, there are at most two vertices of degree $\Delta(G)$. Now we turn to prove Theorem 3.2 in two cases when G has two, and when G has one vertex of degree $\Delta(G)$.

Case 1. G contains two vertices of maximum degree.

Let $d(x) = d(y) = \Delta(G)$. First, there is at most one vertex $z \in G - \{x, y\}$ such that $\lfloor \frac{\Delta(G)}{2} \rfloor + 1 \leq d(z) < \Delta(G)$. Suppose there are $k \geq 2$ vertices of degree at least $\lfloor \frac{\Delta(G)}{2} \rfloor + 1$. By Claim 3.3, $\delta(G) \geq 2$. These imply that $2\varepsilon(G) - 2(v(G) + \frac{3}{2}\Delta(G) - 4) \geq 2\Delta(G) + k(\lfloor \frac{\Delta(G)}{2} \rfloor + 1) + \delta(G)(v(G) - k - 2) - 2(v(G) + \frac{3}{2}\Delta(G) - 4) = k(\lfloor \frac{\Delta(G)}{2} \rfloor - 1) + (\delta(G) - 2)(v(G) - k - 2) - \Delta(G) + 4 \geq (\lfloor \frac{\Delta(G)}{2} \rfloor - 1)k - \Delta(G) + 4 \geq 2(\lfloor \frac{\Delta(G)}{2} \rfloor - 1) - \Delta(G) + 4 > 0$, a contradiction to the fact that $\varepsilon(G) \leq v(G) - \frac{3}{2}\Delta(G) - 4$.

Secondly, there exists a vertex $z \in V(G) - \{x, y\}$ such that $d(z) \geq \lfloor \frac{\Delta(G)}{2} \rfloor + 1 \geq 3$. If every vertex in $V(G) - \{x, y\}$ is of degree at most $\lfloor \frac{\Delta(G)}{2} \rfloor$, then $V(G) - \{x, y\}$ is an independent set of vertices in G by Claim 3.5. Thus, $d(w) = 2$ for $w \in G - \{x, y\}$. Clearly, G is totally $(\Delta(G) + 1)$-colourable, a contradiction. Thus z exists. Since $d(z) \geq 3$, there exists a vertex $z' \in N(z) - \{x, y\}$. As we showed above that z is the only vertex in $V(G) - \{x, y\}$ of degree $\geq \lfloor \frac{\Delta(G)}{2} \rfloor + 1$, it follows that $d(z') \leq \lfloor \frac{\Delta(G)}{2} \rfloor$. By Claim 3.5, $d(z) + d(z') \geq \Delta(G) + 2$. It follows that $\varepsilon(G) = \sum_{v \in V(G)} d(v) \geq 2\Delta(G) + (\Delta(G) + 2) + 2(v(G) - 4) = 2(v(G) + \frac{3}{2}\Delta(G) - 4) + 2 > 2(v(G) + \frac{3}{2}\Delta(G) - 4)$, a contradiction. This proves Case 1.

Case 2. G contains exactly one vertex of degree $\Delta(G)$.

Let $d(u) = \Delta(G)$ and $\{u_1, u_2, ... u_k\}$ be the vertices of degree 1 in the neighborhood of u. By Claim 3.5, there is no vertex of degree 2 in G otherwise there are two vertices of maximum degree in G. If $V(G) = \{u, u_1, ..., u_k\}$ then the theorem is clearly true, so we can assume that $V(G) \neq \{u, u_1, ..., u_k\}$, so that $v(G) \geq k + 2$ and $k \neq \Delta(G)$. Let $\alpha = \min\{d(x) \mid x \in V(G) - \{u, u_1, u_2, ..., u_k\}\}$. Then $\alpha \geq 3$.

Suppose $k = 0$. Since G has only one vertex of maximum degree and $\alpha \geq 3$, $2(v(G) + \frac{3}{2}\Delta(G) - 4) - (\Delta(G) + 3(v(G) - 1)) = 2\Delta(G) - v(G) - 5 \geq 0$ so $\Delta(G) \geq 6$. We must also have $\delta(G) < 4$, otherwise $2\varepsilon(G) - 2(v(G) + \frac{3}{2}\Delta(G) - 4) \geq \Delta(G) + \delta(G)(v(G) - 1) - 2(v(G) + \frac{3}{2}\Delta(G) - 4) \geq 2(v(G) - \Delta(G) + 2) > 0$, a contradiction. Thus $\delta(G) = 3$. Let v be a vertex of degree 3. By Claim 3.5, v has at least two neighbors of degree $\Delta(G) - 1$.

It follows that $2\varepsilon(G)-2(v(G)+\frac{3}{2}\Delta(G)-4) \geq \Delta(G)+2(\Delta(G)-1)+3(v(G)-3)-2(v(G)+\frac{3}{2}\Delta(G)-4) > 0$, a contradiction. Hence, $k \geq 1$.

Recall that u_1 is a vertex of degree 1. By Theorem 3.1, $G-u_1$ is totally $(\Delta(G-u_1)+2)$-colourable when $\Delta(G-u_1) \leq 5$. It follows that G is totally $(\Delta(G)+1)$-colourable for $\Delta(G) \leq 6$. Hence, we may assume that $\Delta(G) \geq 7$.

Subcase 2.1. $\alpha = 3$.

Let w be a vertex of degree 3. There are at least two vertices in $N(w)$, say w_1 and w_2, of degree $\Delta(G)-1$ by Claim 3.5. Since $d(w_1) = \Delta(G)-1$, $v(G) \geq \Delta(G)+k$. Suppose for the moment that all the vertices in $V(G)-\{u, w_1, w_2\}$ are of degree less than $\lfloor\frac{\Delta(G)}{2}\rfloor+1$, then by Claim 3.5, they are independent. It follows that the vertices of G have only four possible degrees, namely $1, 3, \Delta(G)-1, \Delta(G)$. This implies that $2(v(G)+\frac{3}{2}\Delta(G)-4) \geq 2\varepsilon(G) \geq k+\Delta(G)+2(\Delta(G)-1)+3(v(G)-k-3)$, so $2k+3 \geq v(G) \geq \Delta(G)+k$, and so $k+3 \geq \Delta(G) \geq 7$, so $k \geq 4$. Now we can totally colour G in $\Delta(G)+1$ colours. First, colour u, w_1, w_2 and the possible edges between them. Secondly, we colour the edges zw_i. For each such vertex z of degree 3, there are at least two colours available for zw_1 since $d(w_1) \leq \Delta(G)-1$ and one colour available for zw_2 since $d(w_2) \leq \Delta(G)-1$. There are at least four colours available for zu since $d_{G-\{u_1,u_2,\ldots,u_k\}}(u) \leq \Delta(G)-4$. Edges joining two vertices of degree three can now be coloured since $\Delta(G) \geq 7$. After all the edges incident with the vertices of degree 3 are coloured, we can colour the vertices of degree 3 since $\Delta(G) \geq 7$.

Then we can colour the edges incident to the vertices of degree 1 and the vertices of degree 1. This contradicts the fact that G is a counterexample and implies that G must contain a vertex $x \in V(G) - \{u, w_1, w_2\}$ such that $d(x) \geq \lfloor \frac{\Delta(G)}{2} \rfloor + 1$. Since $\Delta(G) \geq 7$, $d(x) \geq \lfloor \frac{\Delta(G)}{2} \rfloor + 1 \geq 4$ and in turn there exists another vertex $y \in N(X)$ such that $y \in V(G) - \{u, w_1, w_2\}$. By Claim 3.5, $y \neq w$. Applying claim 3.5 to the edge xy, it follows that either (a) $d(y) \leq \lfloor \frac{\Delta(G)}{2} \rfloor$ and $d(x) + d(y) \geq \Delta(G) + 2$, or (b) $d(y) \geq \lfloor \frac{\Delta(G)}{2} \rfloor + 1$ and $d(x) + d(y) \leq \Delta(G) + 1$. Recall that $\Delta(G) \geq k + 1$ and $\Delta(G) \geq 7$. In case (a) we have $2\varepsilon(G) \geq k + \Delta(G) + 2(\Delta(G) - 1) + (\Delta(G) + 2) + 3(v(G) - k - 5) = 2(v(G) + \frac{3}{2}\Delta(G) - 4) + (v(G) + \Delta(G) - 2k - 7) \geq 2(v(G) + \frac{3}{2}\Delta(G) - 4) + (\Delta(G) + k + \Delta(G) - 2k - 7) > 2(v(G) + \frac{3}{2}\Delta(G) - 4)$, a contradiction. In case (b), it follows that $\Delta(G)$ is odd and $d(x) + d(y) = \Delta(G) + 1$. The the condition $\varepsilon(G) \leq v(G) + \frac{3}{2}\Delta(G) - 4$ can be reexpressed in the form of $\varepsilon(G) \leq v(G) + \frac{3}{2}\Delta(G) - \frac{9}{2}$. Similarly we have $2\varepsilon(G) \geq k + \Delta(G) + 2(\Delta(G) - 1) + (\Delta(G) + 1) + 3(v(G) - k - 5) = 2(v(G) + \frac{3}{2}\Delta(G) - \frac{9}{2}) + (v(G) + \Delta(G) - 2k - 7) \geq 2(v(G) + \frac{3}{2}\Delta(G) - \frac{9}{2}) + (\Delta(G) + k + \Delta(G) - 2k - 7) > 2(v(G) + \frac{3}{2}\Delta(G) - \frac{9}{2})$, a contradiction.

Subcase 2.2. $\alpha \geq 4$.

If $\Delta(G) = 7$, then $\varepsilon(G) \leq v(G) + \frac{3}{2}7 - 4 = v(G) + 6.5$ and $v(G) \geq k + \alpha + 1$. Since $\varepsilon(G)$ is an integer, we have $\varepsilon(G) \leq v(G) + 6$. It follows that $2(v(G) + 6) \geq 2\varepsilon(G) \geq k + \Delta(G) + \alpha(v(G) - k - 1)$. Therefore, when $\alpha \geq 5$, $0 \leq 2(v(G) + 6) - (k + \Delta(G) + \alpha(v(G) - k - 1)) = (\alpha - 1)k + (5 + \alpha) - (\alpha - 2)v(G) \leq (\alpha - 1)k + (5 + \alpha) - (\alpha - 2)(k + \alpha + 1) = k + 7 - \alpha(\alpha - 2) \leq k - 8$. But clearly, $k-8<0$ since $\Delta(G)=7$, a contradiction. If $\Delta(G)=7$ and $\alpha=4$, $2(v(G)+6)\geq 2\varepsilon(G)\geq k+7+4(v(G)-k-1)$. That is, $k\geq\lceil\frac{2}{3}v(G)\rceil-3\geq\lceil\frac{2}{3}(\Delta(G)+1)\rceil-3=\lceil\frac{2}{3}\times 8\rceil-3=3$. If G contains no vertex of degree 6 then we can delete two vertices of degree 1 and the result graph G' has maximum degree 5. By Theorem 3.1, it is totally 7-colourable and this total colouring can be extended to a total colouring of G, a contradiction. Thus, G contains a vertex of degree 6 and $v(G)\geq k+7$. It follows that $2\varepsilon(G)\geq k+7+6+4(v(G)-k-2)=2(v(G)+6)+(2v(G)-3k-7)$. Note that $2v(G)-3k-7\geq 2(k+7)-3k-7=7-k>0$ since $\Delta(G)=7$. It follows that $2\varepsilon(G)\geq 2(v(G)+6)+(2v(G)-3k-7)>2(v(G)+6)\geq 2\varepsilon(G)$, a contradiction. Hence, we may assume that $\Delta(G)\geq 8$ in the following.

First we show that $G-\{u,u_1u_2,...,u_k\}$ contains three independent edges. Since $\alpha=\min\{d(x)\mid x\in V(G)-\{u,u_1,u_2,...,u_k\}\}\geq 4$, we can easily find that there are two independent edges $\{x_1y_1,x_2y_2\}$ in $G-\{u,u_1u_2,...,u_k\}$. By Theorem 3.1, $G'=G-\{u_1,u_2...,u_k\}$ is totally 7-colourable if $\Delta(G-\{u_1,u_2...,u_k\})\leq 5$ and it follows that G can be totally coloured with $\Delta(G)+1$ colours where $\Delta(G)\geq 8$. Hence, we may assume that $\Delta(G-\{u_1,u_2,...,u_k\})\geq 6$. This implies that there exist two vertices z_1 and z_2 in $V(G)-\{x_1,x_2,y_1,y_2,u,u_1,u_2,...,u_k\}$. If $z_1z_2\in E(G)$ or if $\{u,x_1,x_2,y_1,y_2\}\not\subseteq N(z_1)\bigcup N(z_2)$, we have the three independent edges as desired. So we assume $z_1z_2\notin E(G)$ and $N(z_1)\cup N(z_2)\subseteq\{u,x_1,x_2,y_1,y_2\}$. Since $d(z_1)\geq 4$ and $d(z_2)\geq 4$, we may assume that $\{x_1,y_1\}\subseteq N(z_1)$ and $y_1\in N(z_2)$. Now we obtain three independent edges $\{x_1z_1,x_2y_2,y_1z_2\}$ in $G-\{u,u_1,u_2,...,u_k\}$. Let $\{x_1y_1,x_2y_2,x_3y_3\}$

be the three independent edges. By Claim 3.5, $d(x_i) + d(y_i) \geq \Delta(G)+2$ for $i = 1, 2, 3$, which implies $2\varepsilon(G) \geq (\Delta(G)+3(\Delta(G)+2) + \alpha(v(G) - k - 7) + k) \geq 4\Delta(G) + k + 6 + 4(v(G) - k - 7) = 4v(G) + 4\Delta(G) - 3k - 22$. Since $\varepsilon(G) \leq v(G) + \frac{3}{2}\Delta(G) - 4$, we have the following inequality:

$$3k \geq 2v(G) + \Delta(G) - 14 \qquad (*).$$

From $(*)$ along with $v(G) \geq \Delta(G) + 1$ we obtain $3k \geq 2(\Delta(G)+1)+\Delta(G)-14 = 3(\Delta(G)-4)$, i.e. $k \geq \Delta(G)-4$. This implies that there are at least 2 vertices from $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ that are not in $N(u)$ so that $v(G) \geq \Delta(G) + 3$. Again from $(*)$ along with $v(G) \geq \Delta(G) + 3$, we have $3k \geq 2(\Delta(G) + 3) + \Delta(G) - 14 = 3(\Delta(G) - 3) + 1$ so that $k \geq \Delta(G) - 2$. Thus, $v(G) \geq \Delta(G) + 5$. Similarly, we can prove that $k \geq \Delta(G) - 1$, which then implies that $k = \Delta(G) - 1$ since $k \neq \Delta(G)$, and $v(G) \geq \Delta(G) + 6$. Finally, by $(*)$, we have that $3k \geq 2(\Delta(G) + 6) + \Delta(G) - 14 = 3(\Delta(G) - 1) + 1 \geq 3k + 1$, a contradiction. This completes the proof of the theorem. □

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