Some Formulas for Generalized Stirling Numbers

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Abstract

In this paper we look at generalizations of Stirling numbers which arise for abitrary integer sequences and their $k-$th powers. This can be seen as a complementary strategy to the unified approach suggested in [9]. The investigations of [3] and [14] present a more algebraically oriented approach to generalized Stirling numbers.
In the first and second section of the paper we give the corresponding formulas for the generalized Stirling numbers of the second and and the first kind respectively. In the third section we briefly discuss some examples and special cases, and in the last section we apply the square case to facilitate a counting approach for set partitions of even size.

1 Some Identities for Generalized Stirling Numbers of the Second Kind

Assume that $a = (a_0, a_1, ..., a_n, ...)$ is a sequence of integers and that $d_{n,r}^{(a)}$ are the associated Stirling numbers of the second kind. As an example we let $a = (0, 1, 2, 3, ...)$ be the natural sequence which gives the ordinary Stirling numbers of the second kind.
The Stirling numbers of the second kind with respect to the sequence a denoted by $d_{n,r}^{(a)} = d_{n,r}^{(a,1)}$ are defined by the equation:

$$x^n = \sum_{r=0}^{n} d_{n,r}^{(a)}(x - a_0)(x - a_1)\dots(x - a_{r-1}), \tag{1}$$

with the abbreviating notation that

$$[x]_r^{(a,1)} = (x - a_0)(x - a_1)\dots(x - a_{r-1})$$

and the convention that the empty product $[x]_0^{(a,1)} = 1$ is equal to one. Clearly $d_{n,n}^{(a)} = 1$ for $n \geq 0$. Also, $d_{n,r}^{(a)} = 0$ for $r > n \geq 0$ or $n, r < 0$. Using the generalization of this notation with the expressions

$$[x]_r^{(a,k)} = (x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_{r-1}^k) \tag{2}$$

and the convention that the empty product $[x]_0^{(a,k)} = 1$ is equal to one. Generalizing the definition of the $d_{n,r}^{(a)}$, for any integer $k \geq 1$ we define the Stirling numbers of the second kind associated to the k^{th} power $d_{n,r}^{(a,k)}$ with respect to the sequence a by

$$x^{nk} = \sum_{r=0}^{n} d_{n,r}^{(a,k)}(x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_{r-1}^k) \tag{3}$$

for $n \geq r \geq 0$ and $d_{n,r}^{(a,k)} = 0$ for $r > n \geq 0$ or $n, r < 0$. Note that $d_{n,n}^{(a,k)} = 1$ for $n \geq 0$, and $d_{n,0}^{(a,k)} = a_0^{kn}$.

It is known that the $d_{n,r}^{(a,1)}$ satisfy a number of recurrence relations. Listed below is the corresponding recurrence relation between the $d_{n,r}^{(a,k)}$ which contains $k = 1$ as a special case. The next proposition gives the most basic relation.

Proposition 1 *For all $n, k \geq 1$ and $r \geq 0$*

$$d_{n,r}^{(a,k)} = d_{n-1,r-1}^{(a,k)} + a_r^k d_{n-1,r}^{(a,k)}.$$

Proof. By the definition of the $d_{n,r}^{(a,k)}$ note that

$$\sum_{i=0}^{n} d_{n,i}^{(a,k)}(x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_{i-1}^k) = x^{kn} = x^k x^{(n-1)k}$$

$$= \sum_{i=0}^{n-1} d_{n-1,i}^{(a,k)} x^k (x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_{i-1}^k)$$

$$= \sum_{i=0}^{n-1} d_{n-1,i}^{(a,k)}(x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_{i-1}^k)(x^k - a_i^k + a_i^k)$$

$$= \sum_{i=0}^{n-1} d_{n-1,i}^{(a,k)}(x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_i^k)$$

$$+\sum_{i=0}^{n-1} a_i^k d_{n-1,i}^{(a,k)}(x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_{i-1}^k)$$

$$=\sum_{i=1}^{n} d_{n-1,i-1}^{(a,k)}(x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_{i-1}^k)$$

$$+\sum_{i=0}^{n-1} a_i^k d_{n-1,i}^{(a,k)}(x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_{i-1}^k)$$

and the result follows.

q.e.d.

In the remainder of this section we will abbreviate $d_{n,r}^{(a,1)}$ as $d_{n,r}^{(a)}$. This is no loss of generality since we may replace any sequence $(a_0, a_1, a_2, a_3, \dots)$ by the sequence of its k-th powers $(a_0^k, a_1^k, a_2^k, a_3^k, \dots)$. We present the generalization of a formula for the $d_{n,r}^{(a)}$ which has several applications.

Theorem 1 *For all $n \geq r \geq 1$ and $k \geq 1$*

$$\begin{aligned} d_{n,r}^{(a)} &= \sum_{j=0}^{r} \frac{(-1)^j (a_{r-j})^n}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i) \cdot \prod_{m=r-j+1}^{r}(a_m - a_{r-j})} \\ &= \sum_{j=0}^{r} \frac{a_{r-j}^n}{\prod^r_{\substack{i=0\\ i\neq r-j}} (a_{r-j} - a_i)} \quad . \end{aligned}$$

In the following we present one proof of this formula by direct calculation, and one proof invoking the theory of symmetric functions, and then we go on drawing some consequences of the formula involving partial fraction expansion of the generating series.

Proof. For all $n \geq r \geq 1$ and $k \geq 1$, let

$$S(n, r; a) = \sum_{j=0}^{r} \frac{(-1)^j a_{r-j}^n}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i) \prod_{m=r-j+1}^{r}(a_m - a_{r-j})} \quad .$$

Then it follows from the definitions that

$$S(n, 1; a) = a_1^{n-1} = d_{n,1}^{(a)} \quad \text{for all } n \geq 1. \tag{4}$$

For $n \geq r \geq 2$

$$S(n, r; a) = \sum_{j=0}^{r} \frac{(-1)^j a_{r-j}^n}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i) \prod_{m=r-j+1}^{r}(a_m - a_{r-j})}$$

$$= \frac{a_r^n}{(a_r - a_0)(a_r - a_1)\dots(a_r - a_{r-1})}$$

$$+ \sum_{j=1}^{r} \frac{(-1)^j a_{r-j}^n}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i)\prod_{m=r-j+1}^{r}(a_m - a_{r-j})}$$

$$= \frac{a_r^n}{(a_r - a_0)(a_r - a_1)\dots(a_r - a_{r-1})}$$

$$+ \sum_{j=1}^{r} \frac{(-1)^j a_{r-j}^{n-1} a_{r-j}}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i)\prod_{m=r-j+1}^{r}(a_m - a_{r-j})}$$

$$= \frac{a_r^n}{(a_r - a_0)(a_r - a_1)\dots(a_r - a_{r-1})}$$

$$+ \sum_{j=1}^{r} \frac{(-1)^j a_{r-j}^{n-1}(a_r + a_{r-j} - a_r)}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i)\prod_{m=r-j+1}^{r}(a_m - a_{r-j})}$$

$$= \frac{a_r^n}{(a_r - a_0)(a_r - a_1)\dots(a_r - a_{r-1})}$$

$$+ a_r \sum_{j=1}^{r} \frac{(-1)^j a_{r-j}^{n-1}}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i)\prod_{m=r-j+1}^{r}(a_m - a_{r-j})}$$

$$+ \sum_{j=1}^{r} \frac{(-1)^j a_{r-j}^{n-1}(a_{r-j} - a_r)}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i)\prod_{m=r-j+1}^{r}(a_m - a_{r-j})}$$

$$= a_r \sum_{j=0}^{r} \frac{(-1)^j a_{r-j}^{n-1}}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i)\prod_{m=r-j+1}^{r}(a_m - a_{r-j})}$$

$$+ \sum_{j=1}^{r} \frac{(-1)^{j-1} a_{r-j}^{n-1}}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i)\prod_{m=r-j+1}^{r-1}(a_m - a_{r-j})}$$

$$= a_r \sum_{j=0}^{r} \frac{(-1)^j a_{r-j}^{n-1}}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i)\prod_{m=r-j+1}^{r}(a_m - a_{r-j})}$$

$$+ \sum_{j=0}^{r-1} \frac{(-1)^j a_{r-1-j}^{n-1}}{\prod_{i=0}^{r-j-2}(a_{r-1-j} - a_i)\prod_{m=r-j}^{r-1}(a_m - a_{r-1-j})}$$

$$= a_r S(n-1, r; a) + S(n-1, r-1; a).$$

Since $S(n,r;a)$ satisfies the usual triangular recurrence relation, and the boundary conditions (4), this implies by known results (see e.g. [6], [3]) that $S(n,r;a) = d_{n,r}^{(a)}$ for all $n \geq r \geq 1$ and any sequence

$$a = (a_0, a_1, \ldots, a_k, \ldots).$$

q.e.d.

In terms of generating series we use the expansion

$$\prod_{i=0}^{n-1}(1 - a_i x)^{-1} = \sum_{r\geq 0} h_{n+r,n} x^r \tag{5}$$

of the complete symmetric functions $h_{n+r,n}$ in the variables $a_0, a_1, a_2, \ldots, a_{n-1}$. This is combined with the partial fraction expansion

$$\prod_{i=0}^{n-1}(1 - a_i u)^{-1} = \sum_{i=0}^{n-1} \frac{\beta_i}{1 - a_i u} \tag{6}$$

Expanding this and comparing the powers of u we get a system of linear equations in the quantities β_i.

Theorem 2 *The $d_{n,r}^{(a)}$ have the rational generating function*

$$\varphi_r^{(a)} = \sum_{n\geq r} d_{n,r}^{(a)} u^n = \frac{u^r}{(1 - a_0 u)(1 - a_1 u)\ldots(1 - a_r u)}.$$

Proof. This is a direct consequence of an identity for the expression

$$\varphi_r^{(a)} = \frac{u^r}{(1 - a_0 u)(1 - a_1 u)\ldots(1 - a_{r-1} u)}$$

By expanding $\varphi_r^{(a)}$ into partial fractions we see that it is sufficient to determine the quantities p_j, for $j = 0, 1, 2, \ldots, r$ in the following equation:

$$u^r = p_0 \prod_{i=1}^{r} (1 - a_i u) u^r + \ldots + p_j \prod_{i=0, i\neq j}^{r} (1 - a_i u) u^r + \ldots + p_r \prod_{i=0}^{r-1} (1 - a_i u) u^r.$$

We find

$$p_j = \frac{1}{\prod_{i=0,i\neq j}^{r} (1 - \frac{a_i}{a_j})}.$$

This is easily verified by letting $u = \frac{1}{a_j}$ with $j = 0, 1, 2, \ldots, r$ in turn. Hence we obtain that

$$\varphi_r^{(a)} = \sum_{j=0}^{r} \frac{(-1)^j a_{r-j}^r u^r}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i)\prod_{m=r-j+1}^{r}(a_m - a_{r-j})(1 - a_{r-j}u)}$$

$$= \sum_{j=0}^{r} \frac{(-1)^j a_{r-j}^r u^r}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i)\prod_{m=r-j+1}^{r}(a_m - a_{r-j})} \sum_{n\geq r} a_{r-j}^{n-r} u^{n-r}$$

$$= \sum_{n\geq r}\{\sum_{j=0}^{r} \frac{(-1)^j a_{r-j}^n}{\prod_{i=0}^{r-j-1}(a_{r-j} - a_i)\prod_{m=r-j+1}^{r}(a_m - a_{r-j})}\} u^n$$

$$= \sum_{n\geq r} d_{n,r}^{(a)} u^n \quad \text{(by Theorem 1).}$$

q.e.d.

In the second part of [11] in $ 8 , formulas 1) and 2) two expressions are given which can be interpreted as being the right and the left hand side of the formula in theorem 2. The author of [11] then proceeds with the equivalent of a statement of theorem 2 and its proof in the special case $(a_0 = 1, a_1 = 2, a_3 = 2, \ldots)$. We have not been able to find in [11] or in [8] a formula that amounts to theorem 2 in the case of any general sequence $(a_0, a_1, a_2, \ldots)$, or any other sequence except $(1, 2, 3, \ldots)$.

We now give one explicit representation and one recursion formula for the numbers $d_{n,r}^{(a)}$.

Proposition 2 *For all $n \geq r \geq 0$ and any sequence a we have*

$$d_{n,r}^{(a)} = \sum_{k_0+k_1+\ldots+k_r=n-r} a_r^{k_r} \cdot a_{r-1}^{k_{r-1}} \cdot \ldots \cdot a_1^{k_1} \cdot a_0^{k_0}. \qquad (7)$$

Proof of (7).
By theorem 2,

$$\varphi_r^{(a)} = \frac{u^r}{(1 - a_0 u)(1 - a_1 u)\ldots(1 - a_r u)} = \sum_{n\geq r} d_{n,r}^{(a)} u^n.$$

$$\Rightarrow \frac{\varphi_r^{(a)}}{u^r} = \frac{1}{(1 - a_0 u)(1 - a_1 u)\ldots(1 - a_r u)} = \sum_{n\geq r} d_{n,r}^{(a)} u^{n-r}$$

$$= (1-a_0u)^{-1}(1-a_1u)^{-1}\dots(1-a_ru)^{-1}$$
$$= (\sum_{c_0\geq 0}(a_0u)^{c_0})(\sum_{c_1\geq 0}(a_1u)^{c_1})\dots(\sum_{c_r\geq 0}(a_ru)^{c_r})$$
$$= \sum_{c_0,\dots,c_r\geq 0}(a_0^{c_0}a_1^{c_1}\dots a_r^{c_r})u^{c_0+c_1+\dots+c_r}$$
$$= \sum_{n\geq r}\sum_{\substack{c_0+c_1+\dots+c_r=n-r\\ c_0,c_1,\dots,c_r\geq 0}}(a_0^{c_0}a_1^{c_1}\dots a_r^{c_r})u^{n-r} \tag{8}$$

and the result follows by equating coefficients.
We remark that in the case $a_0 = 0$, which happens for instance in the case of the usual Stirling numbers, the inner sum in (8) can be rewritten as

$$\sum_{n\geq r}\sum_{\substack{c_1+\dots+c_r=n-r\\ c_1,\dots,c_r\geq 0}}(a_1^{c_1}\dots a_r^{c_r})u^{n-r}$$

Proposition 3 *For all $n \geq r \geq 1$ and any sequence a we have*

$$d_{n,r}^{(a)} = \sum_{l=r}^{n} d_{l-1,r-1}^{(a)}\cdot a_r^{n-1}. \tag{9}$$

Proof of (9).

$$\varphi_r^{(a)} = \sum_{n\geq r} d_{n,r}^{(a)}u^n = \frac{u^r}{(1-a_0u)(1-a_1u)\dots(1-a_ru)}$$
$$= (\frac{u^{r-1}}{(1-a_0u)(1-a_1u)\dots(1-a_{r-1}u)})\frac{u}{1-a_ru}$$
$$= \frac{u}{1-a_ru}\varphi_{r-1}^{(a)} = u(1-a_ru)^{-1}\varphi_{r-1}^{(a)} = u\sum_{m\geq 0}(a_ru)^m\varphi_{r-1}^{(a)}$$
$$= u\sum_{m\geq 0}(a_ru)^m\sum_{l\geq r-1}d_{l,r-1}^{(a)}u^l = u\sum_{m\geq 0}(a_ru)^m\sum_{l\geq r}d_{l-1,r-1}^{(a)}u^{l-1}$$
$$= \sum_{m\geq 0}(a_ru)^m\sum_{l\geq r}d_{l-1,r-1}^{(a)}u^l = \sum_{m\geq 0}\sum_{l\geq r}d_{l-1,r-1}^{(a)}a_r^m u^{l+m}$$
$$= \sum_{n\geq l}\sum_{l\geq r}d_{l-1,r-1}^{(a)}a_r^{(n-l)}u^n = \sum_{n\geq r}\sum_{l=r}^{n}d_{l-1,r-1}^{(a)}a_r^{(n-l)}u^n.$$

By equating coefficients,

$$d_{n,r}^{(a)} = \sum_{l=r}^{n}d_{l-1,r-1}^{(a)}a_r^{n-l}.$$

q.e.d.

2 Some Identities for Generalized Stirling Numbers of the First Kind

Assume that $a = (a_0, a_1, \ldots, a_i, \ldots)$ is a sequence of integers and that $c_{n,r}^{(a)} = c_{n,r}^{(a,1)}$ are the associated Stirling numbers of the first kind. We also let $c_{n,r}^{(a,k)}$ be the associated Stirling numbers of the first kind corresponding to the sequence $(a_0^k, a_1^k, \ldots, a_i^k, \ldots)$.

More precisely, as noted previously, we make the convention that the empty product is 1, i.e. $c_{0,0}^{(a)} = \grave{c}_{0,0}^{(a)} = 1$, and the Stirling numbers of the first kind $c_{n,r}^{(a)}$ with respect to the sequence a are defined by the equation

$$(x - a_0)(x - a_1)\ldots(x - a_{n-1}) = \sum_{r=0}^{n} c_{n,r}^{(a)} x^r. \tag{10}$$

The *reversely signed* Stirling numbers of the first kind $\grave{c}_{n,r}^{(a)}$ are defined by

$$(x + a_0)(x + a_1)\ldots(x + a_{n-1}) = \sum_{r=0}^{n} \grave{c}_{n,r}^{(a)} x^r.$$

Clearly $c_{n,n}^{(a)} = \grave{c}_{n,n}^{(a)} = 1$ for $n \geq 0$ and $c_{n,0}^{(a)} = (-1)^n \cdot a_0 a_1 \ldots a_{n-1}$ and $\grave{c}_{n,0}^{(a)} = a_0 a_1 \ldots a_{n-1}$ for $n \geq 1$. Also $c_{n,r} = \grave{c}_{n,r}^{(a)} = 0$ for $r > n \geq 0$ or $n, r < 0$.

For any integer $k \geq 1$, similarly as above we make the convention that the empty product is 1, i.e. $c_{0,0}^{(a,k)} = \grave{c}_{0,0}^{(a,k)} = 1$, and we define the Stirling numbers of the first kind associated to the k^{th} power $c_{n,r}^{(a,k)}$ with respect to the sequence a by

$$(x^k - a_0^k)(x^k - a_1^k)\ldots(x^k - a_{n-1}^k) = \sum_{r=0}^{n} c_{n,r}^{(a,k)} x^{rk} \tag{11}$$

and the reversely signed Stirling numbers of the first kind associated to the k^{th} power $\grave{c}_{n,r}^{(a,k)}$ by

$$(x^k + a_0^k)(x^k + a_1^k)\ldots(x^k + a_{n-1}^k) = \sum_{r=0}^{n} \grave{c}_{n,r}^{(a,k)} x^{rk}$$

for $n \geq r \geq 0$ and $c_{n,r}^{(a,k)} = \grave{c}_{n,r}^{(a,k)} = 0$ for $r > n \geq 0$ or $n, r < 0$. Note that $c_{n,n}^{(a,k)} = \grave{c}_{n,n}^{(a,k)} = 1$ for $n \geq 0$ and $c_{n,0}^{(a,k)} = (-1)^n a_0 a_1 \ldots a_{n-1}$ as well as $\grave{c}_{n,0}^{(a,k)} = a_0 a_1 \ldots a_{n-1}$. for $n \geq 1$.

It is known that the $c_{n,r}^{(a,1)}$ satisfy a number of recurrence relations. Listed below is a recurrence relation between the $c_{n,r}^{(a,k)}$ which contains $k=1$ as a special case. The next proposition gives this most fundamental relation.

Proposition 4 *For all $n,k \geq 1$ and $r \geq 1$*

$$c_{n,r}^{(a,k)} = c_{n-1,r-1}^{(a,k)} - a_{n-1}^{k} c_{n-1,r}^{(a,k)}.$$

Proof. From the definition of the $c_{n,r}^{(a,k)}$ note that

$$\begin{aligned}
\sum_{i=0}^{n} c_{n,i}^{(a,k)} x^{ik} &= (x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_{n-1}^k) \\
&= (x^k - a_{n-1}^k) \cdot \sum_{i=0}^{n-1} c_{n-1,i}^{(a,k)} x^{ik} \\
&= \sum_{i=0}^{n-1} c_{n-1,i}^{(a,k)} x^{(i+1)k} - a_{n-1}^k \sum_{i=0}^{n-1} c_{n-1,i}^{(a,k)} x^{ik} \\
&= \sum_{i=1}^{n} c_{n-1,i-1}^{(a,k)} x^{ik} - a_{n-1}^k \sum_{i=0}^{n-1} c_{n-1,i}^{(a,k)} x^{ik}
\end{aligned}$$

and the result follows.

q.e.d.

By the same method we also get

$$\grave{c}_{n,r}^{(a,k)} = \grave{c}_{n-1,r-1}^{(a,k)} + a_{n-1}^{k} \grave{c}_{n-1,r}^{(a,k)}$$

for $n,k \geq 1$ and $r \geq 1$.

We express the product of polynomials

$$\prod_{i=0}^{n-1} (x^{k-1} + a_i x^{k-2} + a_i^2 x^{k-3} + \dots + a_i{}^{k-2} x + a_i{}^{k-1})$$

as the sum

$$\sum_{\nu=0}^{n(k-1)} \gamma_{\nu,n-1}^{(a,k-1)} x^{\nu}$$

where $\gamma_{\nu,n-1}^{(a,k-1)} = 0$ if $\nu < 0$ or $\nu > n(k-1)$, and where the numbers $\gamma_{\nu,n-1}^{(a,k-1)}$ have the explicit form

$$\gamma_{\nu,n-1}^{(a,k-1)} = \sum_{\substack{i_0+i_1+\ldots+i_{n-1}=n(k-1)-\nu \\ 0\le i_0,i_1,\ldots,i_{n-1}\le k-1}} a_0^{i_0} a_1^{i_1} \ldots a_{n-1}^{i_{n-1}}.$$

Thus the following equation is valid:

$$\prod_{i=0}^{n-1}(x^{k-1} + a_i x^{k-2} + a_i^2 x^{k-3} + \ldots + a_i{}^{k-2} x + a_i{}^{k-1}) = \sum_{\nu=0}^{n(k-1)} \gamma_{\nu,n-1}^{(a,k-1)} x^{\nu}.$$

The coefficients $\gamma_{\nu,n-1}^{(a,k-1)}$ defined this way satisfy the following recursive relation.

Proposition 5 $\gamma_{\nu,n-1}^{(a,k-1)}$ *has the k-fold recursive relation*

$$\gamma_{\nu,n-1}^{(a,k-1)} = a_{n-1}^0 \gamma_{\nu-k+1,n-2}^{(a,k-1)} + a_{n-1}^1 \gamma_{\nu-k+2,n-2}^{(a,k-1)} + \ldots + a_{n-1}^{k-1} \gamma_{\nu,n-2}^{(a,k-1)}.$$

Proof. Note that

$$\sum_{\nu=0}^{n(k-1)} \gamma_{\nu,n-1}^{(a,k-1)} x^{\nu} = \prod_{i=0}^{n-1}(x^{k-1} + a_i x^{k-2} + \ldots + a_i^{k-2} x + a_i^{k-1})$$

$$= (x^{k-1} + a_{n-1} x^{k-2} + \ldots + a_{n-1}^{k-2} x + a_{n-1}^{k-1}) \cdot \prod_{i=0}^{n-2}(x^{k-1} + a_i x^{k-2} + \ldots + a_i^{k-2} x + a_i^{k-1})$$

$$= \sum_{\nu=0}^{(n-1)(k-1)} \gamma_{\nu,n-2}^{(a,k-1)} x^{\nu+k-1} + a_{n-1} \sum_{\nu=0}^{(n-1)(k-1)} \gamma_{\nu,n-2}^{(a,k-1)} x^{\nu+k-2} + \ldots$$

$$+ a_{n-1}^{k-2} \sum_{\nu=0}^{(n-1)(k-1)} \gamma_{\nu,n-2}^{(a,k-1)} x^{\nu+1} + a_{n-1}^{k-1} \sum_{\nu=0}^{(n-1)(k-1)} \gamma_{\nu,n-2}^{(a,k-1)} x^{\nu}$$

$$= \sum_{\nu=k-1}^{n(k-1)} \gamma_{\nu-k+1,n-2}^{(a,k-1)} x^{\nu} + a_{n-1} \sum_{\nu=k-2}^{n(k-1)-1} \gamma_{\nu-k+2,n-2}^{(a,k-1)} x^{\nu} + \ldots$$

$$+ a_{n-1}^{k-2} \sum_{\nu=1}^{(n-1)(k-1)+1} \gamma_{\nu-1,n-2}^{(a,k-1)} x^{\nu} + a_{n-1}^{k-1} \sum_{\nu=0}^{(n-1)(k-1)} \gamma_{\nu,n-2}^{(a,k-1)} x^{\nu}$$

and the result follows by equating coefficients.

q.e.d.

The next theorem relates the $c_{n,r}^{(a,1)}$ and the γ_ν with the $c_{n,r}^{(a,k)}$. Here and in the following we abbreviate

$$\gamma_{n(k-1)-\nu,n-1}^{(a,k)} = \gamma_{n(k-1)-\nu}. \tag{12}$$

Theorem 3 *For* $0 \le t \le kn$,

$$\sum_{\substack{i_b+i_c=t \\ 0\le i_b\le n \\ 0\le i_c\le n(k-1)}} c_{n,n-i_b}^{(a,1)} \gamma_{n(k-1)-i_c} = \begin{cases} c_{n,n-\frac{t}{k}}^{(a,k)} & \textit{if } t \equiv 0 \pmod k \\ 0 & \textit{if } t \not\equiv 0 \pmod k \end{cases}.$$

Proof. First note that

$$(x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_{n-1}^k)$$

$$= (x-a_0)(x-a_1)\dots(x-a_{n-1}) \cdot \prod_{i=0}^{n-1} (x^{k-1} + a_i x^{k-2} + \dots + a_i^{k-2}x + a_i^{k-1})$$

$$= (\sum_{r=0}^{n} c_{n,n-r}^{(a,1)} x^{n-r}) \cdot (\gamma_0 + \gamma_1 x + \dots + \gamma_{n(k-1)-1} x^{n(k-1)-1} + \gamma_{n(k-1)} x^{n(k-1)})$$

$$= \sum_{t=0}^{kn} \sum_{\substack{i_b+i_c=t \\ 0\le i_b\le n \\ 0\le i_c\le n(k-1)}} c_{n,n-i_b}^{(a,1)} \gamma_{n(k-1)-i_c} x^{kn-t}.$$

Also, from the definition of the $c_{n,r}^{(a,k)}$

$$(x^k - a_0^k)(x^k - a_1^k)\dots(x^k - a_{n-1}^k) = \sum_{r=0}^{n} c_{n,n-r}^{(a,k)} x^{k(n-r)}.$$

Note that when $t = lk$ we have $x^{kn-t} = x^{k(n-l)}$.
By equating coefficients

$$\sum_{\substack{i_b+i_c=t \\ 0\le i_b\le n \\ 0\le i_c\le n(k-1)}} c_{n,n-i_b}^{(a,1)} \gamma_{n(k-1)-i_c} = \begin{cases} c_{n,n-\frac{t}{k}}^{(a,k)} & \text{if } t \equiv 0 \pmod k \\ 0 & \text{if } t \not\equiv 0 \pmod k \end{cases}.$$

q.e.d.

Assume $\omega \neq 1$ is a k^{th} root of unity. By factoring all the expressions

$$(x^k - a_i^k) = \prod_{j=0}^{k-1} (x - a_i\omega^j), \tag{13}$$

completely into a product of linear factors, the coefficients γ_ν may be determined explicitly. Accordingly we may factorize

$$\prod_{i=0}^{n-1} (x^{k-1} + a_i x^{k-2} + \ldots + a_i^{k-2} x + a_i^{k-1})$$

into linear factors. Hence

$$\sum_{\nu=0}^{n(k-1)} \gamma_\nu x^\nu = \prod_{j=1}^{k-1} \prod_{i=0}^{n-1} (x - a_i\omega^j).$$

Note that

$$\sum_{1 \leq i_1 < \ldots < i_m \leq k-1} \omega^{i_1 + \ldots + i_m} = (-1)^m, \text{for } m = 1, \ldots, k-1.$$

Then by (10)

$$\sum_{\nu=0}^{n(k-1)} \gamma_\nu x^\nu = \prod_{j=1}^{k-1} \sum_{r=0}^{n} c_{n,n-r}^{(a,1)} \omega^{j^r} x^{n-r} = \prod_{j=1}^{k-1} \sum_{r_j=0}^{n} c_{n,n-r_j}^{(a,1)} \omega^{j^{r_j}} x^{n-r_j}$$

$$= \sum_{\nu=0}^{n(k-1)} \sum_{\substack{r_1+\ldots+r_{k-1}=n(k-1)-\nu \\ 0 \leq r_1,\ldots,r_{k-1} \leq n}} c_{n,n-r_1}^{(a,1)} \cdots c_{n,n-r_{k-1}}^{(a,1)} \omega^{r_1+2r_2+\ldots+(k-1)r_{k-1}} x^\nu.$$

By equating coefficients,

$$\gamma_{n(k-1)-\nu} = \sum_{\substack{r_1+\ldots+r_{k-1}=\nu \\ 0 \leq r_1,\ldots,r_{k-1} \leq n}} c_{n,n-r_1}^{(a,1)} \cdots c_{n,n-r_{k-1}}^{(a,1)} \omega^{r_1+2r_2+\ldots+(k-1)r_{k-1}}.$$

We now determine the coefficients $\gamma_{n(k-1)-\nu}$ for the cases $k = 2$ and 3. If $k = 2$,

$$\gamma_{n-\nu} = c_{n,n-\nu}^{(a,1)} \omega^\nu = c_{n,n-\nu}^{(a,1)} (-1)^\nu. \tag{14}$$

If $k = 3$ and $\nu \equiv 0 \pmod 2$, then

$$\gamma_{2n-\nu} = \sum_{\substack{r_1+r_2=\nu \\ 0\le r_1,r_2\le n}} c_{n,n-r_1}^{(a,1)} c_{n,n-r_2}^{(a,1)} \omega^{r_1+2r_2}$$

$$= \sum_{\substack{r_1=r_2 \\ r_1+r_2=\nu \\ 0\le r_1,r_2\le n}} c_{n,n-r_1}^{(a,1)} c_{n,n-r_2}^{(a,1)} \omega^{r_1+2r_2} + \sum_{\substack{r_1<r_2 \\ r_1+r_2=\nu \\ 0\le r_1,r_2\le n}} c_{n,n-r_1}^{(a,1)} c_{n,n-r_2}^{(a,1)} (\omega^{r_1+2r_2} + \omega^{r_2+2r_1})$$

$$= \sum_{\substack{r_1=\frac{\nu}{2} \\ 0\le r_1\le n}} (c_{n,n-r_1}^{(a,1)})^2 \omega^{3r_1} + \sum_{\substack{r_1<r_2 \\ r_1+r_2=\nu \\ 0\le r_1,r_2\le n}} c_{n,n-r_1}^{(a,1)} c_{n,n-r_2}^{(a,1)} (\omega^{r_1+2r_2} + (\omega^{r_1+2r_2})^2)$$

$$= (c_{n,n-\frac{\nu}{2}}^{(a,1)})^2 + 2\cdot \sum_{\substack{r_1<r_2 \\ r_1+r_2=\nu, r_1+2r_2\equiv 0(3) \\ 0\le r_1,r_2\le n}} c_{n,n-r_1}^{(a,1)} c_{n,n-r_2}^{(a,1)} - \sum_{\substack{r_1<r_2 \\ r_1+r_2=\nu, r_1+2r_2\not\equiv 0(3) \\ 0\le r_1,r_2\le n}} c_{n,n-r_1}^{(a,1)} c_{n,n-r_2}^{(a,1)}.$$

If $k = 3$ and $\nu \equiv 1 \pmod 2$, then

$$\gamma_{2n-\nu} = 2\cdot \sum_{\substack{r_1<r_2 \\ r_1+r_2=\nu, r_1+2r_2\equiv 0(3) \\ 0\le r_1,r_2\le n}} c_{n,n-r_1}^{(a,1)} c_{n,n-r_2}^{(a,1)} - \sum_{\substack{r_1<r_2 \\ r_1+r_2=\nu, r_1+2r_2\not\equiv 0(3) \\ 0\le r_1,r_2\le n}} c_{n,n-r_1}^{(a,1)} c_{n,n-r_2}^{(a,1)}.$$

The correspondimg expressions for $k = 4$ have been computed in the first chapter of ([12]).

3 Applications for Particular Sequences

The above results can be applied to the various standard sequences of integers, generalizing the standard sequence $a = (0, 1, 2, 3, \ldots, n, \ldots)$ which led to the usual Stirling numbers. In [12] the case of the k–th power sequence $a = (0^k, 1^k, 2^k, \ldots, n^k, \ldots)$ was investigated, and the corresponding formulas were stated in detail. More over certain extension not shown were obtained, e.g. several representations of these generalized Stirling numbers as determinants of Vandermonde like matrices, as well as some further formulas.

The case of the natural square sequence $(0, 1, 4, 9, 16, \ldots, n^2, \ldots)$ which corresponds to the above case $k = 2$ was discussed with a somewhat different approach in [13] in the last chapter, where the corresponding generalized Stirling numbers of the first and second kind were called *central factorial numbers.* As shown in (14) in this case the coefficients $\gamma_{n-\nu}$ are just the unsigned (that is the absolute values of the) ordinary Stirling numbers for the sequence $(0, 1, 2, 3, \ldots)$.

Of course there is a variety of other examples available, by using some other standard sequences like the sequence of binomial coefficients

$$(\binom{k}{k}, \binom{k+1}{k}, \binom{k+2}{k}, \ldots, \binom{n+k}{k}, \ldots).$$

This may be generalized further using multinomial coefficients. Other possibilities include the sequence of k-gonal numbers $1 + \frac{kn(n-1)}{2} - (n-1)^2$ for any fixed value of $k \geq 3$, the sequences of Fibonacci and Lucas numbers, the sequence of prime numbers $(2, 3, 5, 7, 11, \ldots)$ and many other sequences.

4 An Application to Counting Partitions of Even Size

Let $a = (0, 1, 2, 3, \ldots)$ be the standard sequence of all natural numbers. In this section we look at the connection between the numbers $d_{n,r}^{(a,2)}$ and the partitions of sets which have an even number of elements.

Proposition 6 *Let $S_{2m,k}(2)$ denote the number of partitions of a set containing $2m$ elements into k* even *parts. Then $S_{2m,k}(2)$ satisfies the recursive relation:*

$$S_{2m,k}(2) = k^2 S_{2m-2,k}(2) + (2k-1) S_{2m-2,k-1}(2).$$

Proof. Let the partitioned set be

$$S = \{1, 2, \ldots, 2m-3, 2m-2, 2m-1, 2m\}$$

and remove the last two elements. After removing the elements from the k parts of the partition of S there are two possibilities, a partition of $T = \{1, 2, \ldots, 2m-2\}$ where either

1. all the parts again have an even number of elements, or
2. two of the parts have an odd number of elements and the rest have an even number of elements.

For the first case, $2m-1$ and $2m$ are together in the same part, and either they form an entire part by themselves or they form a part together with other elements. If $\{2m-1, 2m\}$ is one of the parts there are $S_{2m-2,k-1}(2)$ possible partitions and if $\{2m-1, 2m\}$ is a proper subset of one of the parts then there are $kS_{2m-2,k}(2)$ possible partitions.

For the second case, we first show that the number of partitions of the $(2m-2)$−set T into k parts where $k-2$ parts have an even number of elements and the other two parts have an odd number of elements is equal to N where

$$N = \binom{k}{2} S_{2m-2,k}(2) + (k-1)S_{2m-2,k-1}(2).$$

Let these partitions be referred to as *almost even partitions* of T of length k. The term $\binom{k}{2}S_{2m-2,k}(2)$ corresponds to all those almost even partitions where both odd parts contain 3 or more elements and the term $(k-1)S_{2m-2,k-1}(2)$ corresponds to all those almost even partitions where at least one odd part has just a single element. Next we construct a bijection.

This bijection goes from the set of the almost even partitions of T of length k to the union of two sets. The first set is the set of all triples (Π, P_1, P_2) where Π is a partition of T into k even parts and P_1 and P_2 are two of those k parts, and the number of such triples is $\binom{k}{2}S_{2m-2,k}(2)$. The second set is the set of all pairs (Σ, P_1) where Σ is a partition of T into $k-1$ even parts and P_1 is one of those $k-1$ parts, and there are $(k-1)S_{2m-2,k-1}(2)$ such pairs.

The bijection is defined as follows. Take any almost even partition of T where A_1 and A_2 are the two odd parts and select the minimal element a of $A_1 \cup A_2$. Then remove a from A_i where $i = 1$ or $i = 2$ and place a into the other part A_j where $j = 2$ or $j = 1$. If $\mid A_i \mid \geq 3$ then define $P_1 = A_i \backslash \{a\}, P_2 = A_j \cup \{a\}$ and $P_l = A_l$ for $l = 3, \ldots, k$. Letting this partition of length k be Σ it is clear that Σ together with P_1 and P_2 form a triple (Σ, P_1, P_2) in the first of the two sets described above.

However, if $\mid A_i \mid = 1$ then define $P_1 = A_j \cup a$ and $P_l = A_{l+1}$ for $l = 2, \ldots k-1$. Denoting this partition again by Σ, it is also clear that Σ and P_1 form a pair (Σ, P_1) in the second of the two sets described above.

The number of partitions for the second case is $2N$ because the two elements $2m-1$ and $2m$ can be removed from two of the k parts in two ways.

Taking into account the various cases,

$$S_{2m,k}(2) = 2\binom{k}{2}S_{2m-2,k}(2) + 2(k-1)S_{2m-2,k-1}(2)$$
$$+kS_{2m-2,k}(2) + S_{2m-2,k-1}(2)$$
$$= k^2 S_{2m-2,k}(2) + (2k-1)S_{2m-2,k-1}(2).$$

q.e.d.

Theorem 4 *For all $n \geq r \geq 1$*

$$S_{2n,r}(2) = (1)(3)\dots(2r-1)d_{n,r}^{(a,2)}.$$

Proof. By Proposition 1, $d_{n,r}^{(a,2)} = d_{n-1,r-1}^{(a,2)} + r^2 d_{n-1,r}^{(a,2)}$. Also,

$$S_{2n,r}(2) = r^2 S_{2n-2,r}(2) + (2r-1)S_{2n-2,r-1}(2) \qquad \text{from Proposition 6.}$$

Therefore

$$(1)(3)\dots(2r-1)d_{n,r}^{(a,2)} = (1)(3)\dots(2r-1)d_{n-1,r-1}^{(a,2)} + (1)(3)\dots(2r-1)r^2 d_{n-1,r}^{(a,2)}$$
$$= (2r-1)((1)(3)\dots(2r-3)d_{n-1,r-1}^{(a,2)})$$
$$+r^2(1)(3)\dots(2r-1)d_{n-1,r}^{(a,2)}.$$
$$\Rightarrow S_{2n,r}(2) = 1\cdot 3\cdot\dots\cdot(2r-1)d_{n,r}^{(a,2)}.$$

q.e.d.

Remark: We have become aware of some attempts [5] [15] to generalize Bernoulli numbers in a similar way that we have generalized Stirling numbers in the first two sections of the present paper. It remains to be investigated whether there is a relationship between these two developments.

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