Chromatic number of regular walls

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Abstract

We formalize the intuitive question of coloring the bricks of a wall in such a way that no repetition occurs in any row, nor any vertical line intersects two or more bricks with the same color. We achieve a complete classification up to the least number of required colors, among all dimensions of the walls, and all admitted incidences of the bricks. The involved combinatorial structures (namely, *regular walls*) are a special case of more general structures, which can be interpreted as adjacency matrices of suitable directed hypergraphs. Coloring the bricks is equivalent to coloring the arcs of the corresponding hypergraph. Regular walls seem interesting also for their connections with latin rectangles.

Key-words: brick, chromatic number, coloring, directed hypergraph, non-overlapping, regular wall, wall.

1 Introduction

Regular walls formalize the intuitive concept of a real wall made of bricks. Once given the basic definitions, it will be clear that a regular wall is a precise combinatorial structure (namely, a *wall*) with some further properties. Accordingly, in Section 2 we will provide a formal introduction to walls; in

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particular, we will describe a basic connection between walls and directed hypergraphs. Anyway, before starting with the definitions, we anticipate the core of our work as follows. In the left side of Figure 1 we have depicted a regular wall whose bricks are 2 units long. Some bricks are necessarily partial. The height and length of this wall are 6 and 11 units respectively. The numbers inside bricks may be interpreted as colors. Observe that no color occurs more than once in each row, and that no vertical line intersects two or more bricks equally colored. We are interested in finding the least number of colors required for a coloring which has the two properties mentioned above. Actually, it can be shown that at least 7 colors are needed in this specific case. Thus, the given coloring satisfies our requirement.

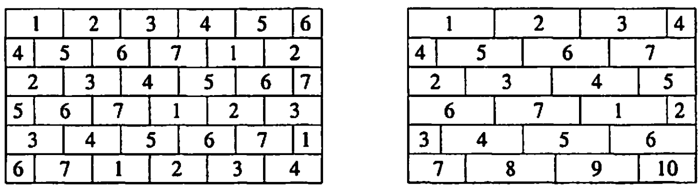

Figure 1: Brick-coloring of regular walls

If the height and length of a regular wall are considered as variables (say m, n), we aim to find a formula which returns the least number of colors required, as a function of m, n. Such investigation is fulfilled in Section 3. In Section 4 we define some more general structures, by modifying the brick incidence as illustrated in the right side of Figure 1 (we anticipate that the given coloring turns out to be not minimal). Also in the general case we achieve a complete classification up to the least number of colors. In particular, we deduce that such number is always very small, with respect to the possible values over all walls. Although the theorem in Section 3 is a particular case of the general theorem in Section 4, the former result provides a suggestive introduction to the techniques developed to obtain the latter result. Finally, in Section 5 we apply some of the above results to a real context.

2 Basic concepts on walls

The intuitive concept of wall, namely a structure made of bricks put together in some proper way, has been formalized in [8], by means of the following two definitions.

Definition 2.1. A *pluriomino* P is a finite set of unit squares in the plane, arranged like a chessboard having some squares cut out, possibly.

In particular, a connected pluriomino with no finite set of cut points is classically defined as a *polyomino* (see for example [7]).

Definition 2.2. A *wall* P is a pluriomino whose squares have been labeled under the condition that every label appears in a unique row of P. Its *degree*, $\delta(P)$, is the greatest number of distinct labels in the same row or column. Every maximal set of squares having the same label is called a *brick*.

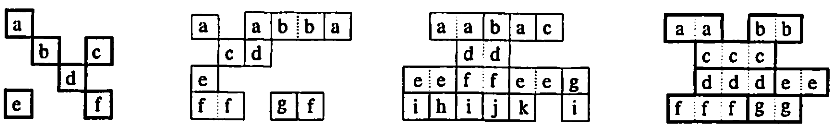

Figure 2: Four walls

Observe that the usual concept of brick is generalized, because every brick of P is allowed to be partitioned into several disconnected pieces lying on the same row. Moreover, some empty spaces may be present. In the first and fourth case of Figure 2 no brick is disconnected, whence the bricks can be easily recognized even with no labeling. We give these two walls a particular name.

Definition 2.3. If a wall contains only connected bricks, then it is termed *coherent*.

Walls have a natural rephrasing in hypergraph theory, for they can be interpreted as adjacency matrices of *non-overlapping directed hypergraphs* (for all details see [8]; for general notions about directed hypergraphs, see for example [1], [5]).

Definition 2.4. A *directed hypergraph* $H = (V(H), E(H))$ consists of a set V, the *nodes*, together with a set $E \subseteq \mathcal{P}(V) \times V$. Each element $e := (A, z)$ is a *hyperarc* (or simply an *arc*). A and z are respectively the *tail* and the *head* of e. The hypergraph H is termed *non-overlapping* if there exists no pair of arcs of H sharing the same head and some node of the tails.

In Figure 3 we have depicted a non-overlapping directed hypergraph, together with the corresponding wall. Observe that the row-indices correspond to the heads.

The notion of *brick-coloring*, which we are going to introduce, can be rephrased as a way of coloring the arcs of the corresponding hypergraph. Such arc-coloring (defined in [8]) is an extension of the existing notion of arc-coloring for digraphs (see [6], [4]).

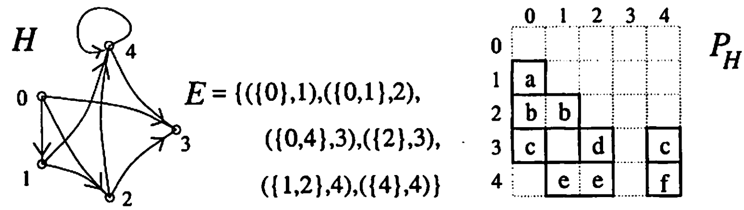

Figure 3: A non-overlapping directed hypergraph and the related wall

Definition 2.5. A *(brick-)coloring* of the wall P is the assignment of a symbol to each square of P, in such a way that: 1) all the squares of any fixed brick get the same symbol; 2) no symbol is present in more than one brick in the same row; 3) no symbol appears more than once in every column. The *chromatic number* of P, denoted by $\rho(P)$, is the least number of symbols required for a coloring of P.

Each coloring of P in $\rho(P)$ colors is termed *optimal*. Notice that $\rho(P) \geq \delta(P)$ for every wall P. Coherent walls play a basic role, with regard to the coloring. Indeed, the following holds.

Proposition 2.6. If P is a coherent wall, then $\rho(P) \leq 2\delta(P) - 1$, and the bound can be attained for every value of the degree. Furthermore, for every couple of integers $r \geq d \geq 2$ there exists a wall Q such that $\delta(Q) = d$, $\rho(Q) = r$.

The above result has been the starting point for analyzing coherent walls with a look to their coloring properties (see [8]). For example, classifying the coherent walls of a fixed degree up to their chromatic number seems a complex problem, which is currently open.

Regular walls arise as a special case of coherent walls. Unlike the general case, an exhaustive classification up to the chromatic number has been done for this particular class. In fact, such classification is the main scope of the present paper.

Definition 2.7. Let m, n, k be positive integers with $k \geq 2$. Then, the (m, n, k)*-regular wall* - in symbols $W^k_{m,n}$ - is the wall defined as follows. We consider a rectangular pluriomino with no missing square, whose height and length are m and n respectively. This pluriomino is regarded as a matrix $(m_{ij}, 1 \leq i \leq m, 1 \leq j \leq n)$. The (i, j)-square is then labeled by

$$l(i,j) := \left(i, \left\lfloor \frac{j-i}{k} \right\rfloor\right) ,$$

where $\lfloor x \rfloor$ denotes the largest integer $n \leq x$.

Thus, in Figure 1 we have depicted $W^2_{6,11}$ and $W^3_{6,10}$. We have subsequently observed (with no proof) that $\rho(W^2_{6,11}) = 7$ and $\rho(W^3_{6,10}) < 10$.

Independently of the interpretation via hypergraphs, regular walls seem quite interesting to investigate. For example they are closely related to latin rectangles, as we will shortly underline in the conclusion.

In the sequel we will use the following symbols. If $x \in \mathbf{R}$, then $\lceil x \rceil$ denotes the smallest integer $n \geq x$. If $a \in \mathbf{Z}$ and $b \in \mathbf{N}^+$, then $re(a, b)$ stands for the remainder of the euclidean division $a : b$. That is, $re(a, b) := a - b\lfloor a/b \rfloor$.

3 The chromatic number of $W^2_{m,n}$

In this section we classify all the $(m, n, 2)$-regular walls, up to their chromatic number, as follows.

Theorem 3.1. If $m \geq 2$, then

$$\rho(W^2_{m,n}) = \begin{cases} m & \text{if} \quad n \leq 2\left\lfloor \frac{m}{2} \right\rfloor - 1 \\ m+1 & \text{if} \quad 2\left\lfloor \frac{m}{2} \right\rfloor \leq n \leq 2m+1 \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{if} \quad n \geq 2m+2 \end{cases} .$$

Furthermore,

$$\rho(W^2_{1,n}) = \left\lceil \frac{n}{2} \right\rceil .$$

Proof. Throughout the proof, every coloring in t colors will be represented by some map $\gamma : \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\} \to \{1, 2, \ldots, t\}$, which assigns a suitable "color" to each square. We start with the case $m \geq 2$.

Firstly we show that the numbers $\{1, 2, \ldots, m\}$ are not enough to color $W^2_{m,n}$ if $n \geq 2\lfloor m/2 \rfloor$. For some fixed coloring γ, we may assume that $\gamma(i, 1) = i$ for all i. If i is odd, then $\gamma(i, 1) = \gamma(i, 2)$, which implies that $\gamma(j, 2)$ is even for every even number j (indeed, all the odd colors up to m are not allowed). Analogously, if i is even then $\gamma(i, 2) = \gamma(i, 3)$, whence $\gamma(j, 3)$ must be odd for every odd number j (indeed, all the available even colors have been given to the $(j, 2)$-squares with j even). By iterating the above argumentation, we deduce that every even row must get all even colors. As there are $\lfloor m/2 \rfloor$ even colors up to m, the largest number of squares in every even row must be $2(\lfloor m/2 \rfloor - 1) + 1$ (the last addendum corresponds to the leftmost brick). Thus, m colors are not enough if $n \geq 2(\lfloor m/2 \rfloor - 1) + 2 = 2\lfloor m/2 \rfloor$.

A slight change to the above argument yields $\rho(W^2_{m,n}) = m$ whenever $n \leq 2\lfloor m/2 \rfloor - 1$. More precisely, we will exhibit a coloring of $W^2_{m,n}$ in m

colors, and the inequalities $\rho(W^2_{m,n}) \geq \delta(W^2_{m,n}) \geq m$ will determine the chromatic number in this case. We firstly assume that m is odd. It is enough to exhibit a coloring of $W^2_{m,2\lfloor m/2 \rfloor -1}$, because we can subsequently restrict such coloring to the first n columns. To this end, we provide two colorings of the odd rows and the even rows separately, and we put them together. Observe that the coloring of the wall formed by the only odd rows is equivalent to the construction of a $\lceil m/2 \rceil \times \lfloor m/2 \rfloor$ latin rectangle, while the other coloring is equivalent to finding a latin square of dimension $\lfloor m/2 \rfloor$. Since $\lceil m/2 \rceil + \lfloor m/2 \rfloor = m$, it is actually possible to construct the required latin rectangles using m colors in all. If m is even, we use the same argument with the only difference that the latin rectangle has one row less; indeed, the odd numbers up to m are as many as the even numbers up to m. As a consequence we obtain a further latin square of dimension $\lfloor m/2 \rfloor = m/2$. Since $m/2 + m/2 = m$, we get the conclusion.

Now we exhibit a coloring of $W^2_{m,2m+1}$ in $m+1$ colors. This result will yield the second assertion of the three listed ones, by restricting the coloring to the first n columns of the wall. We distinguish two cases.

1) m is odd. The bricks of row 1 are then colored, from left to right, with $1, 2, \ldots, m, m+1$. The colors of row 2 begin with $(m+1)/2+1$ and then follow the same order of the upper bricks, passing from $m+1$ to 1 and then increasing up to $(m+1)/2$. Thus, the sequence is $(m+1)/2 + 1, (m+1)/2+2, \ldots, m+1, 1, 2, \ldots, (m+1)/2$. For briefness, we will say that such colors follow the *circular order*. The next odd rows (namely, row $3, 5, \ldots, m$) are colored using the circular order, but their leftmost colors are respectively equal to $2, 3, \ldots, (m+1)/2$. Instead, the leftmost colors of the remaining even rows (namely, row $4, 6, \ldots, m-1$) are given the colors $(m+1)/2+2, (m+1)/2+3, \ldots, (m+1)/2 + (m-1)/2 = m$, and the even rows are colored using the circular order as well. We provide two examples in Figure 4. The two colorings are the restrictions of the colorings applied to $W^2_{5,11}$ and $W^2_{7,15}$ respectively.

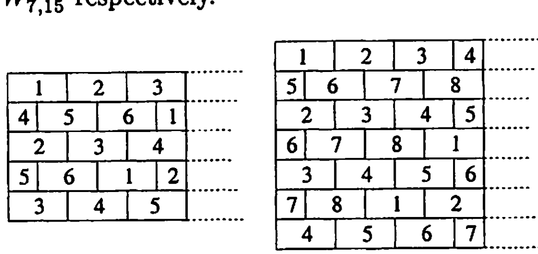

Figure 4: Optimal colorings of $W^2_{5,6}, W^2_{7,7}$

We have to prove that the above coloring γ is legal. Clearly, γ is legal on each single row, because the circularly ordered colors are as many as the bricks of each row. Now, the colors of column 1 range from 1 to m, whence they are distinct. The colors of column 3 are obtained by shifting ahead (with respect to the circular order) the colors of column 1. Thus, they are distinct. A similar argument can be iterated so as to yield legality for each odd column. It remains to check the even columns. The $(i, 2)$-squares with i odd belong to the same corresponding bricks of column 1. Consequently, their colors are $1, 2, \ldots, (m+1)/2$. For i even, the $(i, 2)$-squares are colored by shifting ahead the color of each left adjacent brick. Therefore, these squares get the colors $(m+1)/2+2, (m+1)/2+3, \ldots, m, m+1$. It follows that the second column is legally colored, and an argumentation similar to the one above (concerning shifts) ensures that all the even columns are legally colored.

2) m is even. We manage this case in a similar fashion. The bricks of the first row are colored with $1, 2, \ldots, m+1$. The leftmost bricks of the remaining odd rows are given the colors $2, 3, \ldots, m/2$ (from the top to the bottom), whereas the leftmost partial bricks in the even rows are given the colors $m/2+1, m/2+2, \ldots, m$. Now, reasoning as in the odd case, the legality of such coloring can be proved. The left side of Figure 1 provides an example (actually, the coloring of $W^2_{6,13}$ has been restricted to $W^2_{6,11}$).

The third assertion of the theorem is proved as follows. Assuming that $n \geq 2m+2$, we add $\lfloor n/2 \rfloor - m$ suitable rows to $W^2_{m,n}$ so as to obtain $W^2_{\lfloor n/2 \rfloor, n}$. Since $n \leq 2\lfloor n/2 \rfloor + 1$, the above result guarantees that $W^2_{\lfloor n/2 \rfloor, n}$ can be colored in $\lfloor n/2 \rfloor + 1$ colors. Thus, by restricting the coloring to $W^2_{m,n}$, we get the conclusion because $\rho(W^2_{m,n}) \geq \delta(W^2_{m,n}) = \lfloor n/2 \rfloor + 1$.

The last assertion of the theorem, concerning the case $m = 1$, is easily proved by counting the bricks in the unique row. □

4 Generalizing to $W^k_{m,n}$

The following theorem concerns the general case, where the further variable $k \geq 2$ is considered. We will essentially require the same techniques of the above proof.

Theorem 4.1. Let k be an integer greater than 1. If $m > k$ and $k \not| m$, then

$$\rho(W^k_{m,n}) = \begin{cases} m & \text{if} \quad n \leq m-k \\ m+1 & \text{if} \quad m-k+1 \leq n \leq km+1 \\ \lceil \frac{n-1}{k} \rceil + 1 & \text{if} \quad n \geq km+2 \end{cases} .$$

If $k|m$, the upper bound in the first condition is changed to $m-k+1$

and the lower bound in the second condition becomes $m - k + 2$; the rest is unchanged.

If $1 < m < k$, the first condition is changed to $n = 1$ and the lower bound in the second condition becomes 2; the rest is unchanged.

Finally,

$$\rho(W^k_{1,n}) = \left\lceil \frac{n}{k} \right\rceil .$$

Proof. As announced, we will follow the same steps of the previous proof. In particular, we will represent any coloring γ as a map sending each square to some suitable positive integer. For convenience, the leftmost square of any brick will be termed *initial.* We begin with the case $m > k, k \not| m$.

Firstly, we show that m colors are not enough to color $W^k_{m,n}$ if $n \geq m - k + 1$. For some fixed coloring γ, we assume that $\gamma(i, 1) = i$ for all i. With regard to column 2, the initial squares are all the $(2 + kt, 2)$-squares, for every possible integer t. Thus, if we want to use only m colors, the following condition must hold for every t.

$$\gamma(2 + kt, 1) \equiv \gamma(2 + kt, 2) \bmod k \ , \ \gamma(2 + kt, 1) \neq \gamma(2 + kt, 2) \ .$$

In particular, the $(2 + kt, 2)$-squares need to be colored with the same set of colors previously used to color the $(2 + kt, 1)$-squares. Similarly, for the initial squares of column 3 we have

$$\gamma(3 + kt, 2) \equiv \gamma(3 + kt, 3) \bmod k \ , \ \gamma(3 + kt, 2) \neq \gamma(3 + kt, 3) \ .$$

By iterating the above reasoning, we deduce that every brick of some fixed row i must be colored with some number of the form $i + uk, u \in \mathbf{Z}$. Now, by the hypothesis on n, we get the following contradiction. Let us define μ as $re(m, k)$. By the definition of μ, the allowed colors in row $\mu + 1$ are as many as $(m - \mu)/k$. Since the leftmost brick of row $\mu + 1$ is made of μ squares, the largest number of squares (say s) in this row must satisfy $s \leq \mu + k((m - \mu)/k - 1) = m - k$, which contradicts the choice of n (observe that, by the above calculation, if $n = m - k + 1$ the rightmost brick of row $\mu + 1$ consists of a single square; we will avail of this property in the next step).

As in the case $k = 2$, the above part of the proof can be adapted to yield $\rho(W^k_{m,n}) = m$ whenever $n \leq m - k$. To this end, we will exhibit a coloring of $W^k_{m,m-k}$ in m colors. As row $\mu + 1$ has now a rightmost complete brick, the reader can easily see that every row from the 2-nd to the μ-th has one brick more than row $\mu + 1$, whereas row 1 has the same number of bricks than row $\mu + 1$ (indeed, rows $2, 3, \ldots, \mu$ are obtained by iterated left shifts of row $\mu + 1$, and they all have both a partial rightmost brick and a partial leftmost brick). Since row $\mu+1$ has $(m-\mu)/k = \lfloor m/k \rfloor$ bricks, all the upper rows apart the first have $\lceil m/k \rceil$ bricks. It is equally easy to check that the lower rows, from the $(\mu+2)$-th to the k-th, have all the same number of bricks than row $\mu+1$. Thus, finding a suitable coloring of $W^k_{m,m-k}$ is equivalent to creating $\mu-1$ latin squares of dimension $\lceil m/k \rceil$ - each one corresponding to rows $a, a+k, a+2k, \ldots, a+(\lceil m/k \rceil - 1)k$ for some $a \geq 2, a \leq \mu$ - together with $k-\mu$ latin squares of dimension $\lfloor m/k \rfloor$ - each one corresponding to rows $a, a+k, a+2k, \ldots, a+(\lfloor m/k \rfloor - 1)k$ for some $a \geq \mu+1, a \leq k$ - and a further $\lceil m/k \rceil \times \lfloor m/k \rfloor$ latin rectangle, corresponding to rows $1, 1+k, 1+2k \ldots$. Clearly, no color must occur in more than one configuration. Since

$$(\mu-1)\lceil m/k \rceil + (k-\mu)\lfloor m/k \rfloor + \lceil m/k \rceil = m,$$

we can actually make up such independent colorings, using m colors in all. In the left side of Figure 5 we have illustrated the case $m = 15, n = 11, k = 4$.

Now we exhibit a coloring of $W^k_{m,km+1}$ in $m+1$ colors. As a consequence we will obtain the second assertion of the three listed ones, by restricting the coloring to the first n columns of the wall. As in the case $k = 2$, we assume that the colors of each row follow the circular order $1, 2, 3, \ldots, m, m+1, 1, 2, \ldots$ from left to right. Therefore, in order to define the coloring it is enough to assign one color to each leftmost brick. We define such assignment as a bijection from the set of row-indices $\{1, 2, \ldots, m\}$ to the same set representing colors (thus, we do not use the color $m+1$). We firstly introduce the following total order $\prec$ in $\{1, 2, \ldots, m\}$.

$$i \prec j \text{ if } (k|i-j \text{ and } i<j) \text{ or } re(i-2,k) > re(j-2,k) \ .$$

For example, if $m = 11$ and $k = 4$, we get the ordered list (starting with the minimum)

$$\overbrace{1 \;\; 5 \;\; 9}^{re(i-2,4)=3} \quad \overbrace{4 \;\; 8}^{re(i-2,4)=2} \quad \overbrace{3 \;\; 7 \;\; 11}^{re(i-2,4)=1} \quad \overbrace{2 \;\; 6 \;\; 10}^{re(i-2,4)=0} \ .$$

In particular, the sequence begins with $1, 1+k, 1+2k, \ldots$ in all cases. We are now ready for coloring all the leftmost bricks. We define the coloring as the unique order-preserving bijection, say β, from $(\{1, 2, \ldots, m\}, \prec)$ to the same set ordered by the standard order $<$. Thus, in our example, the sequence of colors from row 1 to row 11 is given by the following table.

row	1	2	3	4	5	6	7	8	9	10	11
color	1	9	6	4	2	10	7	5	3	11	8 .

In the right side of Figure 5 we have illustrated the case $m = 15, n = 12, k = 4$. The given coloring is the restriction of the coloring related to $W^4_{15,61}$. Notice that the wall in the left example has the largest length to make possible a coloring in 15 colors, having fixed the height (15) and the length of each complete brick (4).

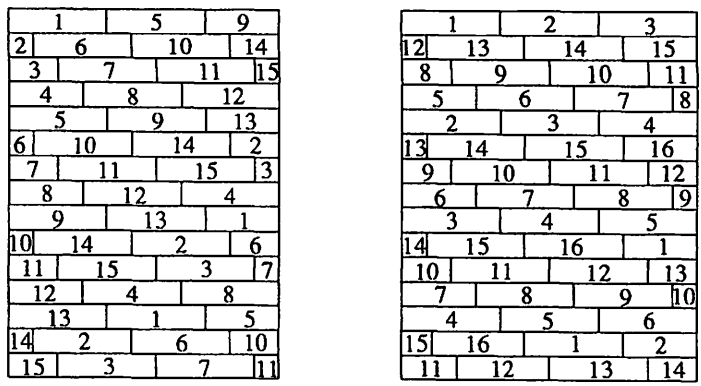

Figure 5: Optimal colorings of $W^4_{15,11}, W^4_{15,12}$

Now we prove the legality of our coloring. As row 2 contains the largest number of bricks among all rows - this number being equal to $((km+1)-1)/k+1 = m+1$ - the circular order does not give rise to repetitions in each row. Due to the circular order again, it is enough to check the columns from the second to the k-th. We begin with column 2. The $(i,2)$-squares with $i \equiv 2 \bmod k$ are colored by adding 1 to the color of each left adjacent brick. By the definition of β, it can be easily checked that the involved left colors to increase are consecutive, and that the greatest of them is equal to m. The other squares of column 2 are not initial, whence their color does not change with respect to column 1. It follows that the increased numbers make up a legal coloring of column 2, because $m+1$ is not present in column 1. Now we pass to column 3. Let us define as $\xi, \xi+1, \ldots, m$ the above sequence of colors to increase in column 1 (in Figure 5 such sequence corresponds to 12,13,14,15, while in the previous row-color table it corresponds to 9,10,11). With regard to column 3, the initial squares are the $(i,3)$-squares with $i \equiv 3 \bmod k$. The involved left colors in column 2 are consecutive, and the greatest of them is equal to $\xi - 1$. Observe that such color is changed in ξ, which is not present in any other square of column 3 (indeed, ξ has been replaced with $\xi+1$ when passing from the (2,1)-square to the (2,2)-square). Thus, the increased numbers yield a legal coloring of column 3. By iterating the above argument up to column k, we deduce the correctness of our coloring.

The third assertion of the theorem is established in the same fashion of the corresponding proof in Section 3. Firstly, we notice that $\rho(W^k_{m,n}) \geq \delta(W^k_{m,n}) \geq \lceil (n-1)/k \rceil + 1$, by counting the bricks of row 2. Now, a coloring in $\lceil (n-1)/k \rceil + 1$ colors is exhibited as follows. We add $\lceil (n-1)/k \rceil - m$ suitable rows to $W^k_{m,n}$ so as to obtain $W^k_{\lceil (n-1)/k \rceil, n}$ (notice that $\lceil (n-1)/k \rceil \geq \lceil (km+1)/k \rceil > m$). Since $n = (n-1)+1 \leq k\lceil (n-1)/k \rceil + 1$, the above result ensures that $W^k_{\lceil (n-1)/k \rceil, n}$ admits a coloring in $\lceil (n-1)/k \rceil + 1$ colors. Thus, by restricting such coloring to $W^k_{m,n}$, we get the conclusion.

We pass to establish the claim in the case that $k|m$. We show that m colors are not enough to color $W^k_{m,n}$ if $n \geq m - k + 2$. Indeed, by the same reasoning of above, we deduce that every brick of some fixed row i must be colored with some number of the form $i + uk, u \in \mathbf{Z}$. Now, by the hypothesis on n, we get the following contradiction. The allowed colors in row 2 are as many as m/k. Since the leftmost (partial) brick of row 2 has length 1, the largest number of squares (say s) in this row satisfies $s \leq 1 + k(m/k - 1) = m - k + 1$, which contradicts the choice of n.

Now we provide a suitable coloring of $W^k_{m,m-k+1}$ in m colors, using essentially the same argument of the case $k \not| m$. In the present case, a quick reasoning could show that each row consists of m/k bricks. The required coloring is therefore equivalent to k latin squares - colored with all different numbers - of dimension m/k, each one corresponding to rows $a, a+k, a+2k, \ldots, a+(m/k-1)k$ for some positive $a \leq k$. Since $k \cdot m/k = m$, we can make up such independent colorings using m colors in all.

The rest of the proof follows the same argument of the case $k = 2$.

Let us now assume that $1 < m < k$. Then, the wall consisting of the first two columns of $W^k_{m,n}$ must be colored in $m+1$ colors, because column 2 has a unique initial brick. On the other hand, the order-preserving bijection β works in this case as well (the related sequence of colors of the leftmost bricks, from row 1 to row m, is $1, m, m-1, m-2, \ldots, 3, 2$). Therefore, if $2 \leq n \leq km + 1$ we can color $W^k_{m,n}$ in $m + 1$ colors. The rest of the proof is not altered.

Finally, if $m = 1$ we simply count the bricks of the unique row. □

We can use Theorem 4.1 for describing the following relationship between the chromatic number and the degree of any regular wall.

Corollary 4.2. If P is a regular wall, then

$$\delta(P) \leq \rho(P) \leq \delta(P) + 1.$$

More precisely, if $P = W^k_{m,n}$ then $\rho(P) = \delta(P)$ unless $m \not| k, m - k + 1 \leq n \leq k(m-1) + 1$ or $m > 1, m|k, m - k + 2 \leq n \leq k(m-1) + 1$. In these cases, $\rho(P) = \delta(P) + 1$.

Proof. If $m = 1$, then $\rho = \lceil n/k \rceil = \delta$. Let us now assume that $m \geq 2$. The argument used for proving the above theorem implies in particular the following identities.

$$\delta(W^k_{m,n}) = \max(\text{n. of bricks in row 2 , n. of bricks in any column}) =$$

$$= \max\left(\left\lceil \frac{n-1}{k} \right\rceil + 1 \text{ , } m\right) .$$

For more briefness, we define the symbol ε as 1 if $k|m$, as 0 otherwise. We distinguish four cases.

1) $n \leq m - k + \varepsilon$. Then, the inequality $\lceil (n-1)/k \rceil + 1 \leq m$ implies that $\delta = m$, and by Theorem 4.1 m colors are enough. Thus, $\rho = \delta$.

2) $m - k + \varepsilon + 1 \leq n \leq k(m-1) + 1$. We still have that $\delta = m$. Though m colors are not enough any more, we can make up a coloring in $m+1$ colors, because $n \leq km + 1$. Thus, $\rho = \delta + 1$.

3) $k(m-1) + 2 \leq n \leq km + 1$. We can still use $m+1$ colors, but now $\lceil (n-1)/k \rceil + 1 = m + 1$. Thus, $\rho = \delta$.

4) $n \geq km + 2$. We have that $\rho = \lceil (n-1)/k \rceil + 1 = \delta$. □

5 An application

In this short section we aim to give a practical meaning to the chromatic number, by interpreting the optimal coloring of a regular wall in a real context.

We suppose that s astronauts must be subjected to a test which shall provide some basic information on their biorhythms. Every test consists of a sequence of n actions to be performed, each one lasting the same time, say t minutes. For instance, such sequence could start with *studying, running, eating, walking in the cold, ...* . The tests take place simultaneously in s special rooms. During every test, an observer makes a report on the astronaut's conditions. Each observer is paid for observing k consecutive actions, in kt minutes. To make the procedure more reliable, each astronaut performs the sequence k times, in k distinct days, so that every fixed sub-sequence of k consecutive actions can be observed by some person, during his/her kt working minutes; as a consequence, some observers may be requested to work less than the usual working time, so as to cover the whole daily report (otherwise, either some initial actions or some final ones are not observed). A daily sequence of actions $\{A_i\}$ is illustrated in the following scheme (in our example, each person p_i observes at most 3 consecutive actions).

$$\underbrace{A_1}_{p_1} \quad \underbrace{A_2\ A_3\ A_4}_{p_2} \quad \underbrace{A_5\ A_6\ A_7}_{p_3} \quad \dots \quad \underbrace{A_{20}\ A_{21}\ A_{22}}_{p_8} \quad \underbrace{A_{23}\ A_{24}}_{p_9}$$

The test for a single astronaut can be described by the regular wall $W^k_{k,n}$, whose bricks (resp. columns, rows) represent the observers' working times (resp. the actions, the days). Moreover, the s tests for all astronauts can be put together sequentially, so as to form the wall $W^k_{ks,n}$.

To increase impartiality still more, every action is assumed to be observed at most once by any fixed observer. Thus, if each observer is associated to some distinct color, every legal coloring of $W^k_{ks,n}$ is equivalent to a correct scheduling for the observers' working times. In particular, $\rho(W^k_{ks,n})$ is the least number of observers required for the whole test. For example, ks observers are enough only if the number of actions does not exceed $ks - k + 1$.

6 Conclusion

The evaluation of $\rho(P)$, for some given wall P, is deeply related to the possibility of finding some algorithm which returns an optimal coloring. In general, it seems quite difficult to provide a relatively fast algorithm for some assigned class of walls. For example, it turns out that the problem of checking whether some wall is colorable in k colors (where k is a parameter greater than 2) is *NP-complete*, and the same result still holds if we simply consider walls of degree 2 (see [9]; for the basic notions about NP-completeness, see for example [2]).

Unlike the general case, the chromatic classification of regular walls has a very low complexity. In other words, the algorithm which returns ρ runs in a quite short time, with respect to the input size $\{m, n, k\}$. Indeed, using Theorem 4.1, it is enough to make a quick distinction by cases.

We think that the most notable property of regular walls, with respect to the chromatic number, is the behavior of $\rho(W^k_{m,n})$ when such number passes from m to $m+1$ as a function of the length n. Indeed, a relatively large number of columns has to be cut out of $W^k_{m,km+1}$, in order to decrease the chromatic number from $m+1$ to m. This phenomenon can also be described by noting that in some cases - against our intuition - we need m colors even if the columns are less than the m rows. For instance, a regular wall with $m = 11, n = 10, k = 2$ requires 12 colors, though every odd row contains only 5 bricks.

Our wrong intuition is conditioned by the well-known coloring properties of latin rectangles. In fact, coloring the bricks of $W^k_{m,n}$ can be interpreted as coloring the squares of a suitable chessboard, with no horizontal nor vertical repetition, and with no repetition on some further "paths". Figure 6 provides an example. Bricks have been substituted by circles. The three kinds of path correspond to the different kinds of incidence among the bricks intersecting some fixed column.

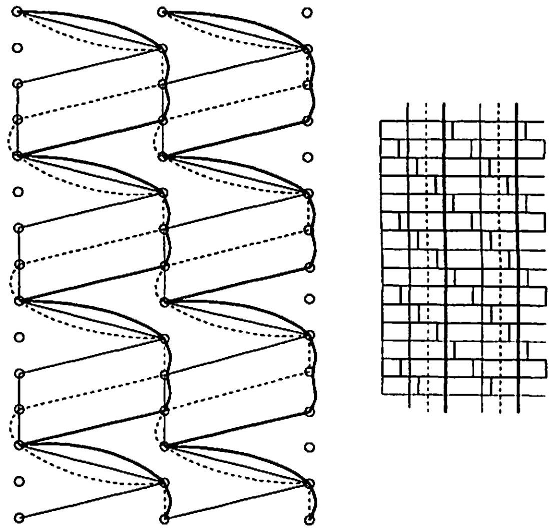

Figure 6: The incidence structure of $W^4_{15,n}$, $n \geq 9$

Therefore, we may consider regular walls with a fixed $k > 1$ as a particular subclass of latin rectangles. Perhaps some nice properties of latin rectangles are still valid in this context, whereas some general properties of regular walls might be successfully used to get some more knowledge of latin rectangles. Notice that an $m \times n$ latin rectangle is equivalent to an optimally colored $(m, n, 1)$-regular wall, provided one extends the definition in the obvious way, for $k = 1$.

Finally, in view of the relationship between walls and directed hypergraphs, it might be interesting to apply the present results and suggestions to the subclass of directed hypergraphs which are represented by regular walls.

References

[1] G. Ausiello, A. D'Atri and D. Saccà, *Minimal Representation of Directed Hypergraphs*, SIAM J. Computing **15**, 1986, pp. 418-431.

[2] D.P. Bovet and P. Crescenzi, *Introduction to the Theory of Complexity*, Prentice Hall, UK, 1994.

[3] V.W. Bryant, *Aspects of Combinatorics, a Wide-Ranging Introduction*, Cambridge University Press, 1993.

[4] M. Espona and O. Serra, *Cayley Digraphs based on the De Bruijn networks*, SIAM J. Discrete Math. **11**, 1998, pp. 305-317.

[5] G. Gallo, G. Longo, S. Nguyen and S. Pallottino, *Directed Hypergraphs and Applications*, Discrete Applied Math. **42**, 1993, pp. 177-201.

[6] O. Serra and M.A. Fiol, *Arc-colored line digraphs and their groups*, in Graph Theory, Combinatorics, Algorithms, and Applications. Proc. Appl.Math. **54**, Y.Alavi et al. eds., SIAM, 1991, pp. 459-468.

[7] R.P. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge Studies in Advanced Mathematics, Cambridge, 1997.

[8] A. Vietri, *Arc-coloring of directed hypergraphs and chromatic number of walls*, submitted.

[9] A. Vietri, *The complexity of arc-colorings for directed hypergraphs*, submitted.