Non 3-Choosable Bipartite Graphs and The Fano Plane

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Abstract

It is known that the smallest complete bipartite graph which is not 3-choosable has 14 vertices. We show that the extremal configuration is unique.

1 Introduction

A *list colouring* of a graph G is a proper vertex colouring of G in which every vertex v is assigned a colour from a set (or list) $L(v)$ of colours. The graph G is *k-choosable*, or *k-list colourable*, if the condition that every list has size at least k is sufficient to guarantee the existence of a list colouring of G. The *choice number*, or *list chromatic number*, of G is the smallest positive integer k such that G is k-choosable. A recent survey on list colourings is [6].

Both Erdös, Rubin and Taylor [2] and Vizing [5] observed that the choice number of a complete bipartite graph can be arbitrarily large. Erdos, Rubin and Taylor demonstrated that the choice number of $K_{7,7}$ is at least four. Their example consisted of assigning the seven lines in the Fano plane to the seven vertices in each set in the bipartition. It is proved in [4] that the choice number of any bipartite graph with at most thirteen vertices is less than or equal to three.

In this article we indicate how it may be shown that the fourteen vertex example of Erdös, Rubin and Taylor is unique up to renaming the colours.

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The proof is a detailed case analysis. The cases to be considered are determined in Lemma 2.8. Our initial goal was to undertake a combinatorial computing project and examine all possibilities, up to isomorphism. In order to be able to limit the size of the search, results were established regarding the structure of the lists in an example where a list colouring does not exist. Eventually there were enough such results for a proof. In the interest of brevity of this article some of the proofs are only sketched, though a detailed example of each type of argument is given. Complete details of all proofs can be found in [3].

For a discussion of the many other contexts in which the Fano configuration arises, see [1].

2 Preparatory Results

We assume throughout this article that $K_{a,c}$ is a complete bipartite graph, where $a \leq c$, in which lists of size three are assigned to the vertices. We use $\mathbf{A} = \{A_1, A_2, \ldots, A_a\}$ to denote the collection of lists assigned to the vertices in one set of the bipartition, and $\mathbf{C} = \{C_1, C_2, \ldots, C_c\}$ to denote the collection of lists assigned to the vertices in other set of the bipartition. (It follows from Lemma 2.6 below that, in an extremal configuration, no two vertices belonging to the same side of the bipartition are assigned the same list. Thus, set notation is appropriate.) Define $N = |A_1 \cup A_2 \cup \cdots \cup A_a \cup C_1 \cup C_2 \cup \cdots \cup C_c|$. For integers $a \leq b$ we use $[a, b]$ to denote the set $\{a, a+1, \ldots, b\}$. We will use the following results from [4].

Theorem 2.1 ([4]) *Every bipartite graph on at most thirteen vertices is 3-choosable.*

Let $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$ be a collection of sets. A *transversal* of $\mathbf{X}$ is a set $T \subseteq X_1 \cup X_2 \cup \cdots \cup X_n$ such that $T \cap X_i \neq \emptyset$ for $i = 1, 2, \ldots, n$.

Lemma 2.2 ([4]) *Suppose the complete bipartite graph $K_{a,c}$ has the collections* $\mathbf{A}$ *and* $\mathbf{C}$ *as the lists assigned to vertices in the sets in the bipartition. The following are equivalent:*

1. *$K_{a,c}$ is not list colourable from the lists in* $\mathbf{A}$ *and* $\mathbf{C}$.
2. *Every transversal of* $\mathbf{A}$ *has a subset that is a list in* $\mathbf{C}$.
3. *Every transversal of* $\mathbf{C}$ *has a subset that is a list in* $\mathbf{A}$.
4. *Each subset of $[1, N]$ is either disjoint from some list in* $\mathbf{A}$ *or has a subset that is a list in* $\mathbf{C}$.

Lemma 2.3 ([4]) *If $K_{a,c}$ is not 3-choosable, then $N \geq 5$.*

Theorem 2.4 **([4])** *If $K_{a,c}$ is not 3-choosable, then $a \cdot \binom{N-3}{l} + c \cdot \binom{N-3}{l-3} \geq \binom{N}{l}$ for all $0 \leq l \leq N$ where $\binom{N-k}{l} = 0$ for $l > N-3$ and $\binom{N-3}{l-3} = 0$ for $l < 3$.*

Corollary 2.5 **([4])** *If $K_{a,c}$ is not 3-choosable, then $a + c \geq \frac{2\binom{N}{l}}{\binom{N-3}{l}+\binom{N-3}{l-3}}$ for all $0 \leq l \leq N$ where $\binom{N-k}{l} = 0$ for $l > N-3$ and $\binom{N-3}{l-3} = 0$ for $l < 3$.*

We will make repeated, often implicit, use of the following fact.

Proposition 2.6 **([4])** *Suppose G is a bipartite graph on fourteen vertices and there is an assignment of lists for which there is no list colouring of G. Then every colour appears in lists on both sides of the bipartition, and no list is assigned to two vertices on the same side of the bipartition.*

We will say that the pair (x, y) is in the set X when $\{x, y\} \subseteq X$.

Lemma 2.7 **([4])** *Let G be a bipartite graph in which lists have been assigned to the vertices of G. Let $N = \left|\bigcup_{v\in V} L(v)\right|$. Suppose there is no list colouring of G and some pair of colours does not appear together in a list. Then there is a collection of lists $L'(v), v \in V$, such that $|L'(v)| = |L(v)|$ for every $v \in V$, $\left|\bigcup_{v\in V} L'(v)\right| = N-1$, and G is not list colourable from the lists $L'(v), v \in V$.*

Lemma 2.8 *Suppose $a + c = 14$. If there is a collection of lists of size three for which there is no list colouring of $K_{a,c}$, then there is such a collection in one of the following situations:*

1. *$N = 7$ and $a = 7$*

2. *$N = 8$ and $a =$ 5, 6 or 7.*

3. *$N = 9$ and $a =$ 5, 6 or 7.*

Proof. Suppose $K_{a,c}$ is not 3-choosable. The fourteen lists of size three together contain 42 not necessarily distinct pairs. Since $\binom{10}{2} = 45$, by Lemma 2.7 it may be assumed that $N \leq 9$.

By Lemma 2.3, $N \geq 5$. By Proposition 2.6 no list is assigned to two vertices on the same side of the bipartition. However, the inequality from Corollary 2.5 is satisfied for neither the pair $N = 5$ and $l = 2$, nor the pair $N = 6$ and $l = 3$. Therefore, $N \geq 7$. Further, using $N = 7$ and $l = 3$ in the inequality from Theorem 2.4 we obtain $a \geq 7$. Using $N = 8$ and $l = 3$, we obtain $a \geq 5$, and using $N = 9$ and $l = 4$, we obtain $a \geq 5$. Finally, $a \leq c$ and $a + c = 14$ implies that $a \leq 7$ for any value of N. The result follows. □

Lemma 2.9 *Suppose $a + c = 14$. If any element is in more than $a - 3$ lists of* **A** *or $c - 3$ lists of* **C**, *then there is a list colouring of $K_{a,c}$.*

Proof: Suppose $K_{a,c}$ is not 3-choosable, and that **A** and **C** are collections of lists from which there is no list colouring of $K_{a,c}$. By Lemma 2.8, we have $5 \le a \le 7$ and $7 \le c \le 9$.

Now suppose some element is in $a - 2$ lists of **A**. Without loss of generality, assume $1 \in A_1 \cap A_2 \cap \cdots \cap A_{a-2}$. Then for any $x \in A_{a-1}$ and $y \in A_a$, $\{1, x, y\}$ is a transversal of **A**. By Lemma 2.2, every transversal of **A** has a subset that is a list in **C**. Hence, for every choice of x and y the elements in $\{1, x, y\}$ are all distinct and $\{1, x, y\}$ is a list in **C** . Hence, $c = 9$. However, all nine lists in **C** contain 1. Therefore, $\{1\}$ is a transversal of **C**, and, by Lemma 2.2, $K_{a,c}$ can be properly coloured, a contradiction. Thus no element is in more than $a - 3$ lists of **A**.

Suppose some element is in $c - 2$ lists of **C**. It can be similarly shown that **C** has nine transversals of size three. Since $a \le 7$, this contradicts the fact that every transversal of **C** has a subset that is a list in **A**. The result follows. □

Lemma 2.10 *Suppose $7 \le N \le 9$. If some element is in more than three lists of* **A** *(or* **C***), then there is a list colouring of $K_{7,7}$.*

Proof: Suppose **A** and **C** are collections of lists from which there is no list colouring of $K_{7,7}$. By Lemma 2.2, every transversal of **A** (respectively **C**) has size at least three, and every transversal of **A** (respectively **C**) of size three is a list in **C** (respectively **A**).

Now suppose that some element is in four lists of **A**. Without loss of generality, assume $1 \in A_i$ for all $i \in [1, 4]$. By Lemma 2.9, $1 \notin A_5 \cup A_6 \cup A_7$. Hence, $|A_5 \cup A_6 \cup A_7| \le 8$, and $|A_5 \cap A_6| + |A_5 \cap A_7| + |A_6 \cap A_7| \ge 1$. Also note that $A_5 \cap A_6 \cap A_7 = \emptyset$, since **A** has no transversal of size two.

We claim that $|A_i \cap A_j| \le 1$ for any $i, j \in [5, 7]$ where $i \ne j$. To see this, assume that $|A_5 \cap A_6| = 2$. Then, for any $x \in A_5 \cap A_6$ and any $y \in A_7$, $\{1, x, y\}$ is a transversal of **A**. Since **A** has no transversal of size two, there are six distinct transversals of this type. Since each of these six transversals is a list in **C**, there are six lists in **C** that contain 1. Lemma 2.9 then asserts that a list colouring exists, a contradiction. Hence, $1 \le |A_5 \cap A_6| + |A_5 \cap A_7| + |A_6 \cap A_7| \le 3$.

Case 1: Suppose $|A_5 \cap A_6| + |A_6 \cap A_7| + |A_5 \cap A_7| = 3$. Then any two of A_5, A_6 and A_7 have exactly one element in common. For any distinct $i, j, k \in \{5, 6, 7\}$, the set $\{1, x, y\}$ where $x \in A_i \cap A_j$ and $y \in A_k - (A_i \cap A_j)$, is a transversal of **A**. Each such transversal is a list in **C**. Hence, at least six lists in **C** contain 1. Lemma 2.9 then asserts that a list colouring exists, a contradiction.

Case 2: Suppose $|A_5 \cap A_6| + |A_6 \cap A_7| + |A_5 \cap A_7| = 2$. Without loss of generality, assume that $|A_5 \cap A_6| = 1$ and $|A_5 \cap A_7| = 1$. Since A_5, A_6 and A_7 have no common element, we may assume that $A_5 = \{2, 3, 4\}$, $A_6 = \{2, 5, 6\}$ and $A_7 = \{3, 7, 8\}$. Then $\{1, 2, 3\}$, $\{1, 2, 7\}$, $\{1, 2, 8\}$, $\{1, 3, 5\}$ and $\{1, 3, 6\}$ are all transversals of **A**. This gives five lists of **C**, each containing 1. Lemma 2.9 then asserts that a list colouring exists, a contradiction.

Case 3: Suppose $|A_5 \cap A_6| + |A_6 \cap A_7| + |A_5 \cap A_7| = 1$. Without loss of generality, we may assume $A_5 = \{2, 3, 4\}$, $A_6 = \{2, 5, 6\}$ and $A_7 = \{7, 8, 9\}$. Then $\{1, 2, 7\}$, $\{1, 2, 8\}$ and $\{1, 2, 9\}$ are transversals of **A** and therefore lists in **C**. Suppose these are C_1, C_2 and C_3, respectively.

For any $x \in A_5 - \{2\}$, $y \in A_6 - \{2\}$ and $z \in A_7$, the set $\{1, x, y, z\}$ is a transversal of **A**. Let S be the set of these twelve distinct transversals. Note that none of C_1, C_2 or C_3 is a subset of any set in S, and no list in **C** is a subset of more than three sets in S. Furthermore, a list C_i is a subset of three sets in S only if it contains 1. Since every set in S has a subset that is a list in **C**, each of C_4, C_5, C_6 and C_7 must be a subset of exactly three sets in S. Therefore, $1 \in C_i$ for all $i \in [4, 7]$. However, this means that $\{1\}$ is a transversal of **C** which is a contradiction.

Hence, if some element appears in more than three lists of **A** or more than three lists of **C**, then G can be properly coloured. □

3 Nine Colours

Lemma 3.1 *Suppose $N = 9$. If some pair appears in more than one list in* **A***, then there is a list colouring of $K_{5,9}$.*

Proof. We argue the contrapositive. Suppose that **A** and **C** are families of lists for which there is no list colouring of $K_{5,9}$. Then, by Lemma 2.2, every transversal of **A** has size at least three, and every transversal of **A** of size three is a list in **C**.

By Lemma 2.9, no element appears in more than two lists of **A**. Since every element appears in at least one list of **A**, there are six elements in exactly two lists each and three elements in exactly one list each. Say every element in $[1, 6]$ is in exactly two lists of **A**.

Suppose some pair, say $(1, 2)$, is in two lists, say A_1 and A_2. Then there are two elements of $[3, 6]$ each of which is in neither A_1 nor A_2. Assume $3 \in A_3 \cap A_4$ and 4 is in two of A_3, A_4 and A_5.

Suppose $4 \in A_3 \cap A_4$. Then for each $x \in A_1 \cap A_2$ and $y \in A_3 \cap A_4$, and $z \in A_5$, $\{x, y, z\}$ is a transversal of **A**. There are twelve such transversals. This contradicts the fact that every transversal of size three is a list in **C**.

Suppose $4 \in A_3 \cap A_5$. Then $\{1, 3, 4\}$ and $\{2, 3, 4\}$ are transversals of **A**. It is also the case that for each $x \in A_5 - \{4\}$, $\{1, 3, x\}$ and $\{2, 3, x\}$ are transversals of **A**. And, for each $y \in A_4 - \{3\}$, both $\{1, 4, y\}$ and $\{2, 4, y\}$ are transversals of **A**. This gives a total of ten transversals of **A** of size three. This contradicts the fact that every transversal of **A** of size three is a list in **C**. □

Lemma 3.2 *Suppose $N = 9$. If some element appears in more than four lists of* **C**, *then there is a list colouring of $K_{5,9}$.*

Proof Sketch: Again, we argue the contrapositive. Suppose that **A** and **C** are families of lists for which there is no list colouring of $K_{5,9}$. Then, by Lemma 2.2, every transversal of **C** has size at least three, and every transversal of **C** of size three is a list in **A**.

By Lemma 2.9, no element is in more than six lists in **C** and no element is in more than two lists of **A**.

If some element x is in exactly six lists of **C** then **C** has at least three transversals of size three containing x (and none of size two). This implies that there are three lists in **A** containing x, contradicting Lemma 2.9.

Suppose some element is in exactly five lists of **C**, say $x \in C_1 \cap C_2 \cap \cdots \cap C_5$. Then, without loss of generality, there is an element $y \in C_6 \cap C_7$. Consider the transversals of **C** of size three or four and containing the pair (x, y). It follows from Lemmas 2.9 and 3.1 that one of these transversals of **C** has no subset in **A**. □

Lemma 3.3 *If $N = 9$, then there is a list colouring of $K_{5,9}$.*

Proof Sketch: Suppose there is no list colouring of $K_{5,9}$. By Lemma 2.9 and Proposition 2.6 every element is in exactly one list of **A**. It now follows from Lemmas 2.2, 3.1 and 3.2 that the lists of **A** are completely determined up to symmetry, and some four of these lists have the property that each pair has exactly one element in common, while the fifth is disjoint from each of these four. It then can be observed that **A** has nine distinct transversals of size three, which completely determines **C**, and then that **C** has a transversal of size three which is not a list in **A**. Hence $K_{5,9}$ can be list coloured, a contradiction. □

Lemma 3.4 *Suppose $N = 9$. If some element is in more than four lists of* **C**, *then there is a list colouring of $K_{6,8}$.*

Proof: Similar to Lemma 3.2. □

Lemma 3.5 *Suppose $N = 9$ and each element appears in exactly two lists of* **A**. *If some element appears in more than three lists of* **C** *then there is a list colouring of $K_{6,8}$.*

Proof Sketch: Suppose that every element appears in exactly two lists of **A**, but there is no list colouring of $K_{6,8}$.

Suppose that some element is in four lists of **C**. Without loss of generality, assume $1 \in C_1 \cap C_2 \cap C_3 \cap C_4$. By Lemma 3.4, 1 is not in any other list of **C**. Since **C** has no transversal of size two, $C_5 \cap C_6 \cap C_7 \cap C_8 = \emptyset$. Furthermore, for any distinct $i, j, k \in [5, 8]$, $C_i \cap C_j \cap C_k = \emptyset$. Otherwise, $\{C_5, C_6, C_7, C_8\}$ would have three transversals of size two, resulting in three transversals of **C** of size three all containing 1. Since every element is in exactly two lists of **A**, this contradicts the fact that every transversal of **C** of size three is a list in **A**.

Since $C_5 \cup C_6 \cup C_7 \cup C_8 \subseteq [2, 9]$, some pair of C_5 through C_8 must have a common element. Without loss of generality, assume $2 \in C_5 \cap C_6$. Therefore, $2 \notin C_7 \cup C_8$.

Examining the possible transversals of **C** containing the pair $(1, 2)$ yields the contradiction that there is a transversal of **C** without a subset in **A**, contrary to Lemma 2.2. □

Lemma 3.6 *If $N = 9$, then for any collection of lists* **A** *and* **C**, *there is a list colouring of $K_{6,8}$.*

Proof Sketch: Assume that **A** and **C** are collections of lists for which there is no list colouring of $K_{6,8}$. Since $N = 9$, either some element appears in three lists of **A**, or every element appears in exactly two lists of **A**.

Case 1: Suppose there is an element that appears in three lists of **A**. Without loss of generality, assume $1 \in A_1 \cap A_2 \cap A_3$. By Lemma 2.9, $A_4 \cup A_5 \cup A_6 \subseteq [2, 9]$. Then two of A_4, A_5 and A_6 have an element in common. Without loss of generality, assume $2 \in A_4 \cap A_5$. If $2 \in A_6$, then **A** has a transversal of size two, a contradiction. Hence, we may assume that $A_6 = \{3, 4, 5\}$. Then $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 2, 5\}$ are transversals of **A**. We may assume these are lists C_1, C_2 and C_3, respectively.

Suppose $3 \in A_4$. Then for any $x \in A_5 - \{2\}$, $\{1, 3, x\}$ is a transversal of **A**. Since there are no transversals of size two, $3 \notin A_5$, so there are two such transversals. If both of these lists are in **C**, there are five lists in **C** containing 1, contrary to Lemma 3.4. Hence, $3 \notin A_4$. It can be similarly shown that $4, 5 \notin A_4$ and $3, 4, 5 \notin A_5$.

Suppose there is some $x \in (A_4 \cap A_5) - \{2\}$. Then $\{1, x, 3\}$, $\{1, x, 4\}$ and $\{1, x, 5\}$ are transversals of A and therefore lists in **C**. These three lists together with C_1, C_2 and C_3 give six lists in **C** all containing 1, contrary to Lemma 3.4. Hence, $A_4 \cap A_5 = \{2\}$.

Now, for any $x \in A_4 - \{2\}$ and $y \in A_5 - \{2\}$, $\{1, 3, x, y\}$ is a transversal of **A**. Let S be the set of these four distinct transversals. Note that none of C_1, C_2 or C_3 is a subset of any set in S.

If none of C_4 through C_8 contains the pair $(1,3)$, then for each choice of x and y, either $\{1,x,y\}$ or $\{3,x,y\}$ is a list in $\mathbf{C}$. This means C_4 through C_8 contain two lists with 1 or three lists with 3. Thus there are five lists of $\mathbf{C}$ containing 1 or five lists of $\mathbf{C}$ containing 3, contrary to Lemma 3.4. Therefore, at least one of C_4 through C_8 contains the pair $(1,3)$. Without loss of generality, assume $(1,3)$ is in C_4.

Now, let S' be set of elements of S that have no subset in $\{C_1, C_2, C_3, C_4\}$. Since C_4 is a subset of at most two sets in S, $|S'| \geq 2$. By Lemma 3.4, none of C_5 through C_8 contains 1. Since each set in S' has a subset that is a list in $C' = \{C_5, C_6, C_7, C_8\}$, at least two lists in C' contain 3. Say $3 \in C_5 \cap C_6$. Hence, $2 \in C_1 \cap C_2 \cap C_3$ and $3 \in C_4 \cap C_5 \cap C_6$. Hence, for any $x \in C_7$ and $y \in C_8$, $\{2,3,x,y\}$ is a transversal of $\mathbf{C}$. Let T be the set of these transversals. Since $1 \notin C_7 \cup C_8$, none of A_1, A_2 or A_3 is a subset of any transversal of the form $\{2,3,x,y\}$. Since the pair $(2,3)$ is in no list of $\mathbf{A}$, there is no choice of x and y such that $x = y$. Hence, $|T| = 9$, and a list of $\mathbf{A}$ is a subset of at most one set in T. Hence, for each choice of x and y, either $\{2,x,y\}$ or $\{3,x,y\}$ is a list in $\mathbf{A}$. This contradicts the fact that $\mathbf{A}$ contains six lists.

Case 2: (*sketch*) Suppose every element appears in exactly two lists of $\mathbf{A}$. Without loss of generality, assume $1 \in A_1 \cap A_2$, $2 \notin A_1 \cup A_2$ and $2 \in A_3 \cap A_4$. Then for any $x \in A_5$ and $y \in A_6$, $\{1,2,x,y\}$ is a transversal of $\mathbf{A}$. Let $\mathbf{C}_{1,2}$ be the number of lists in $\mathbf{C}$ containing the pair $(1,2)$.

We claim that $\mathbf{C}_{1,2} = 3$. Suppose to the contrary that $\mathbf{C}_{1,2} \leq 2$.

Suppose $|A_5 \cap A_6| = 2$. Then, without loss of generality, $A_5 = \{3,4,5\}$ and $A_6 = \{3,4,6\}$. By considering transversals of $\mathbf{A}$, the conclusion that 3 belongs to four subsets in $\mathbf{C}$ is reached, contary to Lemma 3.5.

Suppose $|A_5 \cap A_6| = 1$. The situation where $|A_1 \cap A_2| = 2$ or $|A_3 \cap A_4| = 2$, is covered by the argument above, so assume $|A_1 \cap A_2| = |A_3 \cap A_4| = 1$. Let $A_1 = \{1,u_1,u_2\}, A_2 = \{1,v_1,v_2\}, A_3 = \{2,w_1,w_2\}, A_4 = \{2,x_1,x_2\}, A_5 = \{3,y_1,y_2\}$ and $A_5 = \{3,z_1,z_2\}$. Then, for $i,j \in \{1,2\}$ we have $u_i \neq v_j$, $w_i \neq x_j$, and $y_i \neq z_j$. The set $\{1,2,3\}$ is a transversal of $\mathbf{A}$, hence it is a list in $\mathbf{C}$. Let $S = \{\{1,2,y_i,z_j\}, \{1,3,w_i,x_j\}, \{2,3,u_i,v_j\} : 1 \leq i,j \leq 2\}$. Then $|S| = 12$. Each element of S is a transversal of $\mathbf{A}$, so each has a subset that is a list in $\mathbf{C}$. This gives sufficient structural information about the lists in $\mathbf{C}$ to conclude that either $\{1,2,3\}$ or $\{1,2,3,p\}$ is a transversal of $\mathbf{C}$, for some $p \in [4,9]$. No such transversal has a subset that is a list in $\mathbf{A}$, a contradiction.

Suppose $|A_5 \cap A_6| = 0$. Then $\mathbf{A}$ has nine transversals of the form $\{1,2,x,y\}$, where $x \in A_5$ and $y \in A_6$. One of these has no subset that is a list in $\mathbf{C}$, contrary to Lemma 2.2.

Therefore, $C_{1,2} \geq 3$. Without loss of generality, assume $C_1 = \{1,2,3\}$, $C_2 = \{1,2,4\}$ and $C_3 = \{1,2,5\}$. By Lemma 3.5, $1,2 \notin C_4 \cup C_5 \cup \cdots \cup C_8$.

This proves the claim.

It now follows that each pair from $\{1,2\} \times [6,9]$ must appear in some list in **A**. Hence, $A_1 \cup A_2 \subsetneq \{1\} \cup [6,9]$ and $A_3 \cup A_4 \subsetneq \{2\} \cup [6,9]$. Since each element is in exactly two lists of **A**, this means $A_5 \cup A_6 \subsetneq \{3,4,5\}$. This implies $A_5 = A_6$ which is a contradiction. Hence, there is a list colouring of $K_{6,8}$. □

Lemma 3.7 *Suppose $N = 9$. If any pair appears in more than two lists of* **A** *(or* **C***), then there is a list colouring of $K_{7,7}$.*

Proof: Suppose some pair, say $(1,2)$, is in each of A_1, A_2 and A_3. Since these three lists are all distinct, assume that $A_1 = \{1,2,3\}$, $A_2 = \{1,2,4\}$ and $A_3 = \{1,2,5\}$. By Lemma 2.10, we may assume no element appears in more than three lists of **A**. Then $A_i \subsetneq [3,9]$ for $i \in [4,7]$. Furthermore, by Lemma 2.2, every transversal of **A** of size three is a list in **C**.

Let $Z = [1,5] \times [6,9]$. None of the lists A_1, A_2 and A_3 contains a pair from Z, and every other list in **A** or **C** contains either no pairs or exactly two pairs from Z. Therefore, **C** contains at most fourteen pairs from Z. By Lemma 2.7, we may assume that every pair in Z is in some list of **A** or **C**. Since Z contains twenty pairs, **A** contains at least six pairs from Z. Without loss of generality, we may assume that A_4, A_5 and A_6 each contain two pairs from Z. Since each of these lists is a subset of $[3,9]$, $A_i \cap [3,5] \neq \emptyset$ for $i \in [4,6]$.

Case 1: Suppose none of 3, 4 or 5 is in two of the lists A_4, A_5, A_6 and A_7. Then $|A_i \cap [3,5]| = 1$ for $i \in [4,6]$ and $A_7 \subsetneq [6,9]$. Without loss of generality, assume $A_7 = \{7,8,9\}$. Then exactly six pairs from Z are in **A** and fourteen pairs are in **C**. Therefore, no pair from Z is in more than one list of **A** or **C**. On the other hand, it follows from considering transversals of **A** of size three that some pair from Z belongs to more than one list in **C**. Therefore, there is a list colouring of $K_{7,7}$.

Case 2 (*sketch*): Suppose one of 3, 4 or 5 is in two of the lists A_4, A_5, A_6 and A_7. Without loss of generality, assume that $3 \in A_4 \cap A_5$. Since 3 is in at most three lists in **A**, $3 \notin A_6 \cup A_7$. The remainder of the proof is similar to that of Lemma 3.1, except transversals of size three and four are used.
□

Corollary 3.8 *If the elements x and y each appear in three lists of* **A** *(respectively* **C***), but the pair (x,y) is in no list of* **A** *(respectively* **C***), then there is a list colouring of $K_{7,7}$.*

Lemma 3.9 *Suppose $N = 9$. If there is no list colouring of $K_{7,7}$, then for any collection of lists* **A** *and* **C** *there is a list A' in* **A** *such that*

1. $|A' \cap C_i| \leq 1$ *for all* $i \in [1,7]$

2. *A' is disjoint from exactly three lists in* **A**.

Proof Sketch: By Lemma 2.7, we may assume that every pair of elements appears in some list of **A** or **C**. Since each list contains three pairs, the lists in **C** together contain at most twenty-one of these pairs. Hence, the lists of **A** contain fifteen pairs that are not found in **C**. This means some list in **A** contains three pairs not found in **C**. Let A_1 be such a list. Hence, $|A_1 \cap C_i| \leq 1$ for all $i \in [1, 7]$.

Without loss of generality, assume that $A_1 = \{1, 2, 3\}$.

Case 1 (*sketch*): Suppose that A_1 is disjoint from at most two lists in **A**. Then one of the five element transversals of **A** containing 1, 2, and 3, has no subset that is a list in **C**, a contradiction

Case 2: Suppose that four lists of **A** are disjoint from A_1. These four lists along with A_1 do not contain any pair from the set $X = [1, 3] \times [4, 9]$. The remaining two lists in **A**, say A_2 and A_3, each contain at most two pairs from X. By Lemma 2.7 we may assume that every pair from X is in some list of **A** or **C**. Since the lists of **C** collectively contain at most fourteen of the pairs from X, then each of A_2 and A_3 contain exactly two pairs and C_i contains exactly two pairs for each $i \in [1, 7]$. Furthermore, no pair from X is in two lists of **A** or **C**.

For each $i \in [1, 3]$, define X_i to be the set $\{i\} \times [4, 9]$. Since each list in **C** contains a pair from X, $|C_i \cap A_1| \geq 1$ for all $i \in [1, 7]$. Hence $|C_i \cap A_1| = 1$ for all $i \in [1, 7]$. Therefore, each list in **C** contains either no pairs or exactly two pairs from X_1. Hence, the lists of **C** collectively contain an even number of pairs from X_1. Since X_1 contains six pairs, the lists A_2 and A_3 together contain an even number of pairs from X_1. Similar statements can be made regarding X_2 and X_3.

The collection **A** contains exactly four pairs from X. Suppose **A** contains four pairs from X_1, say, and none from X_2 and X_3. Then, **C** contains two pairs from X_1 and six from each of X_2 and X_3. Without loss of generality, $1 \in C_1, 2 \in C_2 \cap C_3 \cap C_4, 3 \in C_5 \cap C_6 \cap C_7$, and $2, 3 \notin C_1$. Thus, there exist two transversals of **C** of the form $\{x, 2, 3\}$, where $x \in C_1 - \{1\}$. But only one list in **A** contains the pair $(2, 3)$, so this is impossible. Thus, **A** must contain exactly two pairs from each of two of the sets X_1, X_2 and X_3. Without loss of generality, assume **A** contains no pair from X_1, and contains two pairs from each of X_2 and X_3.

If each of A_2 and A_3 were to contain one pair from each of X_2 and X_3, then the pair $(2, 3)$ would be in both A_2 and A_3. Since this pair is also in A_1, by Lemma 3.7 a list colouring would exist. Hence, we may assume that A_2 contains two pairs from X_2 and A_3 contains two pairs from X_3.

This means **C** contains six pairs from X_1, and four pairs from each of X_2 and X_3. Since $|C_i \cap \{1, 2, 3\}| = 1$ we may assume that $1 \in C_1 \cap C_2 \cap C_3$, $2 \in C_4 \cap C_5$ and $3 \in C_6 \cap C_7$. Since no two lists in **C** contain the same pair from X, $C_6 \cap C_7 = \{3\}$. Hence, there are four distinct transversals of $\mathbf{C}$ of the form $\{1, 2, x, y\}$ where $x \in C_6 - \{3\}$ and $y \in C_7 - \{3\}$. By Lemma 2.2, we may assume that each of these transversals has a subset that is a list in $\mathbf{A}$. However, A_1 is not a subset of any such transversal. Neither are any of the four lists in $\mathbf{A}$ disjoint from A_1, nor is A_3. This implies A_2 is a subset of all four transversals which is impossible.

Hence, A_1 is disjoint from exactly three lists in $\mathbf{A}$. □

Theorem 3.10 *Suppose $N = 9$. Then for any collection of lists* $\mathbf{A}$ *and* $\mathbf{C}$, *$K_{7,7}$ can be properly coloured.*

Proof Sketch: By Lemma 2.7, we assume that every pair is in some list of $\mathbf{A} \cup \mathbf{C}$, and, by Lemma 2.2, that every transversal of $\mathbf{A}$ (respectively $\mathbf{C}$) has a subset that is a list in $\mathbf{C}$ (respectively $\mathbf{A}$). By Lemma 3.9, we can assume that $A_1 = \{1, 2, 3\}$, $A_i \cap \{1, 2, 3\} \neq \emptyset$ for all $i \in [2, 4]$, $A_i \cap \{1, 2, 3\} = \emptyset$ for all $i \in [5, 7]$ and $|C_i \cap \{1, 2, 3\}| \leq 1$ for all $i \in [1, 7]$. This means that $\mathbf{A}$ contains at least four and at most six pairs from the set $X = [1, 3] \times [4, 9]$ and $\mathbf{C}$ contains at least twelve and at most fourteen pairs from X. Hence, at least six lists in $\mathbf{C}$ contain an element from $\{1, 2, 3\}$.

Define X_1, X_2 and X_3 as in the previous proof.

Case 1 (*sketch*): Suppose $C_i \cap \{1, 2, 3\} \neq \emptyset$ for all $i \in [1, 7]$. By Lemma 2.10 and Corollary 3.8, we may assume that $1 \in C_1 \cap C_2 \cap C_3$, $2 \in C_4 \cap C_5$ and $3 \in C_6 \cap C_7$. Hence, $\mathbf{C}$ contains at most four pairs from each of X_2 and X_3, which implies that $\mathbf{A}$ contains at least two pairs from each of X_2 and X_3. Now, consider the transversals of $\mathbf{C}$ of size three or four containing the pair $(1, 2)$. One of these has no subset which is a list in $\mathbf{A}$, a contradiction.

Case 2 (*sketch*): Suppose exactly six lists in $\mathbf{C}$ contain an element from $\{1, 2, 3\}$ and some element in $\{1, 2, 3\}$ is in only one list of $\mathbf{C}$. By Lemma 2.10 and Corollary 3.8, we may assume that $1 \in C_1 \cap C_2 \cap C_3$, $2 \in C_4 \cap C_5$, $3 \in C_6$ and $C_7 \cap \{1, 2, 3\} = \emptyset$. Therefore, at most four pairs from X_2 and at most two pairs from X_3 are in $\mathbf{C}$, which implies that $\mathbf{A}$ contains two pairs from X_2 and four pairs from X_3. Since $3 \in A_1$ and no element appears in $\mathbf{A}$ four times it can be assumed without loss that $3 \in A_3 \cap A_4$. In turn, this implies that $2 \in A_2$ and $A_2 \subseteq \{2\} \cup [4, 9]$. For any $x \in C_6 - \{3\}$ and $y \in C_7$, the set $\{1, 2, x, y\}$ is a transversal of $\mathbf{C}$. At least two of these transversals have no subset that is a list in $\mathbf{A}$, a contradiction.

Case 3: Exactly six lists in $\mathbf{C}$ contain an element from $\{1, 2, 3\}$ and every element in $\{1, 2, 3\}$ is in exactly two lists of $\mathbf{C}$. Without loss of generality, assume $1 \in C_1 \cap C_2$, $2 \in C_3 \cap C_4$, $3 \in C_5 \cap C_6$ and $C_7 \cap \{1, 2, 3\} = \emptyset$. Since $\mathbf{C}$ contains at least twelve pairs from X, $\mathbf{C}$ contains exactly four pairs from each of X_1, X_2 and X_3. Since $\mathbf{A}$ contains at most six pairs from X, then $\mathbf{A}$ contains two pairs from each of X_1, X_2 and X_3. Therefore, $\{1, 2, 3\} \subseteq A_2 \cup A_3 \cup A_4$.

Suppose that the pair $(1,2)$ appears in some list in $\mathbf{A}$ besides A_1. Without loss of generality assume that $1,2 \in A_2$. Since there is only one pair from each X_1 and X_2 in A_2, $1,2 \in A_3 \cup A_4$. Since the pair $(1,2)$ is in both A_1 and A_2, by Lemma 3.7, we may assume that $1 \in A_3$ and $2 \in A_4$. Since $A_i \subseteq [4,9]$ for $i = 5,6,7$ then for some $i \neq j$, $i,j \in [5,7]$, $A_i \cap A_j \neq \emptyset$. Assume that $4 \in A_5 \cap A_6$. Then for any $y \in A_7$, the set $\{1,2,4,y\}$ is a transversal of $\mathbf{A}$. Since no list in $\mathbf{C}$ contains the pair $(1,2)$ and every transversal of $\mathbf{A}$ of size three is a list in $\mathbf{C}$, $4 \notin A_7$. Hence, there are three transversals of $\mathbf{A}$ of the form $\{1,2,4,y\}$. Since no list in $\mathbf{C}$ contains both 1 and 2 then for any $y \in A_7$ there is a list $\{a,4,y\}$ in $\mathbf{C}$ for some $a \in \{1,2\}$. Hence, either two lists of $\mathbf{A}$ contain the pair $(1,4)$ or two lists contain the pair $(2,4)$. This contradicts the fact that no pair from X appears in more than one list. Hence, the pair $(1,2)$ is in no list of $\mathbf{A}$ besides A_1.

We may similarly assume that each of the pairs $(1,3)$ and $(2,3)$ appears in no list of $\mathbf{A}$ besides A_1. Hence, $|A_i \cap \{1,2,3\}| = 1$ for all $i = 2,3,4$. Since $\{1,2,3\} \subseteq A_2 \cup A_3 \cup A_4$, we may assume that $1 \in A_2$, $2 \in A_3$ and $3 \in A_4$.

Recall that $C_7 \subseteq [4,9]$. Without loss of generality, assume $C_7 = \{7,8,9\}$. Now , for any $a \in \{1,2,3\}$, at most two pairs from $\{a\} \times [7,9]$ are in $\mathbf{A}$. Hence, $\mathbf{C}$ contains at least one such pair. This means $C_7 \cap (C_{2i-1} \cup C_{2i}) \neq \emptyset$ for all of $i = 1,2$ and 3. Assume $C_7 \cap C_i \neq \emptyset$ for $i = 1,3,5$.

Let $C_2 = \{1,y,z\}$. Then for any $a \in C_1 \cap C_7$ the sets $\{2,3,a,y\}$ and $\{2,3,a,z\}$ are transversals of $\mathbf{C}$. Since the only list in $\mathbf{A}$ containing both 2 and 3 is $\{1,2,3\}$ there is no choice of a such that $a = y$ or $a = z$. Since A_3 and A_4 are the only lists besides $\{1,2,3\}$ containing 2 and 3 respectively, assume $A_3 = \{2,a,y\}$ and $A_4 = \{3,a,z\}$.

Let $C_4 = \{2,w,x\}$. Then for any $b \in C_3 \cap C_7$, $\{1,3,b,w\}$ and $\{1,3,b,x\}$ are transversals of $\mathbf{C}$. As before, we can conclude that $A_2 = \{1,b,w\}$ and $A_4 = \{3,b,x\}$. From the two representations of A_4, we have $\{a,z\} = \{b,x\}$. Since no pair in X is in both A_3 and C_4 then $\{w,x\} \cap \{a,y\} = \emptyset$. Hence, $b = a$. However, this means that A_2 and C_1 both contain the pair $(1,a)$ which contradicts the fact that no pair from X is in more than one list. □

4 Eight Colours

Lemma 4.1 *Suppose $a + c = 14$ and $N = 8$. If there is no list colouring of $K_{a,c}$ then one of the following holds.*

1. *For any $i \neq j$, $|A_i \cap A_j| \leq 1$.*
2. *For any $i \neq j$, $|C_i \cap C_j| \leq 1$.*
3. *For any i and j, $|A_i \cap C_j| \geq 1$.*

Proof Sketch: It follows from Lemma 2.2 that every 4-subset of $[1,8]$ is either disjoint from exactly one list in **A** or has exactly one subset in **C**, but not both. □

Lemma 4.2 *Suppose $a+c=14$ and $N=8$. If some element is in more than $a-4$ lists of* **A** *(respectively* **C***) then there is a list colouring of $K_{a,c}$.*

Proof: Similar to Lemma 2.9. □

Theorem 4.3 *Suppose $a+c=14$, and $N=8$. Then for any collection of lists* **A** *and* **C***, there is a list colouring of $K_{a,c}$.*

Proof: Suppose $a=5$. Since $N=8$, some element appears in at least two lists of **A**. Hence, there is an element in more than $a-4$ elements of **A** and, by Lemma 4.2, a list colouring exists.

Suppose $a=6$. Since $N=8$, some element appears in three lists of **A**. Therefore, by Lemma 4.2, a list colouring exists.

Suppose $a=7$, and **A** and **C** are collections of lists for which $K_{7,7}$ has no list colouring. Then, by Lemma 4.2, no element is in more than three lists of **A**. Hence, at least five elements are in exactly three lists of **A**. It follows that for some A_i, every element in A_i appears in exactly three lists of **A**. Assume that $A_1=\{1,2,3\}$ and each element in A_1 is in three lists of **A**. Furthermore, by Lemma 4.1, A_1 is the only list in **A** containing any of the pairs $(1,2)$, $(1,3)$ and $(2,3)$. Hence, we may assume without loss of generality that $A_2=\{1,4,5\}$, $A_3=\{1,6,7\}$, $2\in A_4\cap A_5$, $3\in A_6\cap A_7$.

The sets $\{1,2,3\}$, $\{2,3,4,6\}$, $\{2,3,4,7\}$, $\{2,3,5,6\}$ and $\{2,3,5,7\}$ are all transversals of **A**. Therefore, by Lemma 2.2, they have subsets that are lists in **C**. Hence, we may assume that $C_1=\{1,2,3\}$. Since the pair $(2,3)$ is in at most one list of **C**, either $\{2,4,6\}$ or $\{3,4,6\}$ is a list in **C**. Without loss of generality, assume $C_2=\{2,4,6\}$. By Lemma 4.1, it follows that $\{3,4,7\}$, $\{3,5,6\}$ and $\{2,5,7\}$ are lists of **C**. We may assume these are lists C_3 through C_5, respectively.

By Lemma 4.1, $A_4\cap C_3$ and $A_4\cap C_4$ are both non-empty. Since 3 is not in A_4, $A_4\cap\{4,7\}\neq\emptyset$ and $A_4\cap\{5,6\}\neq\emptyset$. It can be similarly shown that $A_5\cap\{4,7\}\neq\emptyset$ and $A_5\cap\{5,6\}\neq\emptyset$. Hence, $A_4, A_5\subseteq\{2,4,5,6,7\}$. Since A_4 and A_5 are the only lists in **A** containing 2, the pair $(2,8)$ is in no list in **A**. However, the pair $(2,8)$ is in none of C_1, C_2 or C_5. Since these are the only lists in **C** containing 2, the pair $(2,8)$ is not in any list of **C**. By Lemma 2.7, this implies a list colouring exists, a contradiction. □

5 Seven Colours

Lemma 5.1 *Suppose $N=7$. Either every element appears in exactly three lists of* **A** *(respectively* **C***) or there is a list colouring of $K_{7,7}$.*

Proof. By Lemma 2.10 no element is in more than three lists in **A**. Since $|\mathbf{A}| = 7$, each element is in exactly three lists in **A**. □

Lemma 5.2 *Suppose $N = 7$ and* **A** *and* **C** *are collections of lists for which $K_{7,7}$ is not list colourable. Then every pair of elements appears in exactly one list of* **A** *(respectively* **C***).*

Proof. By Lemma 2.10, no element is in more than three lists of **A** or three lists of **C**. Since $a = c = 7$, each element must appear in exactly three lists in each of **A** and **C**.

Now suppose the pair $(1, 2)$ is in no list of **A**. We may assume, without loss of generality, that $1 \in A_1 \cap A_2 \cap A_3$, $2 \in A_4 \cap A_5 \cap A_6$ and $A_7 = \{3, 4, 5\}$. Then the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 2, 5\}$ are transversals of **A**, and, by Lemma 2.2, lists in **C**.

Since 6 is in exactly three lists of **A**, it must either be in at least two of A_1, A_2 and A_3 or at least two of A_4, A_5 and A_6. Without loss of generality, assume $6 \in A_1 \cap A_2$. Consider any $x \in A_3 - \{1, 7\}$. Then $x \in [3, 6]$. If $x = 6$ then the set $\{2, 3, 6\}$ is a transversal of **A**, otherwise the set $\{2, 6, x\}$ is a transversal of **A**. In either case, the transversal is not a list in **C** since at most three lists of **A** contain 2. This contradicts Lemma 2.2.

Therefore, every pair of elements appears in some list in **A**. Since there are twenty-one such pairs and each of the seven lists in **A** contain exactly three, each pair appears in exactly one list in **A**. □

Lemma 5.3 *Suppose $N = 7$ and* **A** *and* **C** *are collections of lists for which $K_{7,7}$ is not list colourable. Then* $\mathbf{A} = \mathbf{C}$.

Proof. Without loss of generality, assume $A_1 = \{1, 2, 3\}$. Then, by Lemma 5.2, none of $(1, 2)$, $(2, 3)$ or $(1, 3)$ is in A_i for any $i \in [2, 7]$. By Lemma 2.10, each of 1, 2 and 3 is in exactly three lists. We may therefore assume, without loss of generality, that $1 \in A_2 \cap A_3$, $2 \in A_4 \cap A_5$ and $3 \in A_6 \cap A_7$. Hence, the set $\{1, 2, 3\}$ is a transversal of **A**. By Lemma 2.2, this set must be a list in **C**. Therefore, every list in **A** is also in **C** and it follows that $\mathbf{A} = \mathbf{C}$. □

Theorem 5.4 *Suppose $a + c = 14$. Then, for any assignment of lists of size three to its vertices, there is a list colouring of $K_{a,c}$, unless $a = c = 7$ and the collections of lists* **A** *and* **C** *are the following, up to isomorphism:*

$$\mathbf{A} = \mathbf{C} = \{1, 2, 3\}, \{1, 4, 7\}, \{1, 5, 6\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 5\}, \{3, 6, 7\}$$

Proof. It has been shown that unless $N = 7$ and $a = c = 7$ there is a list colouring of $K_{a,c}$. Hence, assume that $N = a = c = 7$. In **A** we are requiring seven 3-subsets of $[1, 7]$ such that every pair appears in exactly one subset. This is a Steiner Triple System on seven points, and it is well known that there is only one of these up to isomorphism. □

Corollary 5.5 *Let G be a bipartite graph on at most fourteen vertices. Then G is 3-choosable unless $G = K_{7,7}$.*

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