All 2-Regular Leaves of Partial 6-cycle Systems

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Abstract

In this paper, we find necessary and sufficient conditions for the existence of a 6-cycle system of $K_n - E(R)$ for every 2-regular not necessarily spanning subgraph R of K_n.

1 Introduction

An $H-$*decomposition* of the graph G is a partition of $E(G)$ such that each element of the partition induces a subgraph isomorphic to H. In the case where H is an m-cycle, such a decomposition is referred to as an *m-cycle system* of G. An m-cycle system will be formally described as an ordered pair (V,B) where V is the vertex set and B is the set of m-cycles.

Results in this area date back to the previous century, but have received a lot of attention over the past 40 years. There have been many results found on H-decompositions of G for various graphs H and G, but mainly on H-decompositions of K_n. The graphs H that have been of most interest are paths [14], m-stars [13], m-cycles [10, 6, 8], m-wheels [3] and m-nestings [3, 9] (which are two decompositions of K_n, one into m-stars and the other into m-cycles, so that each m-cycle can be paired with an m-star to form a wheel). Recently a paper by Alspach and Gavlas [1] and another by Sajner [11] settled the problem of finding the values of n for which there exists an m-cycle system of K_n and of $K_n - I$, where I is a one-factor. This can alternatively be viewed as a *partial* m-cycle system in which the set of edges not in any m-cycle is either $\emptyset$ or induces a one-factor respectively. These edges not in any m-cycle (or the subgraph they induce) are called the *leave* R.

Continuing with the theme of finding graph decompositions of graphs which are close to complete, one way to extend these results is to assume R, the leave, induces a 2-regular graph, and find the necessary and sufficient conditions for the existence of an m-cycle system of $K_n - E(R)$. This naturally generalizes the previously stated results where the leave was empty. In 1986, Colbourn and Rosa [4] used difference methods to find necessary and sufficient conditions for the existence of a 3-cycle system of $K_n - E(R)$ for *any* 2-regular graph R. In 1996, Buchanan [2] solved this problem for $m = n$, that is, for Hamilton decompositions of $K_n - E(R)$, by using amalgamations. Fu and Rodger [5], using yet a third approach to this problem, namely induction, settled the existence problem for 4-cycle systems of $K_n - E(R)$, for any 2-regular subgraph of K_n. Recently, Leach and Rodger [7] have found necessary and sufficient conditions for the existence of a Hamilton decomposition of the complete bipartite graph $K_{a,b}$ with a 2-regular leave.

In this paper, we extend these results by finding necessary and sufficient conditions for the existence of a 6-cycle system of $K_n - E(R)$, for every 2-regular not necessarily spanning subgraph R of K_n; see Theorem 4.1. The proof is a recursive construction which requires that we first solve cases where $n \leq 17$. At first sight this would seem to require constructing many partial 6-cycle systems since the number of possibilities for R is already large when $n = 17$. However, one of the appeals of this paper is that, when $n = 17$ for example, *all* possible leaves can be obtained from a single, very carefully constructed m-cycle system; see Lemma 2.6. This is achieved by using what we call the *switching process*, and is shown in Figure 1.

Let $G[S]$ denote the subgraph of G induced by S.

2 The Small Cases

A tool that we will need is from a theorem by Sotteau [12]. Sotteau proved a generalization of the following result. It is stated here for 6-cycles only.

Lemma 2.1 *There exists a 6-cycle system of $K_{a,b}$ if and only if:*

1) a and b are even,

2) 6 divides a or b, and

3) min $\{a, b\} \geq 4$.

Next we can find necessary conditions limiting the number of edges in R. These can be seen in Lemma 2.2 and Table 1. Alternatively, since R is 2-regular, these conditions can be described in terms of the number of vertices in K_n that occur in no cycle in R; such vertices are called *isolated vertices of* R. The set of isolated vertices of R is denoted by $\mathcal{I}(R)$. This is also shown in Lemma 2.2 and Table 1.

Lemma 2.2 *Let R be a 2-regular subgraph of the complete graph K_n. If there exists a 6-cycle system of $K_n - E(R)$, then n is odd and the number of edges in $K_n - E(R)$ is divisible by 6, and these hold if and only if n, $|E(R)|$, and $|\mathcal{I}(R)|$ are related as in Table 1.*

n	$12k+1$	$12k+3$	$12k+5$	$12k+7$	$12k+9$	$12k+11$
$\lvert E(R)\rvert \pmod 6$	0	3	4	3	0	1
$\lvert \mathcal{I}(R)\rvert \pmod 6$	1	0	1	4	3	4

Table 1: The number of edges in R and the number of isolated vertices of R required in order that 6 divides $|E(K_n - E(R))|$ when n is odd.

Proof: Clearly once the edges in R are removed each vertex must have even degree in order for $K_n - E(R)$ to have a 6-cycle decomposition. so n is odd. and clearly 6 must divide $|E(K_n - E(R))|$.

Suppose. n is odd and $|E(K_n - E(R))|$ is divisible by 6. Then $|E(R)| \equiv \binom{n}{2} \pmod 6$. thus giving the second line of Table 1. Also. since $|\mathcal{I}(\mathcal{R})| + |V(R)| = n$ and $|V(R)| = |E(R)|$. $|\mathcal{I}(R)| = n - |E(R)|$. thus giving the third line of Table 1. A proof of the converse statement follows similarly. □

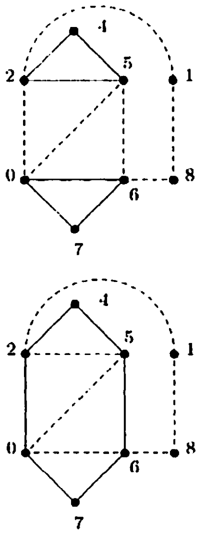
In view of Lemmas 2.1 and 2.2. throughout the rest of this paper we can assume that n is odd. and that

$$|\mathcal{I}(R)| \in \{0.1.3.4\}. \qquad (*)$$

This is possible since by Table 1 we know that $|\mathcal{I}(R)| \equiv 0.1.3.$ or $4 \pmod 6$. so if $|\mathcal{I}(R)| \geq 6$ then we can add 6-cycles to R until the resulting 2-regular graph R' satisfies $|\mathcal{I}(R')| \equiv |\mathcal{I}(R)| \pmod 6$ and $|\mathcal{I}(R')| \leq 4$. Once a set of 6-cycles B with leave R' is found. the required partial 6-cycle system is simply formed by $B \cup (R' \setminus R)$.

Lemma 2.3 *Let $n \in \{1.3.7.9.11\}$ and let R be 2-regular in K_n. If n is odd and $|E(K_n - E(R))|$ is divisible by 6. then there exists a 6-cycle system of $K_n - E(R)$.*

Proof: We consider each value of n in turn. In each case. we construct a 6-cycle system $(\mathbb{Z}_n. B)$. Using Table 1, it is easy to check that the given conditions require that:



We denote this switch by:

$\{(0,6,7),(2,4,5)\}$

$(1,8,6,5,0,2)$ | $(1,8,6,0,5,2)$

↓

$\{(0,2,4,5,6,7)\}$

Figure 1: Combining and separating leaves.

if $n = 1$, then $E(R) = \phi$; if $n = 3$, then R is a 3-cycle; if $n = 7$, then R is a 3-cycle; if $n = 9$, then R is two 3-cycles or R is one 6-cycle; and if $n = 11$, then R is either one 3-cycle and one 4-cycle or is one 7-cycle.

If $n \in \{1, 3\}$, the result is trivial; and for $n \geq 7$, in each case, we define the 6-cycle system $(\mathbb{Z}_n, B)$ as follows.

$\boldsymbol{n = 7}$: $B = \{(0,2,4,5,1,6), (0,4,3,2,6,5), (1,2,5,3,6,4)\}$ is the required 6-cycle system with leave $R = \{(0,1,3)\}$.

$\boldsymbol{n = 9}$: If R is two 3-cycles then $B_1 = \{(1,8,6,5,0,2), (0,3,2,6,4,8), (1,6,3,8,2,7), (1,4,3,7,8,5), (0,1,3,5,7,4)\}$ is the required 6-cycle system with leave $R = \{(0,6,7), (2,4,5)\}$.

If $R = \phi$ then let $B_2 = \{(1,8,6,0,5,2), (0,3,2,6,4,8), (1,6,3,8,2,7), (1,4,3,7,8,5), (0,1,3,5,7,4), (0,2,4,5,6,7)\}$ be the 6-cycle system of K_9 and if R is one 6-cycle then let B_3 be formed from B_2 by removing one 6-cycle.

$\boldsymbol{n = 11}$: If R is one 3-cycle and one 4-cycle then $B_1 = \{(8,7,0,6,2,3), (0,4,7,2,9,5), (0,8,4,1,6,9), (0,3,9,7,5,10), (1,8,6,4,9,10), (1,7,6,10,8,9), (1,3,10,4,2,5), (7,3,5,8,2,10)\}$ is the required 6-cycle system

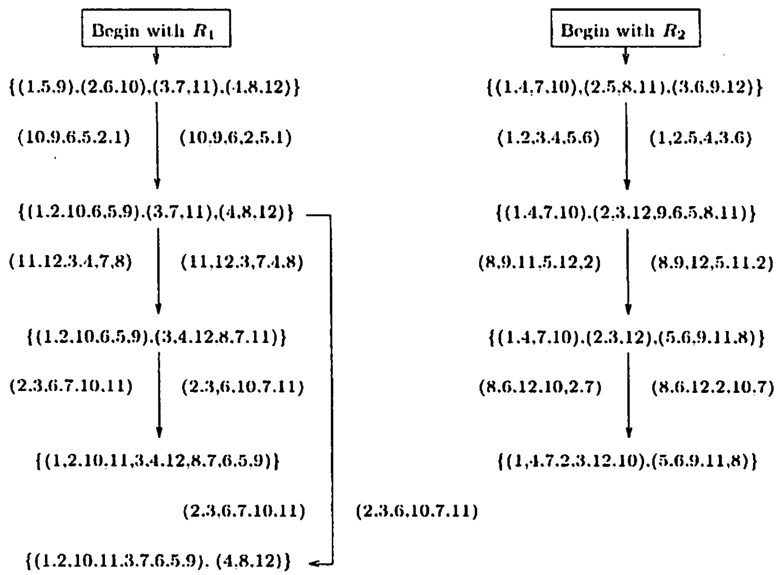


Figure 2: All possible leaves when $n = 13$.

with leave $R = \{(0,1,2),(3,4,5,6)\}$.
If R is one 7-cycle then $B_2 = \{(8,7,0,2,6,3),(0,4,7,2,9,5),(0,8,4,1,6,9),(0,3,9,7,5,10),(1,8,6,4,9,10),(1,7,6,10,8,9),(1,3,10,4,2,5),(7,3,5,8,2,10)\}$ is the required 6-cycle system with leave $R = \{(0,1,2,3,4,5,6)\}$ □

As is evidenced by Lemma 2.3, multiple leaves are possible. In fact, that is usually the case. If the 6-cycles are chosen carefully, the following technique of combining and separating leaves can be used to exhaust many possible 2-regular leaves starting with just *one* partial 6-cycle system. The case $n = 9$ will be used as an example. (This process can be represented pictorially as in Figure 1; notation used to easily describe this switch is also introduced in Figure 1.) In Lemma 2.3, $(\mathbb{Z}_9, B_1)$ has leave $R = \{(0,6,7),(2,4,5)\}$. Since the 6-cycle $(1,8,6,5,0,2)$ is in B_1, the edges $(0,2)$ and $(6,5)$ can be removed from this 6-cycle and added to the leave, and edges

(0, 6) and (2, 5) can be removed from the leave and added to this 6-cycle. The result is a new 6-cycle (1, 8, 6, 0, 5, 2) and the new leave is the 6-cycle (0, 2, 4, 5, 6, 7). This technique will be used extensively to find all possible leaves for $13 \leq n \leq 17$ in the next three results. By carefully selecting the 6-cycles in the initial partial 6-cycle system, *all* leaves can usually be obtained in this way! We call this the *switching process*.

Lemma 2.4 *If* $|E(K_{13}-E(R))|$ *is divisible by 6, then there exists a 6-cycle system* B *of* $K_{13} - E(R)$.

Proof: We start with the 6-cycle systems $(\mathbb{Z}_{13}, B_1)$ and $(\mathbb{Z}_{13}, B_2)$ defined below. Then the other possible 2-regular leaves can be formed from B_1 and B_2 by using the switching process, as is shown in Figure 2.

$B_1 = \{(10, 9, 6, 5, 2, 1), (2, 3, 6, 7, 10, 11), (11, 12, 3, 4, 7, 8), (0, 1, 11, 6, 8, 9), (0, 2, 9, 11, 5, 10), (0, 3, 8, 1, 4, 11), (0, 4, 6, 12, 5, 8), (0, 5, 4, 2, 7, 12), (0, 6, 1, 3, 5, 7), (1, 12, 10, 4, 9, 7), (2, 12, 9, 3, 10, 8)\}$
with leave $R_1 = \{(1, 5, 9), (2, 6, 10), (3, 7, 11), (4, 8, 12)\}$, and
$B_2 = \{(1, 2, 3, 4, 5, 6), (2, 8, 9, 11, 5, 12), (8, 6, 12, 10, 2, 7), (0, 1, 3, 5, 10, 8), (0, 2, 6, 11, 4, 12), (0, 3, 8, 1, 12, 11), (0, 4, 2, 9, 7, 5), (0, 6, 7, 1, 9, 10), (0, 7, 11, 1, 5, 9), (10, 3, 7, 12, 8, 4), (10, 6, 4, 9, 3, 11)\}$
with leave $R_2 = \{(1, 4, 7, 10), (2, 5, 8, 11), (3, 6, 9, 12)\}$.

□

Lemma 2.5 *If* $|E(K_{15}-E(R))|$ *is divisible by 6, then there exists a 6-cycle system* B *of* $K_{15} - E(R)$.

Proof: We start with the 6-cycle system $B = \{(3, 8, 0, 4, 1, 7), (8, 14, 1, 3, 2, 4), (14, 2, 5, 9, 3, 10), (6, 11, 14, 0, 12, 2), (0, 13, 7, 4, 6, 5), (13, 8, 1, 10, 4, 9), (12, 5, 7, 11, 8, 9), (1, 6, 10, 13, 11, 12), (0, 3, 11, 4, 14, 9), (0, 6, 3, 13, 5, 10), (0, 7, 9, 1, 5, 11), (1, 11, 2, 9, 6, 13), (2, 7, 14, 3, 12, 8), (2, 10, 7, 12, 4, 13), (5, 8, 10, 12, 6, 14)\}$ with leave $R = \{(0, 1, 2), (3, 4, 5), (6, 7, 8), (9, 10, 11), (12, 13, 14)\}$. The other possible 2-regular leaves can be formed from B by using the switching process, as is shown in Figure 3. □

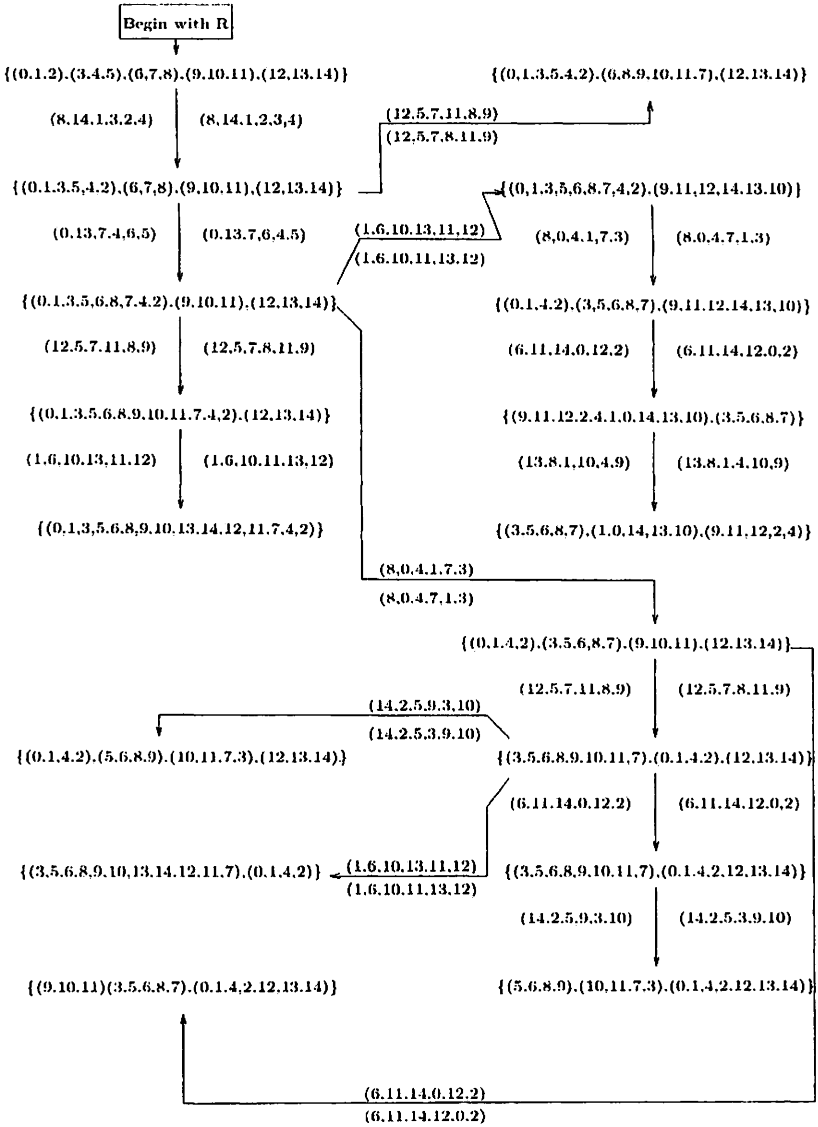


Figure 3: All possible leaves when $n = 15$.

Lemma 2.6 *If $|E(K_{17}-E(R))|$ is divisible by 6, then there exists a 6-cycle system B of $K_{17}-E(R)$.*

Proof: Throughout this lemma, we have $n = 17$. Rather than replacing n with 17, we feel this will give the reader a good feeling for the flavor of the proofs in the general cases. Let $R[v]$ denote the subgraph of R induced by the vertex set V.

First suppose R contains a 4-cycle, say $c = (n-1, n-4, n-2, n-3)$. Let $V(R') = \mathbb{Z}_{n-4}$ and $R' = R[\mathbb{Z}_{n-4}]$. Since $\mathbb{Z}_{n-4} = \mathbb{Z}_{13}$, by Lemmas 2.2 and 2.4, there exists a 6-cycle system $(\mathbb{Z}_{n-4}, B_1)$ of $K_{n-4} - E(R')$. By Lemma 2.1, there exists a 6-cycle system $(\mathbb{Z}_n \setminus \{0\}, B_2)$ of $K_{12,4}$ with bipartition $\{\mathbb{Z}_{n-13} \setminus \mathbb{Z}_1, \mathbb{Z}_{17} \setminus \mathbb{Z}_{13}\}$ of the vertex set; name these so that $c = (n-1, 1, n-3, 2, n-4, 4) \in B_2$. We still have left to place the edges $\{n-1, n-2\}$, $\{n-3, n-4\}$, and $\{0, n-i\}$ for $1 \leq i \leq 4$ into a 6-cycle. Then $(\mathbb{Z}_n, B_1 \cup (B_2 \setminus \{c\}) \cup \{(n-1, 0, n-3, 2, n-4, 4), (0, n-2, n-1, 1, n-3, n-4,)\})$ is a 6-cycle system of $K_n - E(R)$.

If R contains no 4-cycle, then start with the 6-cycle system $(\mathbb{Z}_{17}, B)$ of $K_{17} - E(R)$, where
$B = \{(0,3,11,4,10,8), (0,10,7,9,8,13), (0,12,1,11,2,14), (2,6,3,12,5,16), (4,9,6,16,7,15), (0,5,2,10,15,11), (0,6,15,13,11,16), (2,8,3,13,16,12), (2,7,11,14,12,9), (5,11,6,12,10,14), (0,1,10,16,8,4), (0,2,4,13,5,7), (0,9,11,8,12,15), (1,7,14,9,5,3), (1,6,4,12,7,13), (1,4,7,3,15,9), (1,8,14,3,9,16), (2,13,10,5,8,15), (3,10,6,14,4,16), (5,6,13,14,1,15)\}$; so the leave $R = \{(1,2,3,4,5), (6,7,8), (9,10,11,12,13), (14,15,16)\}$. The other possible 2-regular leaves can be formed from B by using the switching process, as is shown in Figure 4. $\square$

3 Some Building Blocks

In this section, we provide some 6-cycle systems of small graphs which will be used to build 6-cycle systems of $K_n - E(R)$ in Section 4. Let G^c denote

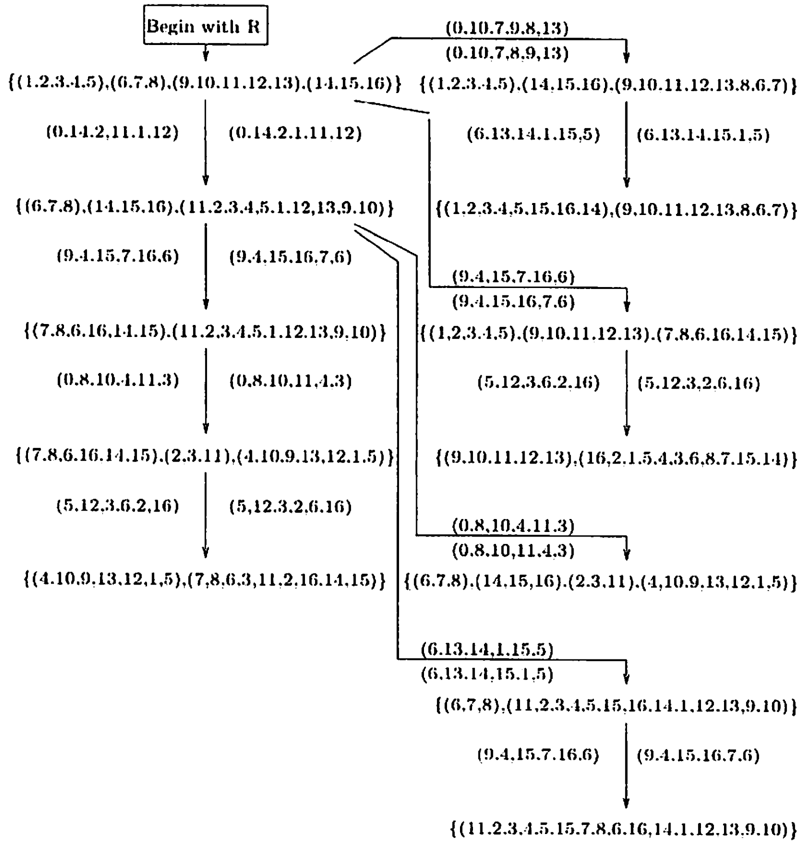


Figure 4: All possible leaves (with no 4-cycles) when $n = 17$.

the complement of a graph G. Also let $G \vee H$ denote the *join* of two vertex disjoint graphs G and H (so $E(G \vee H) = E(G) \cup E(H) \cup \{u, v : u \in V(G), v \in V(H)\}$).

Lemma 3.1 *Define four graphs, each with vertex set* $\mathbb{Z}_9$, *as follows. Let* G_1 *be the graph* $K_3^c \vee K_6 - \{\{0,8\},\{0,7\},\{8,7\},\{1,6\},\{1,5\},\{6,5\},\{2,4\},\{2,3\},\{4,3\}\}$, *let* G_2 *be the graph* $K_3^c \vee K_6 - \{\{0,8\},\{0,7\},\{1,7\},\{1,6\},\{2,5\},\{2,4\},\{8,6\},\{5,3\},\{4,3\}\}$, *let* G_3 *be the graph* $K_3^c \vee K_6 - \{\{0,8\},\{0,7\},\{1,7\},\{1,6\},\{2,4\},\{2,3\},\{8,5\},\{6,5\},\{4,3\}\}$, *and let* G_4 *be* *the graph* $K_3^c \vee K_6 - \{\{0,8\}, \{0,7\}, \{1,7\}, \{1,6\}, \{2,6\}, \{2,5\}\}$, *with* $V(K_3^c) = \mathbb{Z}_3$ *and* $V(K_6) = \mathbb{Z}_9 \setminus \mathbb{Z}_3$ *for each case. There exists a 6-cycle system* B *of* G_β *for* $1 \le \beta \le 4$.

Proof: For $1 \le \beta \le 4$, there exists a 6-cycle system $(\mathbb{Z}_9, B_\beta)$ of G_β defined by

$B_1 = \{(0,4,1,8,6,3), (0,5,2,7,4,6), (1,4,0,3,6,8), (2,6,7,3,5,8)\}$,
$B_2 = \{(0,4,8,7,2,6), (0,3,8,1,4,5), (1,3,6,4,7,5), (2,3,7,6,5,8)\}$,
$B_3 = \{(0,3,6,7,5,4), (0,5,1,4,8,6), (1,3,5,2,7,8), (2,6,4,7,3,8)\}$, and
$B_4 = \{(0,5,8,2,3,6), (0,3,5,7,8,4), (1,4,2,7,6,8), (1,3,7,4,6,5)\}$. □

Lemma 3.2 *Let* G *be the graph* $K_n - E(R)$ *for some 2-regular subgraph* R *of* K_n, *where 6 divides* $|E(K_n - E(R))|$, n *is odd, and* $n \ge 25$. *Furthermore, if* R *contains* x *cycles, then let them have lengths* $l_1, l_2, l_3, \ldots, l_x$. *Suppose that if* R' *is any 2-regular subgraph of* K_{n-12} *such that 6 divides* $|E(K_{n-12}) - E(R')|$ *then there exists a 6-cycle system of* $K_{n-12} - E(R')$. *If there exists a subset* L *of* $\{1, 2, \ldots, x\}$, *say* $\{1, 2, \ldots, y\}$ *such that for each* i *in* L *there exists an integer* j_i *such that*
(i) $3 \le j_i \le 9$ *or* $j_i = 12$, *and*
(ii) $l_i - j_i \ge 3$ *or* $l_i = j_i$
and such that $\sum_{i=1}^{y} j_i = 12$ *(so clearly* $y \le 4$*), then there exists a 6-cycle system of* $K_n - E(R)$.

Proof: Consider a complete graph of order n with vertex set N that contains the 17 vertices $\{0, a_i, b_i, c_i, d_i \mid 1 \le i \le 4\}$ with $|N| = n$. Partition $N \setminus \{0\}$ into four sets A_1, A_2, B_1, and B_2 where: $A_1 = \{a_1, c_1, a_3, c_3\}$; $|A_2| = n - 17$; $\{a_2, c_2, a_4, c_4\} \subseteq A_2$; $|B_1| = |B_2| = 6$; $\{b_1, d_1, b_2, d_2\} \subseteq B_1$; and $\{b_3, d_3, b_4, d_4\} \subseteq B_2$ (see Figure 5).

Let $P_1 = \{A_1, B_1\}, P_2 = \{A_2, B_1\}, P_3 = \{A_1, B_2\}$, and $P_4 = \{A_2, B_2\}$. For $1 \le i, j \le 2$, let $(A_i \cup B_j, B^*_{i+2j-2})$ be a 6-cycle system of $K_{|A_i|,|B_j|}$;

this exists by Lemma 2.1. Without loss of generality, for $1 \le i \le 4$ we may assume each set B_i^* of 6-cycles contains the 6-cycle $C_i = (a_i, b_i, c_i, d_i, x_i, y_i)$ for some x_i, y_i.

Since R contains x cycles of lengths $l_1, l_2, \ldots, l_x$, let $L = \{1, \ldots, y\}$, and $z = |\{i \in L \mid j_i < l_i\}|$. We may assume $j_i < l_i$ for $1 \le i \le z$. Using the assumption in the statement of this lemma, we can obtain a partial 6-cycle system $(\{0\} \cup A_1 \cup A_2, B_5^*)$ with leave R_i' for $1 \le i \le z$, and $y+1 \le i \le x$, where each cycle in R_i' has length $l_i - j_i$ if $1 \le i \le z$, and length l_i for $y+1 \le i \le x$. We may assume that for $1 \le i \le z$, R_i' contains the edge $\{a_i, c_i\}$.

By Lemma 2.4, there exists a partial 6-cycle system $(\{0\} \cup B_1 \cup B_2, B_6^*)$ with leave R_i'' where each cycle has length j_i for $1 \le i \le y$. We may assume that for $1 \le i \le z$, R_i'' does not contain vertex 0 and does contain the edge $\{b_i, d_i\}$.

Finally, consider the partial 6-cycle system $(K_n, B_1^* \cup B_2^* \cup B_3^* \cup B_4^* \cup B_5^* \cup B_6^*)$. For $1 \le i \le z$, first replace the 6-cycle C_i by the 6-cycle $(a_i, c_i, b_i, d_i, x_i, y_i)$. Then replace the leave cycles R_i' and R_i'' by the single leave cycle on edges $(E(R_i') \cup E(R_i'') \cup \{\{a_i, b_i\}, \{c_i, d_i\}\}) \setminus \{\{a_i, c_i\}, \{b_i, d_i\}\}$ of length l_i. □

4 The Main Result

The complete solutions for $n \le 17$ found in Lemmas 2.3 - 2.6 form the basis for an inductive argument which we need to split into the two following results. The first considers the harder cases where $n \equiv 1, 3$ or $5 \pmod{12}$ where $|\mathcal{I}(R)| \le 1$. The second case uses this result to handle the cases where $n \equiv 7, 9$ or $11 \pmod{12}$. These are then gathered together to prove the main theorem, Theorem 4.1.

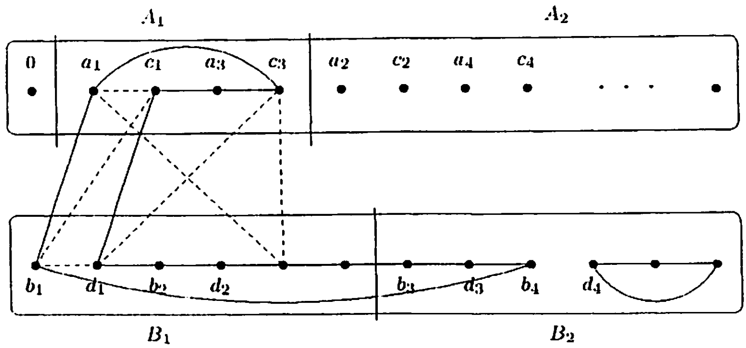


Figure 5: Using Lemma 3.2 with $l_1 = 13, j_1 = 9$, and $l_2 = j_2 = 3$.

Proposition 4.1 *Suppose $n \equiv 1, 3$, or 5 (mod 12), and $n \neq 5$. Assume that for all α satisifying $7 \leq \alpha < n$, and for any set of vertex disjoint cycles in K_α that satisfy*

1') $|E(K_\alpha - E(R'))|$ is divisible by 6, and

2') α is odd,

there exists a 6-cycle system of $K_\alpha - E(R')$. If R is a set of vertex disjoint cycles of K_n such that 6 divides $|E(K_n - E(R))|$ then there exists a 6-cycle system of $K_n - E(R)$.

Proof: In view of Lemmas 2.4, 2.5, and 2.6, we can assume that $n \geq 25$ and in view of assumption (*) we know that $|\mathcal{I}(R)| \in \{0, 1\}$ (see Table 1).

Case 1: Suppose R contains cycles whose lengths add to 9. There are four possible choices for R, namely $R \in \{R_1, R_2, R_3, R_4\}$ (the vertices are named explicity for our construction of B; see Figure 6) where:

$R_1 = \{(0, n-1, n-2), (1, n-3, n-4), (2, n-5, n-6)\}$,

$R_2 = \{(n-1, 0, n-2, 1, n-3), (n-4, 2, n-5, n-6)\}$,

$R_3 = \{(n-1, 0, n-2, 1, n-3, n-4), (n-5, 2, n-6)\}$, and

$R_4 = \{(n-1, 0, n-2, 1, n-3, 2, n-4, n-5, n-6)\}$.

Form a partial 6-cycle system $(\mathbb{Z}_n, B)$ of $K_n - E(R)$ as follows. Define

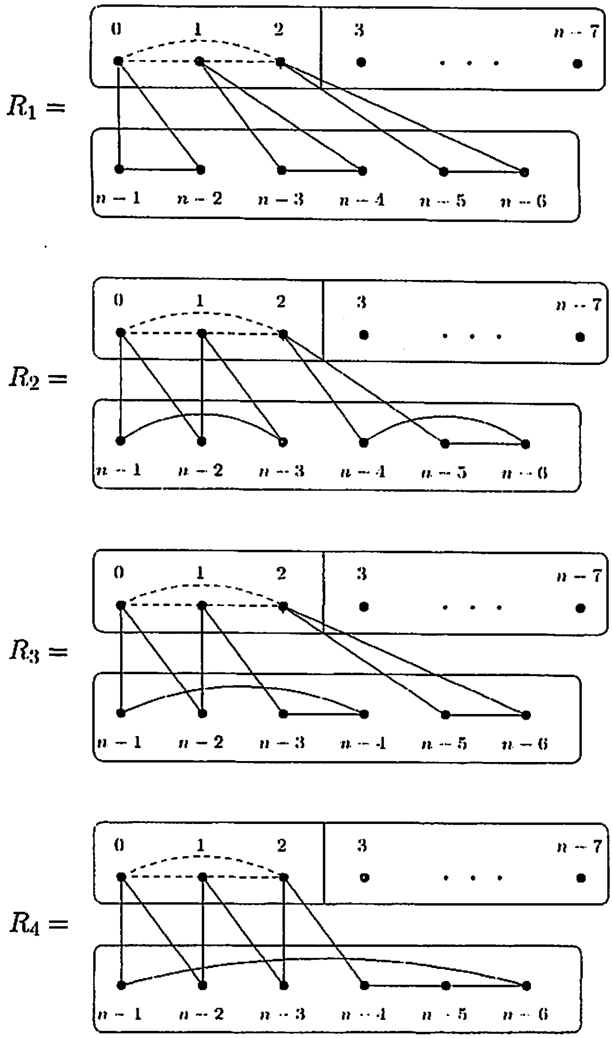


Figure 6: Graphs containing R_i isomorphic to G_i.

$R'_i = R[\mathbb{Z}_{n-6}]$, and let $\alpha = n-6 \geq 7$. Then for $1 \leq i \leq 4$, $|E(R'_i)| = |E(R)|-9 \equiv |E(R)|-3 \pmod 6$. Since we assumed that 6 divides $|E(K_n - E(R))|$, by Table 1, we have that 6 divides $|E(K_\alpha - E(R'_i))|$. Clearly since n is odd, $\alpha = n-6$ is odd. So we can apply the assumption in this proposition to obtain a 6-cycle system $(\mathbb{Z}_{n-6}, B_1)$ of $|E(K_\alpha - E(R'_i))|$.

Since $n \geq 25$, clearly $n-9 \geq 4$, and clearly $n-9$ is even and $|\mathbb{Z}_n \setminus \mathbb{Z}_{n-6}|$ is 6. So by Lemma 2.1 there exists a 6-cycle system $(\mathbb{Z}_n, B_2)$ of $K_{n-9,6}$ with bipartition $\{\mathbb{Z}_{n-6} \setminus \mathbb{Z}_3, \mathbb{Z}_n \setminus \mathbb{Z}_{n-6}\}$ of the vertex set.

Finally, by Lemma 3.1, there exists a 6-cycle system $((\mathbb{Z}_n \setminus \mathbb{Z}_{n-6}) \cup \mathbb{Z}_3, B_3)$ of $K_9 - (E(R_i) \cup \{\{0,1\},\{1,2\},\{2,0\}\}) \cong G_i$. (Notice that the edges in the 3-cycle $(0,1,2)$ are in a 6-cycle in B_1.)

Since $B_1 \cup B_2 \cup B_3$ are cycles covering all edges in K_n except those in R, we have that $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(R)$.

Case 2: Suppose R contains a set of cycles R_1 whose lengths add to 12. Let $V(R_1) = \mathbb{Z}_n \setminus \mathbb{Z}_{n-12}$. Define $R' = R[\mathbb{Z}_{n-12}]$, and let $\alpha = n-12$. Then $|E(R')| = |E(R)| - 12$. Since we assumed that 6 divides $|E(K_n - E(R))|$, by Table 1, we have that 6 divides $|E(K_{n-12} - E(R'))|$. Clearly since n is odd, $n-12$ is odd. So we can apply the assumption in this proposition to obtain a 6-cycle system $(\mathbb{Z}_{n-12}, B_1)$ of $|E(K_{n-12} - E(R'))|$.

Since $n \geq 25$, clearly $n-13 \geq 4$, and clearly $n-13$ is even and $|\mathbb{Z}_n \setminus \mathbb{Z}_{n-12}|$ is divisible by 6, so by Lemma 2.1 there exists a 6-cycle system $(\mathbb{Z}_n, B_2)$ of $K_{n-13,12}$ with bipartition $\{\mathbb{Z}_{n-12} \setminus \mathbb{Z}_1, \mathbb{Z}_n \setminus \mathbb{Z}_{n-12}\}$ of the vertex set.

Finally, by Lemma 2.4, there exists a 6-cycle system $((\mathbb{Z}_n \setminus \mathbb{Z}_{n-12}) \cup \mathbb{Z}_1, B_3)$ of $K_{13} - E(R_1)$.

Since $B_1 \cup B_2 \cup B_3$ are cycles covering all edges in K_n except those in R, we have that $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(R)$.

Case 3: Suppose R contains a set of cycles R_1 whose lengths add to 15. Let $V(R_1) = \mathbb{Z}_n \setminus \mathbb{Z}_{n-15}$. Let $C = (n-15, n-14, n-13)$. Define the 2-regular graph $R' = R[\mathbb{Z}_n \setminus \mathbb{Z}_{n-15}] \cup C$ and let $\alpha = n - 15$. Then $|E(R')| = |E(R)| - 15 + 3 = |E(R)| - 12$. Since we assumed that 6 divides $|E(K_n - E(R))|$, by Table 1 we have that 6 divides $|E(K_{n-12} - E(R'))|$. Since n is odd clearly $n - 12$ is odd. So we can apply the assumption in this proposition to obtain a 6-cycle system $(\mathbb{Z}_{n-12}, B_1)$ of $K_{n-12} - E(R')$.

Since $n \geq 25$, clearly $n - 15 \geq 4$ and clearly $n - 15$ is even, and $|\mathbb{Z}_n \setminus \mathbb{Z}_{n-12}|$ is clearly divisible by 6, so by Lemma 2.1 there exists a 6-cycle system $(\mathbb{Z}_n, B_2)$ of $K_{n-15,12}$ with bipartition $\{\mathbb{Z}_{n-15}, \mathbb{Z}_n \setminus \mathbb{Z}_{n-12}\}$ of the vertex set.

Finally by Lemma 2.5, there exists a 6-cycle system $(\mathbb{Z}_n \setminus \mathbb{Z}_{n-15}, B_3)$ of $K_{15} - E(R)$. (Notice that the edges in C do not occur in a 6-cycle in B_1; each of these edges occur in a cycle in $R \setminus R'$ or in a 6-cycle in B_3.) Since $B_1 \cup B_2 \cup B_3$ is a collection of 6-cycles covering all edges in K_n except those in R. We have that $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle ststem of $K_n - E(R)$.

Case 4: Having dealt with these three cases, we need to determine what possiblities are left.

Let $N(R) = (n_3, n_4, \ldots, n_\beta)$, where β is the length of the longest cycle in R, and where n_i is the number of i-cycles in R. If R contains a cycle of length $l_1 \geq 15$, by choosing $x = 1$ and $j_1 = 12$ apply Lemma 3.2. We can now assume that $\beta \leq 14$. Similarly, we can assume that: by Case 1, $n_3 \leq 2$ and $n_9 = 0$; by Case 2, $n_4 \leq 2$, $n_6 \leq 1$, and $n_{12} = 0$; by Case 3, $n_5 \leq 2$ and $n_{15} = 0$; by Lemma 3.2, $n_7 \leq 2$ (if $n_7 = 3$ then choose $y = 3$ and $j_1 = j_2 = j_3 = 4$), and similarly by Lemma 3.2 $n_8 \leq 1$, $n_{10} \leq 1$, $n_{11} \leq 1$, $n_{13} \leq 1$, and $n_{14} \leq 1$. So we can now assume that $N(R)$ (expressed with $\beta = 14$) is majorized by $N = (2,2,2,1,2,1,0,1,1,0,1,1)$ (the sequence $N(R)$ is *majorized* by $(s_1, s_2, \ldots, s_\beta)$ if $n_i \leq s_i$ for $1 \leq i \leq \beta$). To form the required 6-cycle system $(\mathbb{Z}_n, B)$ of $K_n - E(R)$, we will use Lemma 3.2 and define B by considering various cases in turn.

Case 4.1: Suppose R contains exactly one cycle of length $l_1 \in \{13, 14\}$. Then since $|E(R)| \geq 24$, R contains another cycle say of length l_2. If $l_2 \geq 6$, let $j_1 = 9$ and $j_2 = 3$, and if $l_2 \leq 5$, let $j_1 = 12 - l_2$ and $j_2 = l_2$. In either case apply Lemma 3.2 to form B.

Case 4.2: Suppose R contains a longest cycle of length $l_1 = 11$. Then since $|E(R)| \geq 24$ and since R contains at most two 3-cycles, R contains another cycle with length $l_2 \geq 4$. If $l_2 \geq 7$, let $j_1 = 8$ and $j_2 = 4$. If $l_2 \leq 6$, let $j_1 = 12 - l_2$ and $j_2 = l_2$. In either case apply Lemma 3.2 to form B.

Case 4.3: In view of Cases 4.1 and 4.2, we can now assume that $\beta \leq 10$. Suppose R contains a cycle with length $l_1 = 10$. Then since $|E(R)| \geq 24$ and $n_{10} = 1$, R must contain at least 3 cycles. Suppose R contains two cycles with lengths l_2 and l_3. If $l_2, l_3 \leq 4$, let $j_1 = 12 - l_2 - l_3$, $j_2 = l_2$, and $j_3 = l_3$. If $5 \leq l_2 \leq 7$, let $j_1 = 12 - l_2$ and $j_2 = l_2$. If $l_2 \geq 8$, let $j_1 = 7$ and $j_2 = 5$. In any case apply Lemma 3.2 to form B.

Case 4.4: We can now assume $\beta \leq 8$. Suppose R contains one cycle of length $l_1 = 8$. Since $|E(R)| \geq 24$, R contains more cycles.

If $n_7 \geq 1$, then let $l_2 = 7$, $j_1 = 8$ and $j_2 = 4$.

If $n_7 = 0$ and $n_6 = 1$ then let $l_2 = 6$. There must be a third cycle $l_3 \in \{3, 4, 5\}$.

If $l_3 = 5$, then let $j_1 = 4$, $j_2 = 3$, and $j_3 = 5$.

If $l_3 = 4$, then let $j_1 = 5$, $j_2 = 3$, and $j_3 = 4$.

If $l_3 = 3$, then let $j_1 = 3$, $j_2 = 6$, and $j_3 = 3$.

If $n_7 = n_6 = 0$, then in view of Lemma 3.2, it must be that $N(R) = (2, 0, 2, 0, 0, 1)$; let $j_1 = 4$, $l_2 = j_2 = 5$, and $l_3 = j_3 = 3$.

In any case, apply Lemma 3.2 to form B.

Case 4.5: We can now assume $\beta \leq 7$. Suppose R contains two cycles of length $l_1 = l_2 = 7$. Since $l_1 + l_2 < 24 \leq |E(R)|$ then there exists a third cycle in R of length l_3. If $n_6 = 1$, then let $j_1 = 3$, $j_2 = 3$, and $l_3 = j_3 = 6$. If $n_5 \geq 1$, then let $j_1 = 3$, $j_2 = 4$, and $l_3 = j_3 = 5$. If $n_4 \geq 1$, then let $j_1 = j_2 = j_3 = l_3 = 4$. Since $|E(R)| \geq 24$, it is impossible for $n_6 = n_5 = n_4 = 0$ since $7 + 7 + 3 + 3 < 24$. Apply Lemma 3.2 to form B.

Case 4.6: Suppose R contains at most one 7-cycle. By Case 1: either $n_4 = 0$ or $n_5 = 0$: and either $n_3 = 0$ or $n_6 = 0$. Also, $N(R)$ is majorized by N, so $3n_3+6n_6 \leq 6$ and $4n_4+5n_5 \leq 10$. Therefore $|E(R)| \leq 7+6+10 < 24$. But we know that $|E(R)| \geq 24$, a contradiction. So this is not possible.

Therefore all possibilities of R have been exhausted, and the result is proved.

□

Proposition 4.1 exhausts the cases where $n \equiv 1, 3$, or 5 (mod 12), $n \neq 5$. We now complete the proof by considering the cases $n \equiv 7, 9$, or 11 (mod 12).

Proposition 4.2 *Suppose $n \equiv 7, 9, 11$ (mod 12). Assume that for all α satisifying $7 \leq \alpha < n$, and for any set R' of vertex disjoint cycles in K_α that satisfy*

1′) $|E(K_\alpha - E(R'))|$ is divisible by 6, and

2′) α is odd,

there exists a 6-cycle system of $K_\alpha - E(R')$. If R is a set of vertex disjoint cycles of K_n such that 6 divides $|E(K_n - E(R))|$ then there exists a 6-cycle system of $K_n - E(R)$.

Proof: In view of Lemma 2.3, we can assume that $n \geq 19$.

From Table 1, we see that since $n \equiv 7, 9$, or 11 (mod 12) R has at least 3 isolated vertices; so we can assume that $n-4$, $n-5$, and $n-6$ are isolated vertices. We will consider 3 cases in turn. In each case R' will be formed so that $|E(R')| = |E(R)| - 3$ (so also $I(R') = I(R) - 3$), and α is chosen to be $n - 6 \geq 13$. Since we assume that 6 divides $|E(K_n - E(R))|$, by Table 1 we have that 6 divides $|E(K_{n-6} - E(R'))|$ (since $I(R') = I(R) - 3$), so (1′) is satisfied. Also $n - 6$ is odd since n is odd so (2′) is satisfied. We can apply the assumption in this proposition to obtain a 6-cycle system $(\mathbb{Z}_{n-6}, B_1)$ of $K_{n-6} - E(R')$.

Case 1: Suppose R contains a 3-cycle. Call the 3-cycle $C = (n-1, n-2, n-3)$. Let $R' = R[\mathbb{Z}_{n-6}]$. Then $|E(R')| = |E(R)| - 3$.

Since $n \geq 19$, clearly $n - 7 \geq 4$, and clearly $n - 7$ is even and since $|\mathbb{Z}_n \setminus \mathbb{Z}_{n-6}|$ is divisible by 6, by Lemma 2.1 there exists a 6-cycle system $(\mathbb{Z}_n, B_2)$ of $K_{n-7,6}$ with bipartition $\{\mathbb{Z}_{n-6} \setminus \mathbb{Z}_1, \mathbb{Z}_n \setminus \mathbb{Z}_{n-6}\}$ of the vertex set.

Finally, by Lemma 2.3 there exists a 6-cycle system $(\mathbb{Z}_n \backslash \mathbb{Z}_{n-6} \cup \{0\}, B_3)$ of $K_7 - E(C)$. Since $B_1 \cup B_2 \cup B_3$ is a collection of 6-cycles covering all edges in K_n except those in R, we have that $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(R)$.

Case 2: Suppose R contains a cycle of length $x \geq 6$; call this x-cycle $C = (0, n-1, n-2, 1, n-3, 2, 3, 4, \ldots, x-4)$. Let R' be formed from $R[\mathbb{Z}_{n-6}]$ by adding edges $\{0, 1\}$ and $\{1, 2\}$. Then $|E(R')| = |E(R)| - 3$.

Since $n \geq 19$, clearly $n - 9 \geq 4$, and clearly $\mathbb{Z}_{n-6} \setminus \mathbb{Z}_3$ is even and $|\mathbb{Z}_n \setminus \mathbb{Z}_{n-6}|$ is divisible by 6, so by Lemma 2.1 there exists a 6-cycle system $(\mathbb{Z}_n, B_2)$ of $K_{n-9,6}$ with bipartition $\{\mathbb{Z}_{n-6} \setminus \mathbb{Z}_3, \mathbb{Z}_n \setminus \mathbb{Z}_{n-6}\}$ of the vertex set.

Finally, by Lemma 2.3, there exists a 6-cycle system $(\mathbb{Z}_n \backslash \mathbb{Z}_{n-6} \cup \mathbb{Z}_3, B_3)$ of $K_9 - E((0, n-1, n-2, 1, n-3, 2))$. (Notice that if $x \neq 6$ then the edge $\{0, 2\}$ occurs in a 6-cycle in B_1.)

Since $B_1 \cup B_2 \cup B_3$ contain cycles covering all edges in K_n except those in R, we have that $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(R)$.

Case 3: Suppose R contains only cycles of length 4 and 5. Since by Table 1 $|E(R)| = 12x + y$ for some $y \in \{3, 6, 7\}$ so $|E(R)| = 4(3z) + z'$ where $z' \geq 2$. Therefore R has at least two 5-cycles. Choose two 5-cycles and call them $C_1 = (0, n-1, n-2, 1, 2)$ and $C_2 = (3, n-3, 4, 5, 6)$. Let R' be formed from $R[\mathbb{Z}_{n-6}]$ by adding edges $\{0, 1\}$ and $\{3, 4\}$. Then $|E(R')| = |E(R)| - 3$.

Since $n \geq 19$, clearly $n - 11 \geq 4$, and clearly $|\mathbb{Z}_{n-6} \setminus \mathbb{Z}_5|$ is even and $|\mathbb{Z}_n \setminus \mathbb{Z}_{n-6}|$ is divisible by 6, so by Lemma 2.1, there exists a 6-cycle system $(\mathbb{Z}_n, B_2)$ of $K_{11,6}$ with bipartition $\{\mathbb{Z}_{n-6} \setminus \mathbb{Z}_5, \mathbb{Z}_n \setminus \mathbb{Z}_{n-6}\}$ of the vertex set.

Finally, there exists a 6-cycle system $((\mathbb{Z}_n \setminus \mathbb{Z}_{n-6}) \cup \mathbb{Z}_5, B_3)$ of $K_5^c \vee K_6 + \{\{0,1\},\{3,4\}\} - \{\{0,n-1\},\{1,n-2\},\{3,n-3\},\{4,n-3\},\{n-1,n-2\}\}$ defined by $B_3 = \{(0,1,n-6,n-5,n-1,n-3), (0,n-6,n-2,n-3,1,n-5), (2,n-6,n-1,n-4,n-2,n-5), (3,n-4,n-3,n-5,4,n-2), (0,n-4,4,n-6,2,n-2), (1,n-4,n-6,4,3,n-1), (2,n-4,n-5,3,n-6,n-3)\}$. We have that $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(R)$. □

Finally, we present the main theorem of this paper.

Theorem 4.1 *Let R be a 2-regular graph in the complete graph K_n. There exists a 6-cycle system of $G = K_n - E(R)$ if and only if:*
1) $|E(K_n - E(R))|$ is divisible by 6,
2) n is odd, and
3) $n \neq 5$.

Proof: Suppose that there exists a 6-cycle system (V, B) of $G = K_n - R$. Since the 6-cycles in B partition the edges of G, 6 must divide $|E(K_n - E(R))|$. For each $v \in V$, the edges in B incident with v are partitioned into pairs by the 6-cycles. Since R is 2-regular there are 0 or 2 edges incident with v in R. Therefore $d(v)$ is even. It follows that since $n = d(v) + 1$, n is odd. Finally, suppose $n = 5$. Clearly there is no 6-cycle on 5 vertices so $B = \phi$. It follows that all of the edges in K_5 must be in R. But the edges remaining do not induce a 2-regular graph. Therefore $n \neq 5$.

To prove the sufficiency, we begin by observing that if $n \in \{1,3,7,9,11,13,15,17\}$ then Lemmas 2.3, 2.4, 2.5, and 2.6 provide a 6-cycle system of $K_n - E(R)$. We can now assume that $n \geq 19$.

If $n \geq 19$ then the sufficiency follows by induction by applying Proposition 4.2 if $n \equiv 7, 9$, or 11 (mod 12) and applying Proposition 4.1 if $n \equiv 1, 3$, or 5 (mod 12). □

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