Closest Approximations to Real Numbers *

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Abstract

A rational number p/q is said to be a closest approximation to a given real number α provided it is closer to α than any other rational number with denominator *at most* q. We determine the sequence of closest approximations to α, giving our answer in terms of the simple continued fraction expansion of α.

Keywords: Continued fraction, convergents, best approximations, closest approximations
AMS Subject Classification (2000): 11J

The study of approximations of real numbers by rationals is an integral part of the Theory of Numbers. The central theme to this study is the representation of real numbers by their simple continued fraction, and the

*To appear in *Ars Combinatoria*

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simplest sequence of rational numbers in the sense of size of denominators is obtained by looking at the sequence of *convergents* to this continued fraction expansion.

A systematic and elementary introduction to the theory of continued fractions is usually available in most books on Number Theory. For the sake of completeness, we introduce some elementary properties of simple continued fractions. All these and more can be found in [3] or in [4], for instance.

Let α denote a real number with continued fraction expansion

$$\alpha \ = < a_0, a_1, \ldots, a_n, \ldots > .$$

Such an expression for α is *unique* except for the case when α is a rational number. When α is rational, such an expansion is necessarily finite, and since a_n may be replaced by $< a_n - 1, 1 >$, unique only provided we insist that the last term a_n be chosen greater than 1 always.

The numerators and denominators of rational numbers

$$\frac{p_n}{q_n} \doteq < a_0, a_1, \ldots, a_n >,$$

termed the *convergents* to α, satisfy the second order recurrence

$$u_n = a_n u_{n-1} + u_{n-2} \ \forall \ n \geq 0, \tag{1}$$

with $p_{-2} = 0 = q_{-1}$ and $p_{-1} = 1 = q_{-2}$. In particular, we have

$$\alpha = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}} \ \forall \ n \geq -1, \tag{2}$$

where $\alpha_n \doteq < a_n, a_{n+1}, \ldots >$.

From the identity

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \ \forall \ n \geq -1, \tag{3}$$

one can now derive

$$\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(\alpha_{n+1} q_n + q_{n-1})} \ \forall \ n \geq 0, \tag{4}$$

which justify the terminology of convergent used for the rational numbers p_n/q_n.

The convergents turn out to be the only *best approximations* to α (see [1]). A reduced rational number p/q is a best approximation to α provided

$$|q\alpha - p| = ||q\alpha|| \le ||q'\alpha|| \ \forall\ 1 \le q' \le q, \tag{5}$$

where $||x||$ is the distance from x to its nearest integer.

Therefore, if p/q is a best approximation to α, then p is the integer closest to $q\alpha$ and if p' is any integer,

$$\left|\alpha - \frac{p}{q}\right| \le \left|\alpha - \frac{p'}{q'}\right| \ \forall\ 1 \le q' \le q \tag{6}$$

Let Q be any positive integer. It follows from (5) that the best approximation among all rational numbers with denominators not exceeding Q is given by the convergent p_n/q_n such that $q_n \le Q < q_{n+1}$. However, if we look to minimize $|\alpha - p/q|$ among all rationals with denominators not exceeding Q, the answer may not always be found among the convergents; see [3], for instance. The purpose of this note is to address this problem.

Definition: Let $\alpha \in \mathbb{R}$. A reduced rational number p/q is said to be a *closest approximation* to α if

$$\left|\alpha - \frac{p}{q}\right| < \left|\alpha - \frac{p'}{q'}\right| \ \forall\ 1 \le q' \le q,\ \frac{p'}{q'} \ne \frac{p}{q}.$$

Thus, if we let

$$d_\alpha(q) \doteq \min_{p \in \mathbb{Z}} \left|\alpha - \frac{p}{q}\right|,$$

p/q is a closest approximation to α provided $d_\alpha(q) < d_\alpha(q')$ for $1 \le q' < q$ for $q > 1$.

Observe that for fixed $\alpha \in \mathbb{R}$ and for a fixed $q \in \mathbb{N}$, the integer p such that $d_\alpha(q) = |\alpha - p/q| = ||q\alpha||/q$ is the integer closest $q\alpha$, which we call the α*−pair* of q. Note that p is the α*−pair* of q if and only if $2|q\alpha - p| \le 1$.

The identity

$$\frac{ap_n + p_{n-1}}{aq_n + q_{n-1}} - \frac{bp_n + p_{n-1}}{bq_n + q_{n-1}} = \frac{(-1)^{n-1}(a-b)}{(aq_n + q_{n-1})(bq_n + q_{n-1})} \tag{7}$$

shows that $\gcd(ap_n + p_{n-1}, aq_n + q_{n-1}) = 1$ for all integers a.
Moreover, for any n and a $(0 \le a \le a_{n+1})$, the $\alpha -$*pair* of $q = aq_n + q_{n-1}$

is $p = ap_n + p_{n-1}$ since

$$2|q\alpha - p| = 2q\left|\frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n-1}} - \frac{ap_n + p_{n-1}}{aq_n + q_{n-1}}\right| = 2\frac{|\alpha_{n+1} - a|}{\alpha'_{n+1}} < 1,$$

where $\alpha'_n \doteq \alpha_n q_{n-1} + q_{n-2}$, *unless* $q_n = 1$. However, if $q_n = 1$, $p_n = a_0$, $q_{n-1} = 0$ and $p_{n-1} = 1$, and the $\alpha - pair$ of $q = a$ is $p = aa_0 + 1$.

Suppose Q is any positive integer. Choose the largest n such that $q_n \leq Q$, and set $k \doteq \lfloor (Q - q_{n-1})/q_n \rfloor$. Then $0 \leq k < a_{n+1}$ and

$$kq_n + q_{n-1} \leq Q < (k+1)q_n + q_{n-1} \tag{8}$$

We claim that any rational number p/q between p_n/q_n and $(kp_n + p_{n-1})/(kq_n + q_{n-1})$ must have denominator $q > Q$.

From (4) and (7) we see that the numerator of

$$\left(\alpha - \frac{p_n}{q_n}\right)\left(\alpha - \frac{kp_n + p_{n-1}}{kq_n + q_{n-1}}\right)$$

equals $-(\alpha_{n+1} - k) < 0$, since $k < a_{n+1} \leq \alpha_{n+1}$. Therefore, α lies between p_n/q_n and $(kp_n + p_{n-1})/(kq_n + q_{n-1})$. Moreover,

$$\left|\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_n(kq_n + q_{n-1})}, \tag{9}$$

whereas from (4) and (7) again we have

$$\begin{aligned}\left|\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}} - \frac{p_n}{q_n}\right| &= \left|\frac{p}{q} - \frac{p_n}{q_n}\right| + \left|\frac{p}{q} - \frac{kp_n + p_{n-1}}{kq_n + q_{n-1}}\right| \\ &\geq \frac{1}{qq_n} + \frac{1}{q(kq_n + q_{n-1})} \\ &= \frac{(k+1)q_n + q_{n-1}}{qq_n(kq_n + q_{n-1})} \end{aligned} \tag{10}$$

Finally, from (8), (9) and (10), we arrive at $q > Q$. This shows that the closest approximation to α must always be one of p_n/q_n and $(kp_n + p_{n-1})/(kq_n + q_{n-1})$, and it remains to determine which of these is the closest approximation.

From (7), we see that the sequence $\left\{\frac{ap_n + p_{n-1}}{aq_n + q_{n-1}}\right\}_{a=0}^{a_{n+1}}$ is monotonic, and since p_n/q_n is closer to α than p_{n-1}/q_{n-1}, we have

$$d_\alpha((a+1)q_n + q_{n-1}) < d_\alpha(aq_n + q_{n-1}) \;\forall\; 0 \leq a \leq a_{n+1} - 1 \tag{11}$$

Now, since $d_\alpha(q_{n+1}) < d_\alpha(q_n) < d_\alpha(q_{n-1})$, there is a *smallest* a between 0 and a_{n+1} for which $d_\alpha(aq_n + q_{n-1}) < d_\alpha(q_n)$. From (2), (4) and (7), such an a satisfies the inequality

$$\left|\frac{\alpha_{n+1} - a}{(aq_n + q_{n-1})(\alpha_{n+1}q_n + q_{n-1})}\right| < \left|\frac{1}{q_n(\alpha_{n+1}q_n + q_{n+1})}\right|,$$

or

$$\frac{\alpha_{n+1} - a}{aq_n + q_{n-1}} < \frac{1}{q_n}, \tag{12}$$

which is the same as $(\alpha_{n+1} - 2a)q_n < q_{n-1}$. The last inequality can be written as

$$2a > \alpha_{n+1} - \frac{q_{n-1}}{q_n} = a_{n+1} + \ < 0, a_{n+2}, a_{n+3}, \ldots > - \ < 0, a_n, a_{n-1}, \ldots, a_1 >, \tag{13}$$

so that the smallest such a equals $\lceil a_{n+1}/2 \rceil$ *unless* a_{n+1} is even and

$$< 0, a_{n+2}, a_{n+3}, \ldots > \ \geq \ < 0, a_n, a_{n-1}, \ldots, a_1 >,$$

in which case it equals $(a_{n+1} + 2)/2$.

We have thus proved the

Theorem: Let $\alpha = \ < a_0, a_1, \ldots, a_n, \ldots >$. Then p/q is a closest approximation to α if and only if p/q equals $(a_0 + \lfloor 1/a_1 \rfloor)/1$ or is of the form $(ap_n + p_{n-1})/(aq_n + q_{n-1})$, where $n \geq 0$ and

$$\begin{cases} (a_{n+1} + 2)/2 \leq a \leq a_{n+1} & \text{if } a_{n+1} \text{ is even and } < 0, a_{n+2}, a_{n+3}, \ldots > \\ & \geq \ < 0, a_n, a_{n-1}, \ldots, a_1 >; \\ \lceil a_{n+1}/2 \rceil \leq a \leq a_{n+1} & \text{otherwise.} \end{cases}$$

There is a simple way to check whether or not the inequality

$$< 0, a_{n+2}, a_{n+3}, \ldots > \ \geq \ < 0, a_n, a_{n-1}, \ldots, a_1 >$$

holds. Suppose $a_{n+1+i} = a_{n+1-i}$ for all i with $0 \leq i \leq k - 1$ but $a_{n+1+k} \neq a_{n+1-k}$. Then the given inequality holds precisely when k is *even* and $a_{n+1+k} \geq a_{n+1-k}$ or when k is *odd* and $a_{n+1+k} \leq a_{n+1-k}$. All this can be written more briefly as equivalent to $(-1)^k(a_{n+1+k} - a_{n+1-k}) \geq 0$.

We close this note with a table of *best* and of *closest* approximations to π.

Closest and Best Approximations to $\pi = < 3, 7, 15, 1, 292, 1, 1, 1, 2, 1, \ldots >$

n	a_{n+1}	p_n	q_n	a	p $(ap_n + p_{n-1})$	q $(aq_n + q_{n-1})$	$\lVert q\pi \rVert$	$\frac{1}{q} \lVert q\pi \rVert$
-1	3	1	0	3	3	1	$0.141592653590\cdots$	$0.141592653590\cdots$
0	7	3	1	4	13	4		$0.108407346410\cdots$
0	7	3	1	5	16	5		$0.058407346410\cdots$
0	7	3	1	6	19	6		$0.025074013077\cdots$
0	7	3	1	7	22	7	$0.008851424871\cdots$	$0.001264489267\cdots$
1	15	22	7	8	179	57		$0.001241776397\cdots$
1	15	22	7	9	201	64		$0.000967653590\cdots$
1	15	22	7	10	223	71		$0.000747583167\cdots$
1	15	22	7	11	245	78		$0.000567012564\cdots$
1	15	22	7	12	267	85		$0.000416183002\cdots$
1	15	22	7	13	289	92		$0.000288305764\cdots$
1	15	22	7	14	311	99		$0.000178512176\cdots$
1	15	22	7	15	333	106	$0.008821280518\cdots$	$0.000083219628\cdots$
2	1	333	106	1	355	113	$0.000030144354\cdots$	$0.000000266764\cdots$
3	292	355	113	146	52163	16604		$0.000000266213\cdots$
3	292	355	113	147	52518	16717		$0.000000262611\cdots$
3	292	355	113	148	52873	16830		$0.000000259056\cdots$
3	292	355	113	149	53228	16943		$0.000000255549\cdots$
3	292	355	113	150	53583	17056		$0.000000252089\cdots$
3	292	355	113	$\vdots$	$\vdots$	$\vdots$		$\vdots$
3	292	355	113	288	102573	32650		$0.000000004279\cdots$
3	292	355	113	289	102928	32763		$0.000000003344\cdots$
3	292	355	113	290	103283	32876		$0.000000002416\cdots$
3	292	355	113	291	103638	32989		$0.000000001494\cdots$
3	292	355	113	292	103993	33102	$0.000019129233\cdots$	$0.000000000578\cdots$

Acknowledgement: The authors are grateful to the referee for his comments and suggestions.

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