Maximal-clique Partitions of Different Sizes

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Abstract

A *maximal-clique partition* of a graph is a family of its maximal complete subgraphs that partitions its edge set. Many graphs do not have a maximal-clique partition, while some graphs have more than one. It is harder to find graphs in which maximal-clique partitions have different sizes. $L(K_5)$ is a well-known example. In 1982, Pullman, Shank, and Wallis [9] asked if there is a graph with fewer vertices than $L(K_5)$ with this property. This paper confirms that there is no such graph.

1 Introduction

For our purpose, graphs are simple. *Cliques* are complete subgraphs of a graph that are not necessarily maximal. The number of vertices in a clique is its *order*. A clique of order n is also called an n-clique or K_n. A 3-clique is also referred to as a *triangle*. The triangle on the set of vertices $\{a, b, c\}$ is represented by $\triangle(a, b, c)$. A *clique partition* of G is a family C of cliques of G such that every edge of G lies in exactly one member of C. If every element in C is maximal, then C is a *maximal-clique partition* of G. The number of cliques in a maximal-clique partition is its *size*.

The subject of clique coverings of graphs has its origins in the problem of representing set intersections by graphs. See Erdös, Goodman, and Pósa [3], Lovasz [5], and Harary [4]. Many authors have investigated this topic, e.g., Orlin [8], Pullman and de Caen [10], [11] and Pullman, Shank, and Wallis [9]. More recently, clique coverings and partitions were studied by Cacetta *et al.* [1], Monson *et al.* [7], and others. The 1995 paper by Monson, Pullman, and Rees [6] is an excellent survey.

Many graphs have no maximal-clique partition. For example, if $n \geq 4$, then the graph obtained by deleting one edge from K_n has none. The *line graph* $L(G)$ of a graph G is the graph whose vertices correspond to edges of G as follows: $e \in V(L(G)) \Leftrightarrow e \in E(G)$ and for any $e_1, e_2 \in V(L(G))$, e_1 is adjacent to e_2 if and only if edges e_1 and e_2 are adjacent in G.

Pullman, Shank, and Wallis [9] found that the line graph of K_5 has exactly two maximal-clique partitions, one of size 10 and one of size 5. They asked whether there is a graph with fewer vertices than $L(K_5)$ with maximal-clique partitions of different sizes. Our main result will confirm that there is none.

Since we are especially interested in this example, in the following we will examine its properties in some detail.

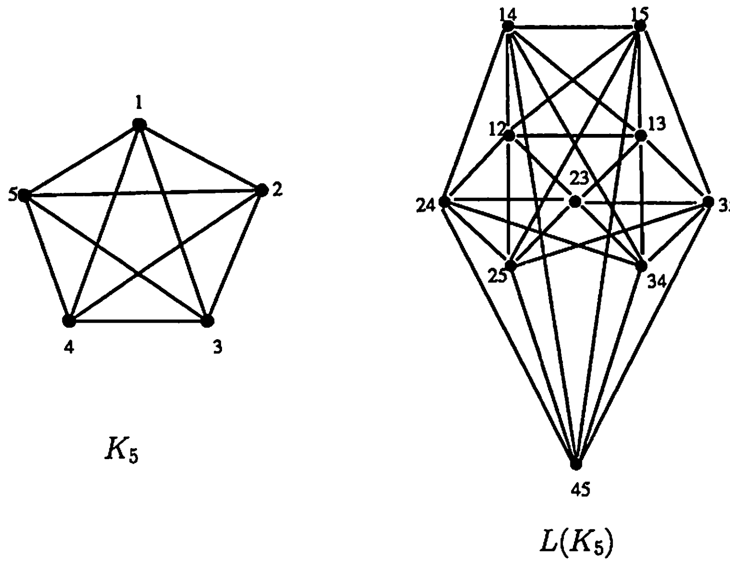

Figure 1: $L(K_5)$: The line graph of K_5

Since K_5 has ten edges, $L(K_5)$ has ten vertices. In Figure 1, each vertex in $L(K_5)$ is labelled according to its inducing edge in G. Then two distinct vertices in $L(K_5)$ are adjacent if and only if their labels share a digit. For each $i = 1, 2, \ldots, 5$, there are four vertices in $L(K_5)$ whose labels contain i; these vertices generate a 4-clique. Thus, we have five 4-cliques partitioning the edge set of $L(K_5)$. Moreover, since $L(K_5)$ has no cliques of order larger than four, the set of those five 4-cliques is a maximal-clique partition of $L(K_5)$. Next, consider the ten 3-sets of $\{1, 2, 3, 4, 5\}$. Let $\{x, y, z\}$ be any 3-set of $\{1, 2, 3, 4, 5\}$. Then vertices xy, yz and xz form a triangle in $L(K_5)$. Consider any vertex outside the triangle $\Delta(xy, yz, xz)$. Its label can contain at most one digit in $\{x, y, z\}$. Thus, it is adjacent to at most two vertices in $\Delta(xy, yz, xz)$. That is, $\Delta(xy, yz, xz)$ is maximal. Hence, a 3-set of $\{1, 2, 3, 4, 5\}$ induces a maximal triangle in $L(K_5)$. Furthermore, the ten maximal triangles induced from the ten 3-sets partition the edge set of $L(K_5)$, forming a maximal-clique partition of $L(K_5)$. Therefore, we have two maximal-clique partitions of $L(K_5)$, one of size 5 and the other of size 10. In general, the intersection graph of the 2-sets of an n-set, $n \geq 5$, (or $L(K_n)$) always has two maximal-clique partitions of different sizes.

2 The result

First, we start by preparing a list of properties of graphs with at least two maximal-clique partitions for the proof of the main theorem. We will apply the classic theorem of de Bruijn and Erdős [2] as formulated in graph theoretic terms by Orlin in [8].

Theorem 1. (N.G. de Bruijn and P.Erdős [2], 1948)
If $\mathcal{C}$ is a clique partition of K_n and $1 < |\mathcal{C}| \leq n$, then either

(i) $\mathcal{C}$ consists of one replica of K_{n-1} and $n-1$ replicas of K_2 or

(ii) $\mathcal{C}$ consists of n replicas of K_{m+1} and $n = m^2 + m + 1$.

If a clique C is covered by maximal cliques in a maximal-clique partition, then the edge set of C is partitioned into smaller cliques by those maximal cliques. Theorem 1 guarantees that an n-clique is decomposed into at least n smaller cliques. Hence, it follows that we need at least n maximal cliques from a maximal-clique partition to cover an n-clique that does not belong to the maximal-clique partition. Now with this idea, we can apply Theorem 1 to get the following lemma and theorems. Three simple facts about clique partitions are listed in the first lemma for easy reference.

Lemma 1.

(i) The intersection of two different maximal cliques of a graph is empty or is a clique of order less than the order of either.

(ii) Any two cliques in the same clique partition of a graph share at most one vertex.

(iii) Let G be a graph with a maximal-clique partition. Let $\mathcal{M}$ be any maximal-clique partition of G. Then each n-clique that does not belong to $\mathcal{M}$ needs at least n different cliques in $\mathcal{M}$ to cover its edges.

Proof. The proofs are simple, but here are the details:

(*i*) Clearly the nonempty intersection of two cliques is a clique. If the intersection is a clique of the same order as one of the original cliques, then one clique is properly contained in the other, contradicting its maximality.

(*ii*) If two cliques in a maximal-clique partition $\mathcal{P}$ share two vertices, they will share the edge joining these vertices, contradicting the fact that $\mathcal{P}$ is a partition.

(*iii*) Let K be an n-clique outside $\mathcal{M}$. Let $\mathcal{P} = \{C \in \mathcal{M} : E(C) \cap E(K) \neq \emptyset\}$ and $\mathcal{P}' = \{K \cap C : C \in \mathcal{P}\}$. Notice that members of $\mathcal{P}'$ are cliques and by Lemma 1(*ii*), pairwise they share at most one vertex, thus $\mathcal{P}'$ is a clique partition of K. Moreover, both types of clique partitions of K in Theorem 1 have size n, which means that K has no clique partitions of size i, where $1 < i < n$. Hence, $|\mathcal{P}| = |\mathcal{P}'| \geq n$. □

Theorem 2. *Let G be a graph with a maximal-clique partition. Let $\mathcal{M}$ be any maximal-clique partition of G, then*

(i) If G has an n-clique K that does not belong to $\mathcal{M}$ and if every pair of maximal cliques in $\mathcal{M}$ covering edges of K shares a vertex in K, then G has at least $2n$ vertices.

(ii) If G has an n-clique K that does not belong to $\mathcal{M}$ and there exist two maximal cliques in $\mathcal{M}$ covering some edges of K but not sharing a vertex in K, then G has at least $n+6$ vertices.

Proof. (i) Let K be an n-clique that does not belong to $\mathcal{M}$ such that every two maximal cliques in $\mathcal{M}$ covering K share a vertex of K. By Lemma 1(iii), K is covered by at least n cliques in $\mathcal{M}$. Let X and Y be two maximal cliques in $\mathcal{M}$ covering K. Then X and Y share a vertex in K. By Lemma 1(i), each of X and Y must contain a vertex that is not contained in K. However, since they share a vertex in K, by Lemma 1(ii), they cannot share a vertex not in K. Hence, each maximal clique that covers K contains a distinct vertex not in K, and there are at least n vertices not in K. These n vertices together with the n vertices in K give at least $2n$ vertices in G.

(ii) Assume G has an n-clique K that does not belong to $\mathcal{M}$, and there exist two maximal cliques, M_1 and M_2, in $\mathcal{M}$ covering edges of K and not sharing a vertex in K. Then each of them contains at least one edge of K, i.e., each of them contains at least two vertices of K. Let ab and $cd \in E(K)$ be contained in M_1 and M_2, respectively. Because M_1 and M_2 do not share a vertex of K, ac, ad, bc and bd cannot be in M_1 or M_2 and they must be in different maximal cliques in $\mathcal{M}$. This yields a 4-clique $\boxtimes(a,b,c,d)$ which is a subset of K covered by six maximal cliques in $\mathcal{M}$.

For convenience, for any $i,j \in \{a,b,c,d\}$, let M_{ij} be the maximal clique in $\mathcal{M}$ covering edge ij of K. Hence, M_1 and M_2 are renamed M_{ab} and M_{cd}, respectively. It is sufficient to show that there exist at least six different extra vertices not in K. By Lemma 1(i), each M_{ij} contains an extra vertex not in K. If all such extra vertices are different, we have six extra vertices as desired. Otherwise, there are two maximal cliques sharing the same extra vertex outside K. By Lemma 1(ii), they cannot share another vertex. Without loss of generality, say M_{ab} and M_{cd} share the same extra vertex v_1. For all $i,j \in \{a,b,c,d\}$, let $P_{ij} = M_{ij} \cap K$. It follows that vertices in $\{v_1\} \cup V(P_{ab}) \cup V(P_{cd})$ form a clique properly containing the clique formed by $\{v_1\} \cup V(P_{ab})$. Since M_{ab} is maximal, it cannot be the clique composed of vertices in $\{v_1\} \cup V(P_{ab})$. Thus, M_{ab} must contain another vertex, say v_2. Similarly, vertices in $\{v_1\} \cup V(P_{ab}) \cup V(P_{cd})$ form a clique properly containing the clique formed by $\{v_1\} \cup V(P_{cd})$; hence, M_{cd} contains another vertex, say v_3. If $v_3 = v_2$ then v_1 and v_2 are two vertices, both of which are in M_{ab} and M_{cd}, contradicting Lemma 1(ii). Hence v_3 is not equal to v_2. Now, let v_4 and v_5 be extra vertices of M_{ac} and M_{ad}, respectively, that is, $v_4 \in M_{ac} \setminus K$ and $v_5 \in M_{ad} \setminus K$. Since edges ac and ad

share a, by Lemma 1(ii), $v_4 \neq v_5$. Furthermore, ac and ad share a vertex with both ab and cd; hence, $v_4, v_5 \notin \{v_1, v_2, v_3\}$.

Next let u be an extra vertex of M_{bc} outside K. Then u cannot be v_1, v_2, v_3 or v_4 because bc shares a vertex with ab, cd and ac. If u is not v_5, we have six different extra vertices as desired. If u is v_5, vertices in $\{v_5\} \cup V(P_{bc}) \cup V(P_{ad})$ form a clique properly containing the clique formed by $\{v_5\} \cup V(P_{bc})$; hence, M_{bc} must contain at least one more vertex outside $\{v_1, v_2, v_3, v_4, v_5\}$; thus, we get the sixth extra vertex. Hence, G contains at least six extra vertices not in K. These six vertices together with the n vertices of K give at least $n + 6$ vertices in G. □

We know that $L(K_5)$ with ten vertices has two maximal-clique partitions of different sizes. The next theorem confirms that any graph on a smaller number of vertices than 10 cannot have maximal-clique partitions of different sizes. Hence, ten is the minimum number of vertices of graphs with this property.

Theorem 3. *If G is a graph with at least two maximal-clique partitions of different sizes, then $|V(G)| \geq 10$.*

Proof. Suppose the theorem is not true. Let G be a graph of at most nine vertices with at least two maximal-clique partitions of different sizes. Let $\mathcal{M}$ and $\mathcal{N}$ be two maximal-clique partitions of G of different sizes. If $\mathcal{M}$ and $\mathcal{N}$ have nonempty intersection, remove all edges of the cliques in the intersection from G to get a graph G' of at most nine vertices with two maximal-clique partitions $\mathcal{M} \setminus \mathcal{N}$ and $\mathcal{N} \setminus \mathcal{M}$. Because $\mathcal{M}$ and $\mathcal{N}$ are different and they partition the edges of G, neither $\mathcal{M} \setminus \mathcal{N}$ or $\mathcal{N} \setminus \mathcal{M}$ is empty. Hence, without lost of generality, assume that $\mathcal{M}$ and $\mathcal{N}$ have empty intersection. Note that neither $\mathcal{M}$ nor $\mathcal{N}$ can contain a 2-clique, because if a 2-clique is a maximal clique, it must be in every maximal-clique partition. Hence, maximal cliques of G in $\mathcal{M} \cup \mathcal{N}$ are cliques of order at least three. By Theorem 2(i) and (ii), if there is a clique of order at least five that not belong to a maximal-clique partition, G has at least ten vertices. Since G has at most nine vertices, $\mathcal{M}$ and $\mathcal{N}$ can not contain cliques of order at least five. Thus, $\mathcal{M}$ and $\mathcal{N}$ are composed of 3-cliques and 4-cliques.

Let m and n be the numbers of 4-cliques in $\mathcal{M}$ and $\mathcal{N}$, respectively, and let s and t be the numbers of 3-cliques in $\mathcal{M}$ and $\mathcal{N}$, respectively. Since $\mathcal{M}$ and $\mathcal{N}$ are clique partitions of G, counting the number of edges we have $6m + 3s = 6n + 3t$. If $m = n$, then $s = t$. Then $|\mathcal{M}| = m + s = n + t = |\mathcal{N}|$, which contradicts $|\mathcal{M}| \neq |\mathcal{N}|$. Hence, $m \neq n$. We can assume without loss of generality that $m > n$. Then $m \geq 1$.

To prove the theorem, it suffices to prove that $\mathcal{M}$ contains a 4-clique covered by six different maximal cliques in $\mathcal{N}$. If this occurs, we can apply Theorem 2(ii) to conclude that G has at least $4 + 6 = 10$ vertices contradicting $|V(G)| \leq 9$. Then it follows that the theorem is true.

Suppose that every 4-clique in $\mathcal{M}$ is covered by fewer than six maximal cliques in $\mathcal{N}$. Hence, each 4-clique in $\mathcal{M}$ must share at least a triangle with some maximal clique in $\mathcal{N}$. Moreover, we can conclude by Lemma 1(i) that they share exactly a triangle and that such a maximal clique in $\mathcal{N}$ must be a 4-clique. However, if any two 4-cliques in $\mathcal{M}$ share a triangle with the same 4-clique in $\mathcal{N}$, they must share at least two vertices, contradicting Lemma 1(ii). Thus, each 4-clique in $\mathcal{M}$ shares a triangle with a 4-clique in $\mathcal{N}$ and no two 4-cliques in $\mathcal{M}$ share a triangle with the same 4-clique in $\mathcal{N}$. Therefore, the number of 4-cliques in $\mathcal{M}$ is at most the number of 4-cliques in $\mathcal{N}$, or $m \leq n$. This contradicts $m > n$. Hence, we have the desired result and the theorem is proved. □

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