ON THE TREES WHOSE 2-STEP COMPETITION NUMBERS ARE TWO

HAN HYUK CHO*† and SUH-RYUNG KIM*‡
Department of Mathematics Education
Seoul National University, Seoul 151-742, Korea
YUNSUN NAM
Biochip Project Team
Samsung Advanced Instutite of Technology
P.O. Box 111, Suwon 440-600, Korea

Abstract

Since Cohen introduced the notion of competition graph in 1968, various variations have been defined and studied by many authors. Using the combinatorial properties of the adjacency matrices of digraphs, Cho *et al.* [2] introduced the notion of a m-step competition graph as a generalization of the notion of a competition graph. Then they [3] computed the 2-step competition numbers of complete graphs, cycles, and paths. However, it seems difficult to compute the 2-step competition numbers even for the trees whose competition numbers can easily be computed. Cho *et al.* [1] gave a sufficient condition for a tree to have the 2-step competition number two. In this paper, we show that this sufficient condition is also a necessary condition for a tree to have the 2-step competition number two, which completely characterizes the trees whose 2-step competition numbers are two. In fact, this result turns out to characterize the connected triangle-free graphs whose 2-step competition numbers are two.
Key Words: 2-step competition graph, 2-step competition number, trees, triangle-free graphs

1 Introduction

Since Cohen [4] introduced the notion of competition graph in 1968, various variations such as competition common enemy graph (competition resource graph), niche graph, p-competition graph have been defined and studied by many authors (see [6, 7, 8] for surveys of the literature of competition graphs). Recently Cho *et al.* [2] introduced another variant called the "m-step competition graph" of a digraph using the combinatorial properties of the adjacency matrices of digraphs. Given a digraph D, a vertex z of D is

*The authors thank the KOSEF for its support under grant Com^{2}MaC-KOSEF.
†Research supported by Choheung Research Fund through Seoul National University.
‡Corresponding author. *email address*: srkim@snu.ac.kr

called an *m-step common prey* for x and y if there are two directed walks of length m one of which is from x to z and the other from y to z. The *m-step competition graph* of D has the same vertex set as D and an edge between vertices x and y if and only if x and y have an m-step common prey in D. By this definition, this new notion of m-step competition graph generalizes that of competition graph as the 1-step competition graph is the competition graph. Given a graph G, the *m-step competition number* of G is the smallest number k such that G together with k isolated vertices is the m-step competition graph of an acyclic digraph. In their paper, Cho *et al.* [2] found the 2-step competition numbers of complete graphs, paths and cycles. After having found those numbers, it seems to be natural to seek for the 2-step competition numbers of trees. However, it does not appear to be easy to give the 2-step competition number of a tree in general while the competition number of any nontrivial tree is known to be one. Cho *et al.* [1] showed that the 2-step competition number of any graph without isolated vertices should be greater than or equal to two. Based on this observation, they defined $\mathcal{T}(m,k,n)$ as the collection of the trees on n vertices with m-step competition number k, and started to look into the trees belonging to $\mathcal{T}(2,2,n)$.

Cho *et al.* [1] defined two classes $\mathcal{T}_1$ and $\mathcal{T}_2$ of trees: Given a tree T and an edge xy in T, $T - xy$ has exactly two components one of which x belongs to and the other of which y belongs to. We denote the former by T_x and the latter by T_y. In addition, given an edge yz of T_y, we denote by S_y and S_z, respectively, the component of $T_y - yz$ to which y belongs and the component of $T_y - yz$ to which z belongs. We also mean by N_x (resp. N_z) the set of the pendant vertices of T_x (resp. S_z) adjacent to x (resp. z).

Let $\mathcal{T}_1$ be the set of all the trees with property: Any T in $\mathcal{T}_1$ has an edge xy such that (i) $(T_x - N) \simeq T_y$ for some $N \subset N_x$ or (ii) $(T_x - N - \nu) \simeq T_y$ for some $N \subset N_x$ and some pendant vertex ν of T_x not in N_x. See Figure 1 for an illustration for the trees in $\mathcal{T}_1$.

Let $\mathcal{T}_2$ be the set of all the trees with property: Any T in $\mathcal{T}_2$ has an edge xy and an edge yz in T_y such that for some induced subgraphs $S_{y\alpha}$, $S_{y\beta}$ of S_y satisfying $V(S_y) = V(S_{y\alpha}) \cup V(S_{y\beta})$ and $V(S_{y\alpha}) \cap V(S_{y\beta}) = \{y\}$, one of the following is true:

(i) $(T_x - N) \simeq S_{y\beta}$ and $S_{y\alpha} \simeq (S_z - N')$ for some $N \subset N_x$ and some $N' \subset N_z$;

(ii) $(T_x - N - \nu) \simeq S_{y\beta}$ and $S_{y\alpha} \simeq (S_z - N')$ for some $N \subset N_x$, $N' \subset N_z$, and some pendant vertex ν of T_x not in N_x;

(iii) $(T_x - N) \simeq S_{y\beta}$ and $(S_{y\alpha} - \nu) \simeq (S_z - N')$ for some $N \subset N_x$, $N' \subset N_z$, and some pendant vertex ν of $S_{y\alpha}$;

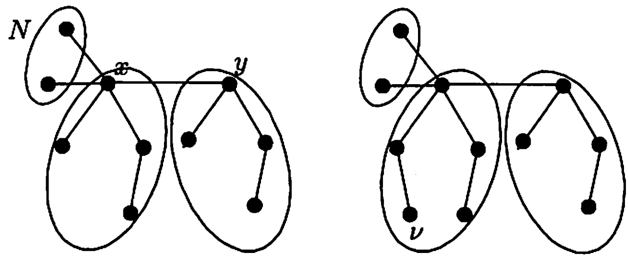

Figure 1: $T, T' \in \mathcal{T}_1$ satisfying $(T_x - N) \simeq T_y$; $(T'_x - N - \nu) \simeq T'_y$

where $|N'| \le \deg_{S_{y,\beta}}(y) + 1$ in each case. See Figure 2 for an illustration for the trees in $\mathcal{T}_2$.

Then they gave the following theorem.

Theorem 1 *If a tree T with n vertices belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, then T belongs to $\mathcal{T}(2,2,n)$.*

2 The main theorem

Cho *et al.* [1] conjectured that the converse of Theorem 1 is also true. The following theorem whose proof is give in the next section shows that their conjecture is true:

Theorem 2 *For a tree T with n vertices, T belongs to $\mathcal{T}(2,2,n)$ if and only if T belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$.*

The above theorem actually gives the characterization for a connected triangle-free graph whose 2-step competition number is two. To see why, we consider a connected triangle-free graph G whose 2-step competition number is two. Then there is an acyclic digraph D with *acyclic labeling* $v_1, v_2, \ldots, v_{n+2}$ whose 2-step competition graph is $G \cup \{v_{n+1}, v_{n+2}\}$. An *acyclic labeling* of the vertex set $V(D)$ of an acyclic digraph D is a labeling of $V(D)$ using the set $\{v_1, v_2, \ldots, v_{n+2}\}$ so that $i < j$ holds whenever there is an arc (v_i, v_j) in D. It is a well-known theorem that every acyclic digraph has an acyclic labeling. Note that v_1, v_2, v_3 cannot be used as 2-step common prey and so there are at most $|V(G)| - 1$ vertices in D which are available for 2-step common prey. Furthermore, since G is connected and

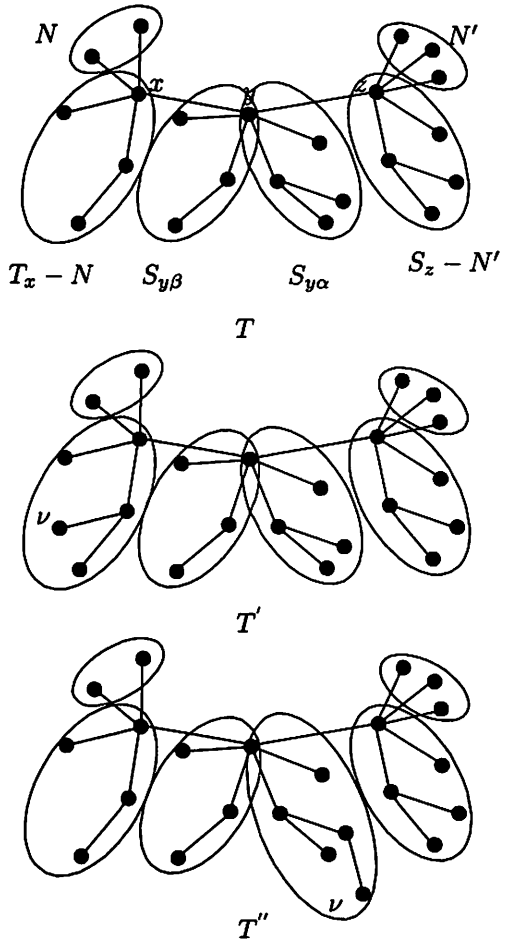

Figure 2: T, T', T'' $\in \mathcal{T}_2$ satisfying ; $(T_x - N) \simeq S_{y\beta}$ and $S_{y\alpha} \simeq (S_z - N')$; $(T'_x - N - \nu) \simeq S_{y\beta}$ and $S_{y\alpha} \simeq (S_z - N')$; $(T''_x - N) \simeq S_{y\beta}$ and $(S_{y\alpha} - \nu) \simeq (S_z - N')$.

triangle-free, a vertex can be used as a 2-step common prey for exactly two distinct vertices and therefore at least $|V(G)|-1$ vertices are used as 2-step common prey in D. Thus, we can conclude that $|E(G)| = |V(G)| - 1$ and G must be a tree. Hence G belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$ by Theorem 2 and we have the following theorem:

Theorem 3 *The 2-step competition number of a connected triangle-free graph G is two if and only if G belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$.*

3 The proof of Theorem 2

In this section, we prove Theorem 2. We first give lemmas which characterize the structure of the elements in $\mathcal{T}(2,2,n)$. As it can easily be seen that K_2 is the only tree that belongs to $\mathcal{T}(2,2,2)$, from now on, we will consider $\mathcal{T}(2,2,n)$ for $n > 2$. Given $T \in \mathcal{T}(2,2,n)$, let D be an acyclic digraph whose 2-step competition graph is $T \cup \{a,b\}$ where a and b are extra isolated vertices. Let $v_1, v_2, \ldots, v_{n+2}$ be an acyclic labeling of $V(D)$. Clearly the labels of a and b are v_{n+1} and v_{n+2}, and we also label the corresponding vertices of T as $v_1, v_2, \ldots, v_n$. Since the edge clique cover number of T is $n-1$ and any of v_1, v_2, v_3 cannot be used as 2-step common prey, all of $v_4, \ldots, v_{n+2}$ should be used as a 2-step common prey. We call an arc (v_i, v_j) in D a *jump-arc* when $i+1 < j$. Since v_4 must be used as a 2-step prey and v_4 cannot be a 2-step prey of v_3, v_4 is a 2-step common prey of v_1 and v_2 and therefore v_1v_2 is an edge of T.

Lemma 4 *Given $T \in \mathcal{T}(2,2,n)$, let D be an acyclic digraph whose 2-step competition graph is $T \cup \{a,b\}$ where a and b are extra isolated vertices. Let $v_1, v_2, \ldots, v_{n+2}$ be an acyclic labeling of $V(D)$. Then the following are true:*

(1) For any v_i, $2 \le i \le n+1$, there exists an arc (v_i, v_{i+1}) in D.

(2) If v_iv_j for $i < j$ is an edge in T, then v_{j+2} is a 2-step common prey of v_i and v_j, and either (v_i, v_{j+1}) or (v_{i+1}, v_{j+2}) is a jump-arc of D.

(3) If there is no incoming jump-arc toward v_j, $2 < j < n+2$, then there should be an incoming jump-arc toward v_{j+1}.

(4) If there is an incoming jump-arc toward $v_j \in V(D)$, then there is no outgoing jump-arc from v_j.

Proof. We use induction on n to prove (1) and (2) together. If $n = 3$, then v_4 should be the 2-step common prey for v_1 and v_2. Thus there exists a directed path of length two in D from v_2 to v_4. The only possible such directed path is $v_2 \to v_3 \to v_4$ and therefore there are arcs (v_2, v_3) and (v_3, v_4) in D. Similarly v_5 should be a 2-step common prey for v_i ($i = 1$ or 2) and v_3. Thus there exists a path of length two from v_3 to v_5, and (v_4, v_5) is an arc in D. For $n = 3$, it can easily be checked that (2) holds. Now suppose that (1) and (2) are true for any tree on less than n vertices whose 2-step competition number is two. Take $T \in \mathcal{T}(2,2,n)$ and let D be an acyclic digraph whose 2-step competition graph is $T \cup \{a, b\}$ where a and b are extra isolated vertices. Since T is connected and D is acyclic, v_{n+2} should be the only 2-step prey of v_n. Thus there exist arcs (v_n, v_{n+1}) and (v_{n+1}, v_{n+2}) in D. Furthermore the degree of v_n in T is one. It follows that $D^* = D - v_{n+2}$ is an acyclic digraph whose 2-step competition graph T^* is a tree on $n-1$ vertices together with isolated vertices v_n and v_{n+1}. By the induction hypothesis, there exists arc (v_i, v_{i+1}) for any i, $2 \leq i \leq n$, and if $v_i v_j$ for $i < j$ is an edge in T^* then there is either jump-arc (v_i, v_{j+1}) or jump-arc (v_{i+1}, v_{j+2}) in D^*. Since it has been shown that (v_{n+1}, v_{n+2}) is in D, (1) follows. Suppose that $v_i v_n$ is an edge in T for some i, $1 \leq i \leq n-1$. Since it has been shown that v_{n+2} is the only 2-step common prey of v_n, there is a directed path $v_i \to v_l \to v_{n+2}$ in D. If $v_l \neq v_{i+1}$ or v_{n+1}, then $v_{l-1} \neq v_i$ or v_n and v_i, v_{l-1}, v_n induce a triangle K_3, which is a contradiction. Thus there is either arc (v_i, v_{n+1}) or arc (v_{i+1}, v_{n+2}) in D and (2) follows.

Now we prove (3). Now suppose that there is no incoming jump-arc toward v_j for any j, $2 < j < n+2$. Since v_{j-1} is adjacent to a vertex v_i in T ($i < j-1$), either (v_i, v_j) or (v_{i+1}, v_{j+1}) is an arc in D by (2). Since there is no incoming jump-arc toward v_j by the supposition, (v_{i+1}, v_{j+1}) is an arc in D and (3) follows.

We prove (4) by contradiction. Suppose both (v_i, v_j) and (v_j, v_k) are jump-arcs in D. Then v_i, v_{j-1}, v_{k-2} induce a triangle K_3 and we reach a contradiction. Thus (4) follows. □

Throughout the rest of this section, we mean an 'acyclic digraph' D to be a digraph with an acyclic labeling v_1, v_2, $\ldots$, $v_{|V(D)|}$ such that $i < j$ whenever (v_i, v_j) is in the digraph. Given $T \in \mathcal{T}(2,2,n)$, we also denote by $\mathcal{D}_T$ the set of minimal digraphs among the acyclic digraphs whose 2-step competition graphs are $T \cup \{a, b\}$ for some extra isolated vertices a and b. Let

$$\mathcal{T}^*(2,2,n) = \{T \in \mathcal{T}(2,2,n) \mid \text{For every } D_T \text{ in } \mathcal{D}_T,\ D_T \text{ has arc } (v_1, v_2).\}$$

Lemma 5 *For a tree T in $\mathcal{T}(2,2,n) - \mathcal{T}^*(2,2,n)$, there exist two adjacent vertices x and y in T such that any vertex other than x and y is adjacent* *to x or y.*

Proof. Since T is not in $\mathcal{T}^*(2,2,n)$, there exists an acyclic digraph D_T in $\mathcal{D}_T$ in which there is no arc (v_1, v_2). From Lemma 4(1), it follows that for any i, $4 \le i \le n+2$, v_i is a 2-step common prey of v_{i-2} and some vertex of an index less than $i-2$. Since v_1 and v_2 are the only possible 2-step predators of v_4, there is an arc (v_1, v_3) in D_T, and v_1 and v_2 are joined in T. By induction on the index of a vertex in D_T, we claim in the following that for any i, $3 \le i \le n+1$, there is a jump-arc from either v_1 or v_2 to v_i. As we have shown above, (v_1, v_3) is a jump-arc. Now suppose that for some k, $3 \le k \le n$, there is a jump-arc from either v_1 or v_2 to v_i for any i, $3 \le i \le k$. Since v_{k+2} is a 2-step common prey of v_k and v_j for some $j < k$, there exists either jump-arc (v_j, v_{k+1}) or jump-arc (v_{j+1}, v_{k+2}) in D_T by Lemma 4(2). Suppose (v_j, v_{k+1}) is in D_T. If $j \ne 1$ and $j \ne 2$, then (v_1, v_j) or (v_2, v_j) is in D_T by the induction hypothesis. Either case contradicts Lemma 4(4). Thus $j = 1$ or $j = 2$ and the claim follows. Now suppose (v_{j+1}, v_{k+2}) is in D_T. Since (v_1, v_3) is a jump-arc, $j \ne 2$ by Lemma 4(4). If $j \ge 3$, then either (v_1, v_{j+1}) or (v_2, v_{j+1}) is in D_T by the induction hypothesis and either case contradicts Lemma 4(4). Thus $j = 1$ and the claim follows. Now take a vertex v_j for $j \ge 3$. Then either (v_1, v_{j+1}) or (v_2, v_{j+1}) is an arc of D_T by the claim. By Lemma 4(1), $v_1 \to v_{j+1} \to v_{j+2}$ or $v_2 \to v_{j+1} \to v_{j+2}$, and therefore v_j is adjacent to v_1 or v_2. □

As Lemma 5 characterizes the trees in $T(2,2,n) - T^*(2,2,n)$, it remains to characterize the trees in $T^*(2,2,n)$. Given $T \in \mathcal{T}^*(2,2,n)$ and $D_T \in \mathcal{D}_T$, we partition the edge set of T in terms of D_T as follows. Let $e = v_i v_j$ $(i < j)$ be an edge in T. By Lemma 4(2) and the minimality of the digraphs in $\mathcal{D}_T$, exactly one of (v_i, v_{j+1}) or (v_{i+1}, v_{j+2}) is an arc of D_T. We say that e is an *α-type edge with respect to D_T* if (v_{i+1}, v_{j+2}) is an arc of D_T. We say that e is a *γ-type edge with respect to D_T* when there is a jump arc (v_i, v_{j+1}) in D_T and either (γ1) or (γ2) is satisfied:

(γ1) $i = 1$;

(γ2) $i = 3$, and (v_2, v_j) is a jump-arc of D_T.

We say that e is a *β-type edge with respect to D_T* if e is neither of α-type nor of γ-type with respect to D_T. By these definitions, we note that v_1 can not be an end point of a β-type edge. We also note that if edge $v_i v_j$ is of α-type with respect to a digraph D_T in $\mathcal{D}_T$, then $v_{i+1} v_{j+1}$ is of β-type with respect to D_T if $j < n$. (See Figure 3 for illustration.)

Lemma 6 *Given $T \in \mathcal{T}^*(2,2,n)$ and $D_T \in \mathcal{D}_T$, let $e = v_i v_j$ for $i < j$ be an edge in T. Suppose that e is a γ-type edge with respect to D_T different* *from v_1v_2. Then v_j is a pendant vertex in T adjacent to v_1 (resp. v_3) if e satisfies ($\gamma1$) (resp. ($\gamma2$)).*

Proof. Since e is different from v_1v_2, $j > 2$. Clearly, v_j is adjacent to v_1 (resp. v_3) if e satisfies ($\gamma1$) (resp. ($\gamma2$)). Moreover there is a jump-arc from v_i to v_{j+1} in D_T by the definition of γ-type edge. Suppose v_j is not a pendant vertex. Then v_j is adjacent to some vertex v_p ($p \neq i$) in T. Then $j < p$ by (1). By Lemma 4(2), either (v_j, v_{p+1}) or (v_{j+1}, v_{p+2}) is a jump-arc in D_T. Since (v_i, v_{j+1}) is a jump-arc, there is no outgoing jump-arc from v_{j+1} by Lemma 4(4). Thus (v_j, v_{p+1}) is an arc in D_T. Then by Lemma 4 (4) there is no incoming jump-arc toward v_j. First assume that $i = 3$. Then by the definition of an γ-type edge there is a jump-arc (v_2, v_j), and we reach a contradiction. Now suppose that $i = 1$. We note that v_{j+1} is used as 2-step common prey since $j > 2$. Then there must be a jump-arc from vertex v_q for some q, $1 < q < j$, to v_{j+1} by Lemma 4(3), which implies that v_{j+2} is a 2-step prey of v_q. Thus, v_1, v_q, and v_j have common prey v_{j+2}, and so they induce a K_3 in T, which is a contradiction. □

Lemma 7 *Given $T \in \mathcal{T}^*(2,2,n)$ and $D_T \in \mathcal{D}_T$, let $e = v_iv_j$ for $i < j$ be an edge in T. Let P be a path from v_i to v_l in T traversing edge e and edge v_kv_l for some $k \in \{1, 2, \ldots, n\}$. Then the following are true:*

(1) $k < l$.

(2) If $e \neq v_1v_2$ and v_kv_l is an α-type (resp. β-type) edge with respect to D_T, then every edge on P is of α-type (resp. β-type).

Proof. We use induction on the length d of P to prove (1). If $d = 2$, then $P = v_iv_jv_l$. If $l < j$, then v_i, v_j, and v_l have v_{j+2} as 2-step common prey by Lemma 4 (2), and induce a triangle K_3 in T, which is a contradiction. Suppose the statement (1) is true when the length of P is less than or equal to $d - 1$ ($d \geq 3$). Now suppose that the length of P is d, and that v_h immediately precedes v_k on P. Then by the induction hypothesis, $h < k$. If $l < k$, then v_{k+2} is a 2-step common prey of v_h, v_l and v_k, and we reach a contradiction. Thus we have $k < l$.

Now we show that (2) holds. Suppose v_kv_l is a β-type edge. Let v_h be the vertex immediately preceding v_k on P. We note that $h < k$ by (1) and either (v_h, v_{k+1}) or (v_{h+1}, v_{k+2}) is an arc in D_T by Lemma 4(2). Suppose that (v_h, v_{k+1}) is not an arc. Then (v_{h+1}, v_{k+2}) is an arc in D_T. Since v_kv_l is a β-type edge, there is arc (v_k, v_{l+1}) in D_T. Then by Lemma 4(4), there is no incoming arc toward v_k. Thus by Lemma 4(3), there must be a jump-arc (v_g, v_{k+1}) for some $g < k$. Since (v_h, v_{k+1}) is not an arc of D_T,

$h \neq g$. Then we reach a contradiction since v_{k+2} is a 2-step common prey of v_h, v_g, and v_k. Thus (v_h, v_{k+1}) is an arc in D_T, and therefore $v_h v_k$ is of γ-type or β-type. Since v_k is not a pendant vertex, $v_h v_k$ is of β-type by Lemma 6. Repeating this argument, we can eventually show that edge $v_i v_j$ is a β-type edge. A similar argument holds for the case in which $v_k v_l$ is an α-type edge. □

Given $T \in \mathcal{T}^*(2,2,n)$ and $D_T \in \mathcal{D}_T$, label as v_i the vertex of T corresponding to v_i of D_T for $i = 1, 2, \ldots, n$. We recall that $v_1 v_2$ is an edge in T. Since every edge of a tree is a cut edge, $T - v_1 v_2$ has exactly two components. From now on, $T_1(D_T)$ and $T_2(D_T)$ denote the two components of $T - v_1 v_2$ where $v_1 \in V(T_1(D_T))$ and $v_2 \in V(T_2(D_T))$, respectively. If $v_2 v_3$ is an edge of $T_2(D_T)$, then $T_2(D_T) - v_2 v_3$ also has exactly two components, and $S_2(D_T)$ and $S_3(D_T)$ denote the two components of $T_2(D_T) - v_2 v_3$ where $v_2 \in V(S_2(D_T))$ and $v_3 \in V(S_3(D_T))$, respectively. When $v_2 v_3$ is not an edge of T, let $S_2(D_T)$ and $S_3(D_T)$ denote $T_2(D_T)$ and the trivial graph with vertex v_3, respectively.

Graphs $T_{1\alpha}(D_T)$, $S_{2\alpha}(D_T)$, $S_{2\beta}(D_T)$, $S_{3\beta}(D_T)$ are defined as follows:

$T_{1\alpha}(D_T)$ is the union of a trivial graph with vertex set $\{v_1\}$ and the subgraph of T induced by the α-type edges in $T_1(D_T)$;

$S_{2\alpha}(D_T)$ is the union of a trivial graph with vertex set $\{v_2\}$ and the subgraph of T induced by the α-type edges in $S_2(D_T)$;

$S_{2\beta}(D_T)$ is the union of a trivial graph with vertex set $\{v_2\}$ and the subgraph of T induced by the β-type edges in $S_2(D_T)$;

$S_{3\beta}(D_T)$ is the union of a trivial graph with vertex set $\{v_3\}$ and the subgraph of T induced by the β-type edges in $S_3(D_T)$.

Let $T_{1\alpha*}(D_T)$ be $T_{1\alpha}(D_T) - v_n$ if $v_n \in V(T_{1\alpha}(D_T))$, and $T_{1\alpha}(D_T)$ otherwise. Finally let $S_{2\alpha*}(D_T)$ be $S_{2\alpha}(D_T) - v_n$ if $v_n \in V(S_{2\alpha}(D_T))$, and $S_{2\alpha}(D_T)$ otherwise. (See Figure 3 for an illustration.)

Suppose that $T \in \mathcal{T}^*(2,2,n)$ and $D_T \in \mathcal{D}_T$ are given. For convenience sake, let $T_1 = T_1(D_T)$, $T_2 = T_2(D_T)$, $S_2 = S_2(D_T)$, $S_3 = S_3(D_T)$, $S_{2\alpha} = S_{2\alpha}(D_T)$, $S_{2\alpha*} = S_{2\alpha*}(D_T)$, $S_{2\beta} = S_{2\beta}(D_T)$, and $S_{3\beta} = S_{3\beta}(D_T)$. Now we present the following lemmas:

Lemma 8 *Let N' be the set of the ends other than v_3 of γ-type edges with respect to D_T satisfying (γ2). Then $|N'| \leq deg_{S_{2\beta}}(v_2) + 1$.*

Proof. Suppose that $v_3 v_j$ is a γ-type edge for some $j \geq 5$. Then by the definition of a γ-type edge, there exist arcs (v_2, v_j) and (v_3, v_{j+1}). Thus,

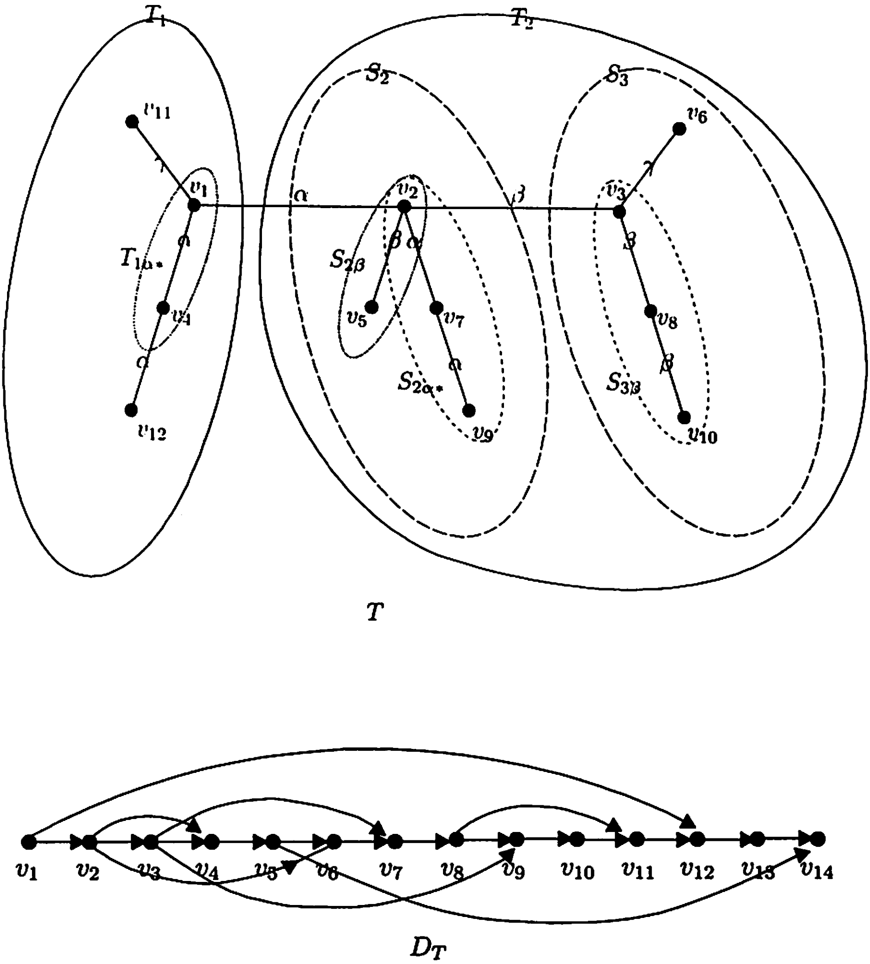

Figure 3: A tree T and an acyclic digraph D_T in $\mathcal{D}_T$. The edges of T are labeled in accordance with the definitions of α-type, β-type, γ-type edges.

edge v_2v_{j-1} is of β-type by the definition of a β-type edge. Then, by the definition of $S_{2\beta}$, v_2v_{j-1} is an edge of $S_{2\beta}$ if $j \geq 5$. Hence

$$|\{v_3v_j \mid v_3v_j \text{ is of } \gamma\text{-type and } j \geq 5\}| \leq deg_{S_{2\beta}}(v_2).$$

Since

$$N' \subset \{v_j \mid v_3v_j \text{ is of } \gamma\text{-type and } j \geq 5\} \cup \{v_4\},$$

the lemma follows. □

Lemma 9 *The followings are true:*

(1) Every non-γ-type edge in T_1 is an α-type edge, and $T_{1\alpha*}$ is a tree.

(2) There is no γ-type edge in S_2, and if v_2v_3 is not an edge of T, then every edge in T_2 is a β-type edge.

(3) $S_{2\alpha*}$ and $S_{2\beta}$ are trees.

(4) Every non-γ-type edge in S_3 is a β-type edge, and $S_{3\beta}$ is a tree.

Proof. Take a non-γ-type edge v_iv_j $(i < j)$ in T_1. Then $2 < i$. Since T_1 is a tree, there is a path P in T_1 from v_1 to v_j tranversing edge v_iv_j. Since the edge v_1v_g on P is of α-type, so is v_iv_j by Lemma 7(2). Thus every non-γ-type edge in T_1 is an α-type edge. Hence by Lemma 6, $T_{1\alpha}$ is connected. Since v_n is a pendant vertex, $T_{1\alpha*}$ is a tree. Hence (1) follows.

We show that (2) holds. Since S_2 contains neither v_1 nor v_3, there is no γ-type edge in S_2. We claim in the following that if v_2v_3 is not an edge of T, then every edge in T_2 is a β-type edge. Suppose v_2v_3 is not an edge of T. Then (v_2, v_4) is not an arc of D_T. By Lemma 4(2), (v_1, v_3) is an arc of D_T, and so, by Lemma 4(4), there cannot be a jump-arc outgoing from v_3. Thus there is no γ-type edge in T_2. If v_2 is adjacent to v_j in T_2, then v_2v_j is a β-type edge since there is no outgoing jump-arc from v_3 in D_T. For every edge v_iv_j in T_2, $2 < i < j$, there exists a path in T_2 from v_2 to v_j traversing v_iv_j, and by Lemma 7(2) v_iv_j is a β-type edge. Thus every edge in T_2 is a β-type edge if v_2v_3 is not in T_2.

Now we show that (3) holds. First suppose v_2v_3 is not an edge of T. Then $S_{2\beta} = T_2$ and $S_{2\alpha*} = v_3$ by (2), and clearly $S_{2\alpha*}$ and $S_{2\beta}$ are trees. Next suppose v_2v_3 is an edge of T. We first show that $S_{2\beta}$ is a tree. Suppose $f = v_iv_j$ $(2 < i < j)$ is an β-type edge in S_2. Then there is a unique path P in T_2 from v_2 to v_i since T_2 is a tree. By Lemma 7(2), every edge in P is a β-type edge. Thus P is in $S_{2\beta}$, and therefore every vertex in $S_{2\beta}$ is connected to v_2 in $S_{2\beta}$. Hence $S_{2\beta}$ is a tree. By a similar argument, we can show that $S_{2\alpha}$ is a tree. Since v_n is a pendant vertex, $S_{2\alpha*} = S_{2\alpha} - v_n$ is also a tree.

Finally we prove that (4) holds. It immediately follows from (2) when v_2v_3 is not an edge. Now let v_2v_3 be an edge of T. If $(v_2, v_4) \notin A(D_T)$, then $(v_1, v_3) \in A(D_T)$ since v_1v_2 is an edge of T. But, then $(v_3, v_5) \notin A(D_T)$ by Lemma 4(4), contradicting Lemma 4(2). Thus, $(v_2, v_4) \in A(D_T)$. Suppose v_3v_g is a non-γ-type edge in S_3. Then $(v_2, v_g) \notin A(D_T)$ or $(v_3, v_{g+1}) \notin A(D_T)$. Since there cannot be an outgoing jump-arc from v_4 in D_T, (v_3, v_{g+1}) is an arc of D_T by Lemma 4(2). Therefore v_3v_g is a β-type edge. Now for a non-γ-type edge v_iv_j in S_3, $3 < i < j$, there exists a path in S_3 from v_3 to v_j traversing v_iv_j. The vertex, say w, immediately following v_3 on this path is not a pendant vertex, and hence v_3w is a non-γ-type edge by Lemma 6. Thus by Lemma 4(2), v_iv_j is a β-type edge. Hence every non-γ-type edge in S_3 is of β-type. By applying a similar argument as above, it can be shown that $S_{3\beta}$ is a tree. □

Lemma 10 *It is true that (a) $T_{1\alpha*} \simeq S_{2\beta}$ and (b) $S_{2\alpha*} \simeq S_{3\beta}$.*

Proof. We prove part (a). Let ϕ be a map from $V(T_{1\alpha*})$ to $V(T)$ mapping each vertex $v_i \in V(T_{1\alpha*})$ to $v_{i+1} \in V(T)$. We claim that this map ϕ induces a graph isomorphism between $T_{1\alpha*}$ and $S_{2\beta}$. Suppose $e = v_iv_j$ $(i < j)$ is an edge in $T_{1\alpha*}$. Then $j < n$ and e is of α-type by the definition. Therefore v_{i+1} and v_{j+1} are joined in T and $f = v_{i+1}v_{j+1}$ is of β-type. We now claim that f is in $S_{2\beta}$. If $i = 1$, then clearly f is in $S_{2\beta}$. Suppose $i > 1$. By Lemma 9(1), there is a path P in a tree $T_{1\alpha*}$ from v_1 to v_i. Let $\phi(P)$ be the sequence of vertices whose ith vertex is the image of the ith vertex of P under ϕ. Then $\phi(P)$ is a path from v_2 to v_{i+1}. Since all the edges on P are α-type edges, all the edges on $\phi(P)$ are of β-type. Since P does not pass through v_2, $\phi(v) \neq v_3$ for any vertex v on P. Thus $\phi(P)$ is a path in $S_{2\beta}$ and f is in $S_{2\beta}$. Hence if v_i and v_j are adjacent in $T_{1\alpha*}$, then v_{i+1} and v_{j+1} are adjacent in $S_{2\beta}$.

Conversely, we prove that if $v_{i+1}v_{j+1}$ is an edge in $S_{2\beta}$, then v_iv_j is an edge in $T_{1\alpha*}$. Let $f = v_{i+1}v_{j+1}$ $(i < j)$ be an edge in $S_{2\beta}$. Note that $f \neq v_2v_3$ and (v_{i+1}, v_{j+2}) is a jump-arc of D_T. Suppose $i = 1$. Then $2 < j < n$ since v_{j+1} is in $S_{2\beta}$. Note that $\phi(v_1) = v_2$ and $\phi(v_j) = v_{j+1}$. Since $2 < j$ and v_2v_{j+1} is of β-type, there is a jump-arc (v_2, v_{j+2}) and, by the minimality of D_T, there is no arc (v_1, v_{j+1}). Thus by the definition of an α-type edge, $e = v_1v_j$ is an α-type edge in $T_{1\alpha*}$. Suppose $i > 1$. Since $v_{i+1}v_{j+1}$ is a β-type edge, v_i and v_j have v_{j+2} as 2-step common prey since there is a jump-arc (v_{i+1}, v_{j+2}) in D_T. Thus v_i and v_j are adjacent and $e = v_iv_j$ is an α-type edge. By Lemma 9(3), there exists a path Q in $S_{2\beta}$ from v_2 to v_{i+1}. Since every edge on Q is of β-type, every edge v_xv_y, where

$v_{x+1}v_{y+1}$ is an edge on Q, is of α-type. Since Q does not pass through v_3, it is true that $v_x \neq v_2$ for any vertex v_{x+1} on Q. Thus there is a path from v_1 to v_i in $T_{1\alpha*}$ and e is in $T_{1\alpha*}$. Therefore there is a unique α-type edge v_iv_j in $T_{1\alpha*}$ corresponding to each edge $v_{i+1}v_{j+1}$ $(i < j)$ in $S_{2\beta}$, and ϕ induces a graph isomorphism between $T_{1\alpha*}$ and $S_{2\beta}$.

Now we show that $S_{2\alpha*}$ is isomorphic to $S_{3\beta}$. First, suppose v_2v_3 is not an edge in T. Then, by Lemma 9(2), $S_{2\alpha*}$ is the trivial graph with vertex v_2 and $S_{3\beta}$ is the trivial graph with vertex v_3.

Suppose v_2v_3 is an edge in T. Now consider a map ϕ from $V(S_{2\alpha*})$ to $V(T)$ mapping each vertex $v_i \in V(S_{2\alpha*})$ to $v_{i+1} \in V(T)$. By applying a similar argument for (a), we can claim that this map ϕ induces a graph isomorphism between $S_{2\alpha*}$ and $S_{3\beta}$. □

Now we present the proof of Theorem 2.

Proof of Theorem 2. The 'if' part of the theorem is proved in [1]. We now prove the 'only if' part. Take a tree $T \in \mathcal{T}(2,2,n)$. If $T \in \mathcal{T}(2,2,n) - \mathcal{T}^*(2,2,n)$, then $T \in \mathcal{T}_1$. For, by Lemma 5, there exist two adjacent vertices x and y in T such that any vertex distinct from x and y is adjacent to x or y. We may assume the degree of x is greater than or equal to that of y in T.

Now suppose that $T \in \mathcal{T}^*(2,2,n)$. Take $D_T \in \mathcal{D}_T$. Then by the definition of $\mathcal{T}^*(2,2,n)$, there is an arc (v_1, v_2) in D_T. Let N be the set of the ends other than v_1 of γ-type edges and satisfying $(\gamma 1)$. We also let N' be the set of the ends other than v_3 of γ-type edges and satisfying $(\gamma 2)$. By Lemma 6, any vertex in N is a pendant vertex adjacent to v_1 and any vertex in N' is a pendant vertex adjacent to v_3. Thus $N \subset N_x$ and $N' \subset N_z$ if N_x (resp. N_z) is the set of the vertices adjacent to v_1 (resp. v_3). By the definition of $T_{1\alpha*}(D_T)$ and Lemma 9(1), $T_{1\alpha*}(D_T) = T_1(D_T) - N - v_n$ if v_n is in $T_{1\alpha}(D_T)$ and $T_{1\alpha*}(D_T) = T_1(D_T) - N$ otherwise. First suppose that v_2 and v_3 are not adjacent in T. Then $S_{2\beta}(D_T) = T_2(D_T)$ by the definition of $S_{2\beta}(D_T)$ and Lemma 9(2). Therefore, by Lemma 10, $T \in \mathcal{T}_1$ as we take $x = v_1$, $y = v_2$, $T_x = T_1(D_T)$, $T_y = T_2(D_T)$, and $\nu = v_n$. Now suppose that v_2 and v_3 are joined in T. Then by the definition of $S_{3\beta}(D_T)$ and Lemma 9(4) , $S_{3\beta}(D_T) = S_3(D_T) - N'$. By definition, $S_{2\alpha*}(D_T) = S_{2\alpha}(D_T) - v_n$ if $v_n \in V(S_{2\alpha}(D_T))$ and $S_{2\alpha*}(D_T) = S_{2\alpha}(D_T)$ otherwise. By Lemma 10, one of the following four cases holds:

$$T_1(D_T) - N \simeq S_{2\beta}(D_T) \text{ and } S_{2\alpha}(D_T) \simeq S_3(D_T) - N';$$
$$T_1(D_T) - N - v_n \simeq S_{2\beta}(D_T) \text{ and } S_{2\alpha}(D_T) \simeq S_3(D_T) - N';$$
$$T_1(D_T) - N \simeq S_{2\beta}(D_T) \text{ and } S_{2\alpha}(D_T) - v_n \simeq S_3(D_T) - N';$$
$$T_1(D_T) - N - v_n \simeq S_{2\beta}(D_T) \text{ and } S_{2\alpha}(D_T) - v_n \simeq S_3(D_T) - N'.$$

But v_n cannot belong to $T_{1\alpha}(D_T)$ and $S_{2\alpha}(D_T)$ at the same time, and therefore the fourth case cannot happen. Thus $T \in \mathcal{T}_2$ as we take $x = v_1$, $y = v_2$, $T_x = T_1(D_T)$, $T_y = T_2(D_T)$, $S_y = S_2(D_T)$, $S_z = S_3(D_T)$, $S_{y\beta} = S_{2\beta}(D_T)$, $S_{y\alpha} = S_{2\alpha}(D_T)$, and $\nu = v_n$. Finally, by Lemma 8, $|N'| \leq degs_{S_{y\beta}}(y) + 1$ and the theorem is proved. □

Acknowledgments

The authors would like to thank an anonymous referee for careful proofreading and helpful comments.

References

[1] H. H. Cho, S-R Kim, and Y. Nam, "A Sufficient Condition for a Tree Belonging to $\mathcal{T}(2,2,n)$," *Congressus Numerantium*, **123** (1997), 43-53.

[2] H. H. Cho, S-R Kim, and Y. Nam, "The m-Step Competition Graph of a Digraph," *Discrete Appl. Math.*, **105** (2000), 115-127.

[3] H. H. Cho, S-R Kim, and Y. Nam, "Acyclic Digraphs Whose 2-Step Competition Graphs Are $P_n \cup I_2$," *Bulletin of the Korean Mathematical Society.*, **37** (2000), 649-657.

[4] J. E. Cohen, "Interval Graphs and Food Webs: A Finding and a Problem," Document 17696-PR, RAND Corp. Santa Monica, Calif., 1968.

[5] S-R. Kim, "Competition Graphs and Scientific Laws for Food Webs and Other Systems," Ph.D. Thesis, Rutgers University, 1988.

[6] S-R. Kim, "The Competition Number and Its Variants," *Annals of Discrete Mathematics*, **55** (1993), 313-326.

[7] J. R. Lundgren, "Food Webs, Competition Graphs, Competition-Common Enemy Graphs, and Niche Graphs," in *Applications of Combinatorics and Graph Theory to the Biological and Social Sciences*, (F. S. Roberts, ed.), Springer-Verlag, "IMH Volumes in Mathematics and Its Application," Vol. 17, 1989, pp. 221-243.

[8] C. Wang, "Competition Graphs, Threshold Graphs and Threshold Boolean Functions," Ph.D. Thesis, Rutgers University, 1991.